Recursive constructions for triangulations

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Abstract

Three recursive constructions are presented; two deal with embeddings of complete graphs and one with embeddings of complete tripartite graphs. All three facilitate the construction of $2^{an^2-o(n^2)}$ non-isomorphic face 2-colourable triangulations of K_n and $K_{n,n,n}$ in orientable and non-orientable surfaces for values of n lying in certain residue classes and for appropriate constants a.

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1 Introduction

The existence of a triangular embedding of the complete graph K_n in an orientable surface for $n \equiv 3$ or 7 (mod 12) was established by Ringel [7] in the course of the project to solve the famous Heawood map colouring problem [8]. For each $n \equiv 3 \pmod{12}$ the triangular embedding of K_n described in [7] happens to be face 2-colourable, while the one for $n \equiv 7 \pmod{12}$, $n \geq 19$, is not. Later, Youngs [10] constructed for each $n \equiv 7 \pmod{12}$ a face 2-colourable triangular embedding of K_n, and also a triangulation of K_n for $n \equiv 3 \pmod{12}$, $n \geq 15$, which is not face 2-colourable, thereby complementing the constructions of [7]. For almost the next three full decades Ringel's and Youngs' embeddings were the only two known non-isomorphic triangulations of K_n in an orientable surface for $n \equiv 3$ or 7 (mod 12) and $n \geq 15$, except for the triangulations of K_{19} described in [6].

Face 2-colourability of triangular embeddings of complete graphs is an extra feature that was not necessary for the solution of the Heawood problem as given in [8]. Nevertheless, this property is interesting because it provides a strong link with design theory, thus opening up new research directions. In a face 2-colourable triangular embedding of K_n the faces in each of the two colour classes form a Steiner triple system of order n (STS(n)). The embedding may then be regarded as a simultaneous embedding of the two STS(n)s. We here recall that an STS(n) may be formally defined as an ordered pair $(V, \mathcal{B})$, where V is an n-element set (the points) and $\mathcal{B}$ is a set of 3-element subsets of V (the blocks), such that every 2-element subset of V appears in precisely one block.

In two earlier papers, [3] and [4], a recursive construction was described which generates a face 2-colourable triangulation of K_{3n-2} in an orientable (non-orientable) surface from a face 2-colourable triangulation of K_n in an orientable (non-orientable) surface. In a subsequent paper [1], the construction was generalised in a fashion which established the existence of $2^{an^2-o(n^2)}$ (where a is a constant) non-isomorphic face 2-colourable triangulations of K_n in orientable and non-orientable surfaces for certain residue classes, in particular $n \equiv 7$ or 19 (mod 36).

The current paper presents three further recursive constructions. Two of these deal with embeddings of complete graphs and one with embeddings of complete tripartite graphs. One of the former constructions can be viewed as a further generalisation of the construction given in [3] and [4]. All three enable us to produce $2^{an^2-o(n^2)}$ non-isomorphic face 2-colourable triangulations of K_n and $K_{n,n,n}$ in orientable and non-orientable surfaces for values of n lying in certain residue classes additional to the classes 7 and 19 modulo 36 covered in [1].

We assume that the reader is familiar with basic facts concerning graph embeddings in surfaces, in particular with lifts of embeddings by means of voltage assignments as treated in Chapters 2-4 of [5]. When working with embedded graphs, we shall use the same notation for the vertices and edges of the abstract graph as well as for the embedded graph; no confusion will be likely. For the greater part of this article, surfaces will be orientable; the non-orientable case is briefly discussed in Section 5. The (orientable) genus of a graph G or a surface S will be denoted by $\gamma(G)$ or $\gamma(S)$. A *face 2-colourable* embedding is one which admits a 2-colouring of faces (black and white) such that no two faces of the same colour share an edge.

An embedding is *triangular*, or a *triangulation*, if all faces are bounded by triangles. We will use the acronym *2to-embedding* for a face 2-colourable triangulation on an orientable surface. Two 2to-embeddings are *isomorphic* if there is a bijection between the corresponding vertex sets, preserving all incidences between vertices, edges and faces; in the case when the face colours are preserved as well, the isomorphism is said to be *colour-preserving*. Two 2to-embeddings (which may or may not be isomorphic) of the *same* graph are said to be *differently labelled* if there exists a triangle that bounds a face in one of the embeddings but not in the other.

A *parallel class* in a triangulation on $3t$ vertices is a set of t pairwise vertex-disjoint triangular faces. In the case $n \equiv 3 \pmod{12}$, it is shown in [4] that there exist 2to-embeddings of K_n in which the associated Steiner triple systems are isomorphic to those produced by the classical Bose construction [2]. Consequently these embeddings have a parallel class in each of the two colour classes.

2 Triangulations of complete tripartite graphs

The focus of the current paper is triangular embeddings of complete graphs. However, our main constructions for such embeddings make extensive use of triangulations of complete *tripartite* graphs. Consequently we start our results by giving both a direct and a recursive construction for 2to-embeddings of $K_{n,n,n}$. The former ensures the existence of suitable embeddings, and the latter enables us to produce large numbers of differently labelled embeddings of $K_{n,n,n}$.

Construction 1 *For each positive integer n, there exists a 2to-embedding of the complete tripartite graph $K_{n,n,n}$. Furthermore, if n is odd then the embedding has a parallel class of triangular faces in each of the two colour classes and the faces in each of these colour classes are consistently oriented (i.e. if the tripartition is $\{x_i\}, \{y_i\}, \{z_i\}$, then the faces of one parallel class correspond to 3-cycles of the form $(x_i y_j z_k)$ and those of the other to 3-cycles of the form $(x_p z_q y_r)$).*

Proof. Let ν be the plane embedding of the multigraph L with faces of length 2 depicted in Figure 1.

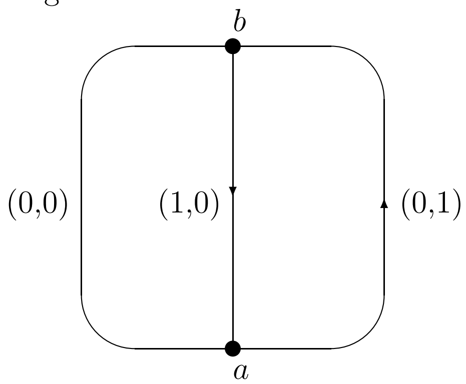

Figure 1. The plane embedding of the graph L.

Figure 1 also shows voltages α on directed edges of L taken in the group $Z_n \times Z_n$, where $Z_n = \{0, 1, \dots, n-1\}$. The edge with no direction carries the zero voltage.

The lifted graph L^α is trivalent and has the vertex set $\{a_{i,j}, b_{i,j} : i, j \in Z_n\}$. The lifted embedding $\nu^\alpha : L^\alpha \to S$ is orientable and has faces of three types. Type 1 faces have boundaries of the form $(a_{i,0}\, b_{i,1}\, a_{i,1}\, b_{i,2}\, a_{i,2}\, b_{i,3} \dots a_{i,n-1}\, b_{i,0})$, where $i \in Z_n$ and subscript arithmetic is modulo n. Type 2 faces have boundaries of the form $(a_{0,j}\, b_{0,j}\, a_{1,j}\, b_{1,j}\, a_{2,j}\, b_{2,j} \dots a_{n-1,j}\, b_{n-1,j})$. Type 3 faces have boundaries of the form $(a_{0,j}\, b_{n-1,j}\, a_{n-1,j-1}\, b_{n-2,j-1}\, a_{n-2,j-2}\, b_{n-3,j-2} \cdots a_{1,j+1}$ $b_{0,j+1})$. Altogether ν^α has $2n^2$ vertices, $3n$ faces and $3n^2$ edges.

Now consider the dual of the lifted graph and embedding. This is orientable and has $3n$ vertices, $2n^2$ faces and $3n^2$ edges. The vertices of the dual are of three Types corresponding to the three Types of faces of ν^α. Since L^α is trivalent, all the faces of the dual are triangles. Once an appropriate orientation is selected, it is easy to verify that all the triangular faces of the dual corresponding to points $a_{i,j}$ have vertices of Types 1, 2 and 3 (in that order) and all those corresponding to points $b_{i,j}$ have vertices of Types 1, 3 and 2 (in that order). Any parallel class of triangles in either of the colour classes will therefore have its triangles consistently oriented. Furthermore, in the dual, vertices of the same Type are never joined and so the dual is tripartite.

In the embedding of L^α, every face of Type 1 has an edge in common with every face of Type 2; a similar observation applies to Types 1 and 3, and to Types 2 and 3. Consequently the dual has an edge between any two vertices of different Types. Consequently the dual is a triangular embedding of $K_{n,n,n}$. Since L^α is bipartite (having bipartition $\{a_{i,j}\}, \{b_{i,j}\}$), the dual is face 2-colourable.

Suppose now that n is odd. Consider all the vertices of L^α which have the form $a_{i,2i}$ ($i \in Z_n$). We claim that no two distinct vertices of this form appear simultaneously on the boundary of a single face. For $a_{i,2i}$ and $a_{j,2j}$ to appear on the boundary of a Type 1 face would require $i = j$. For them to appear on the boundary of a Type 2 face would require $2i \equiv 2j \pmod{n}$ and hence $i = j$. For them to appear on the boundary of a Type 3 face would require $2i - i \equiv 2j - j \pmod{n}$ and hence $i = j$. Thus no pair of vertices from $\{a_{i,2i} : i \in Z_n\}$ appear together on the boundary of a single face in the embedding of L^α. In the dual embedding, the faces corresponding to these vertices have no common vertices and therefore form a parallel class of triangular faces in one of the colour classes. A similar argument applies to the other colour class. □

Remark. The $K_{n,n,n}$ embedding just described has genus $g = (2 - 3n - 2n^2 + 3n^2)/2 = (n-1)(n-2)/2$.

Construction 2 *Suppose that we have k differently labelled 2to-embeddings of $K_{m,m,m}$, all of which have a common parallel class of (consistently oriented) black triangular faces. Then we can construct k^{n^2} differently labelled 2to-embeddings of $K_{mn,mn,mn}$ for each positive integer n. Furthermore*

1. *If the $K_{m,m,m}$ embeddings all have the same black triangular faces (i.e. identically labelled and oriented), then the $K_{mn,mn,mn}$ embeddings constructed also have a common set of black triangular faces.*

2. *If the $K_{m,m,m}$ embeddings all have the same white triangular faces, then the $K_{mn,mn,mn}$ embeddings constructed also have a common set of white*

triangular faces. (Of course (1) and (2) are mutually exclusive if $k > 1$.)

3. *If n is odd then the $K_{mn,mn,mn}$ embeddings constructed all have a common parallel class of (consistently oriented) black triangular faces.*

4. *If n is odd and each $K_{m,m,m}$ embedding has a parallel class of white triangular faces, then the $K_{mn,mn,mn}$ embeddings constructed each have a parallel class of (consistently oriented) white triangular faces.*

Proof. Suppose that $\phi : G \to S$ is a 2to-embedding of $G = K_{n,n,n}$; such an embedding is guaranteed by Construction 1. We assume that G has vertex partition $\{a_j\}, \{b_j\}, \{c_j\}$. Take m disjoint copies of the embedding ϕ : for each $i \in Z_m$ let $G^i = K^i_{n,n,n}$ be the complete tripartite graph with vertex partition $\{a^i_j\}, \{b^i_j\}, \{c^i_j\}$, and let $\phi^i : G^i \to S^i$ be a 2to-embedding such that the natural mapping $f^i : G \to G^i$ which assigns the superscript i to each vertex of G is a colour-preserving and orientation-preserving isomorphism of the embeddings ϕ and ϕ^i. We assume that the surfaces S^i are mutually disjoint and we note that S, and consequently each S^i, has n^2 triangular faces in each of the two colour classes.

Suppose next that $\psi : F \to R$ is any one of the given k 2to-embeddings of $F = K_{m,m,m}$ having a parallel class of consistently oriented black triangular faces. Let a, b, c be vertices of G forming the vertices of a fixed white triangular face T of ϕ. We may, without loss of generality, assume that the clockwise orientation of S induces the cyclic permutation (abc) of vertices on the boundary cycle C of T. For this particular T we take a copy of ψ, say $\psi_T : F^T \to R^T$ which embeds the complete tripartite graph F^T having vertex partition $\{a^i_T\}, \{b^i_T\}, \{c^i_T\}$ and a parallel class of (clockwise) oriented black triangular faces with boundary cycles $(a^i_T c^i_T b^i_T)$ for $i \in Z_m$. Note the difference between the cyclic permutations (abc) on S and $(a^i_T c^i_T b^i_T)$ on R^T. We also assume that R^T is disjoint from all the surfaces S^i.

Now, for each $i \in Z_m$, remove from the embedding ϕ^i the open triangular face $T^i = f^i(T)$. We thereby create in each S^i a hole with boundary cycle C^i, where C^i corresponds to the 3-cycle $(a^i b^i c^i)$ in ϕ^i. To match these holes we also remove from ψ_T the m open triangular faces corresponding to the parallel class, i.e. the faces with boundary curves C^i_T corresponding to the cycles $(a^i_T c^i_T b^i_T)$. Finally, for $i \in Z_m$, we identify the closed curve C^i in the embedding ϕ^i with the curve C^i_T in the embedding ψ_T, in such a way that $a^i \equiv a^i_T, b^i \equiv b^i_T$ and $c^i \equiv c^i_T$.

Applying the above procedure successively to *each* of the n^2 white triangular faces T of S, possibly using a different one of the k embeddings of $K_{m,m,m}$ as the embedding ψ in generating ψ_T for each T, and assuming that all the embeddings ψ_T used are mutually disjoint, we obtain from $\{S^i : i \in Z_m\}$ a new connected oriented triangulated surface which we denote by $\hat{S}$. Roughly speaking, $\hat{S}$ is obtained from $\{S^i : i \in Z_m\}$ by adding n^2 $K_{m,m,m}$ "bridges" raised (for each T) above the white triangular faces $T^i, (i \in Z_m)$. The surface $\hat{S}$ inherits the clockwise orientation from the embeddings ϕ^i and ψ_T as well as a proper 2-colouring of the triangular faces.

Let H be the graph that triangulates $\hat{S}$. Clearly H has vertex set $\{a^i_j, b^i_j, c^i_j : i \in Z_m, j \in Z_n\}$. The graph G contains all edges of the forms $a_j b_k, b_j c_k, c_j a_k$ for $j, k \in Z_n$. Each of these edges is incident with a white triangular face of S and, consequently, each gives rise to m^2 edges $a^i_j b^l_k, b^i_j c^l_k, c^i_j a^l_k$, $(i, l \in Z_m)$ of H. The bridging operation results in m^2n^2 triangular faces in each colour class of $\hat{S}$ and consequently there are precisely $3m^2n^2$ edges in H. It follows that H can have no edges $a^i_j a^l_k, b^i_j b^l_k, c^i_j c^l_k$, and so $H \cong K_{mn,mn,mn}$ with vertex partition $\{a^i_j\}, \{b^i_j\}, \{c^i_j\}$. As an additional check we independently determine the genus of the surface $\hat{S}$. We have $\gamma(\hat{S}) = m\gamma(K_{n,n,n}) + n^2[\gamma(K_{m,m,m}) + m - 1] - (m - 1)$ which reduces to $(mn - 1)(mn - 2)/2 = \gamma(K_{mn,mn,mn})$, as expected.

The foregoing argument applies to any selection of the embedding ψ_T for each of the n^2 triangles T. Since there are k choices for ψ, we may obtain a total of k^{n^2} embeddings of H by this construction. Two different choices of ψ_T, say ψ_1 and ψ_2, having a common parallel class $\{(a^i_T c^i_T b^i_T)\}$ and applied to a single F^T, give differently labelled surfaces R^T, i.e. surfaces having different triangular faces. The resulting embeddings of H corresponding to ψ_1 and ψ_2 will have (some) different triangular faces from one another. Thus the construction gives k^{n^2} differently labelled 2to-embeddings of $K_{mn,mn,mn}$.

1. Let us now suppose that the $K_{m,m,m}$ embeddings all have the same black triangular faces. The black triangular faces in the embeddings of $K_{mn,mn,mn}$ produced by the construction come from two sources:

   (a) the black triangular faces in the embeddings ϕ^i, and

   (b) the black triangular faces in each of the $K_{m,m,m}$ embeddings ψ_T (with the parallel class $\{C^i_T : i \in Z_m\}$ deleted).

   The triangles from (a) are common to all the $K_{mn,mn,mn}$ embeddings produced and, by assumption, those from (b) do not depend on the choices of the embeddings ψ_T. Consequently, all the $K_{mn,mn,mn}$ embeddings produced have the same black triangular faces.

2. Suppose next that the $K_{m,m,m}$ embeddings all have the same white triangular faces. The white triangular faces in the embeddings of $K_{mn,mn,mn}$ produced by the construction all come from the white triangular faces in each of the $K_{m,m,m}$ embeddings ψ_T. By assumption, these do not depend on the choices of the embeddings ψ_T. Consequently, all the $K_{mn,mn,mn}$ embeddings produced have the same white triangular faces.

3. Let us now assume that n is odd. Then by Construction 1, the embedding ϕ may be taken to have a (consistently oriented) parallel class in each of the two colour classes. Let P_b and P_w denote these parallel classes, black and white respectively, and let P_b^i, P_w^i be the corresponding parallel classes in the embedding ϕ^i $(i \in Z_m)$. Then $\bigcup_{i \in Z_m} P_b^i$ will form a (consistently oriented) parallel class of black triangular faces in each of the $K_{mn,mn,mn}$ embeddings produced by the construction.

4. If n is odd and each of the $K_{m,m,m}$ embeddings used to bridge the white triangles $T^i \in P_w^i$ (with P_w^i etc. as in the previous paragraph) has itself got a parallel class of white triangular faces, say Q_T, then $\bigcup_{T \in P_w} Q_T$ will form a parallel class of white triangular faces in any embeddings of $K_{mn,mn,mn}$ produced by the construction. Furthermore, as each Q_T is consistently oriented, the parallel class in the $K_{mn,mn,mn}$ embedding will also be consistently oriented. □

Remark. We may use the construction with $m = 3$ and $k = 2$ by making use of the two differently labelled $K_{3,3,3}$ embeddings given in [1]. This gives 2^{n^2} differently labelled 2to-embeddings of $K_{3n,3n,3n}$. Writing w for $3n$, we may express this by saying that there are (at least) $2^{w^2/9}$ differently labelled 2to-embeddings of $K_{w,w,w}$ for $w \equiv 0 \pmod 3$. Since the two $K_{3,3,3}$ embeddings from [1] have the same black triangles, all the resulting $K_{w,w,w}$ embeddings may be taken to have the same black triangles. Furthermore, if n is odd (equivalently $w \equiv 3 \pmod 6$) then each of the $K_{w,w,w}$ embeddings may be taken to have a common (consistently oriented) parallel class of triangular faces in black.

For our subsequent constructions it is useful to have a large supply of differently labelled 2to-embeddings of $K_{n,n,n}$, all having a common (consistently oriented) parallel class of triangular faces in one of the two colour classes. The previous Remark shows that Construction 2 achieves this for $n \equiv 3 \pmod 6$. An alternative approach for odd n is to take a single 2to-embedding of $K_{n,n,n}$ having a consistently oriented parallel class of triangular faces in a colour class. We apply to this all permutations which fix this parallel class (including its orientation) and which preserve the tripartition. There are $3n!$ such permutations. Suppose that π is one of these permutations and that π fixes a particular realisation of the original embedding of $K_{n,n,n}$. Since π preserves the orientation, the parallel class and the tripartition, π is determined by the image of any single vertex. Consequently, there are at most $3n$ such permutations π. It follows that there are at least $3n!/3n = (n-1)!$ differently labelled 2to-embeddings of $K_{n,n,n}$ all having a common (consistently oriented) parallel class of triangular faces in one of the two colour classes. In fact, for $n = 9, 15, 21$ and 27, this estimate exceeds that given by $2^{n^2/9}$. We can combine these estimates with Construction 2 itself in the manner indicated by way of example in the following Corollary.

Corollary 2.1 *If $w \equiv 0 \pmod 9$ then there are at least 2^{aw^2} differently labelled 2to-embeddings of $K_{w,w,w}$, where $a = \log_2(40320)/81 \approx 0.188879$.*

Proof. In Construction 2, take $m = 9$ and $k = 8! = 40320$. This yields 40320^{n^2} differently labelled 2to-embeddings of $K_{9n,9n,9n}$, for each positive integer n. Putting $w = 9n$ gives the result. □

A further observation concerns Construction 2 itself. It is not necessary for the embeddings ϕ^i described in the first part of the proof to be copies of the same embedding ϕ. All that is needed is that these embeddings should each have the "same" white triangular faces with the "same" orientations. By the term "same" here (and subsequently) we mean that there is a mapping from the vertices of each embedding onto those of each of the other embeddings which preserves the white triangular faces and their orientations. From the Remark following the proof of Construction 2, we have $2^{n^2/9}$ differently labelled 2to-embeddings of $K_{n,n,n}$ for $n \equiv 0 \pmod 3$, and all of these have the same black triangles. If we reverse the colours, we have a plentiful supply of embeddings ϕ^i to which we may reapply the construction. We state the result as Construction 3.

Construction 3 *Suppose that we have k differently labelled 2to-embeddings of $K_{m,m,m}$, all of which have a common parallel class of (consistently oriented)* *black triangular faces. Suppose also that we have N differently labelled 2to-embeddings of $K_{n,n,n}$, all having the same white triangular faces. Then we can construct $N^m k^{n^2}$ differently labelled 2to-embeddings of $K_{mn,mn,mn}$.*

Proof. The proof is very similar to that of the first part of Construction 2. We leave the reader to make the appropriate modifications. The factor N^m reflects the N possible choices for each of the embeddings ϕ^i, $i \in Z_m$. □

Clearly, Constructions 2 and 3 permit estimates of the form 2^{cn^2} for the number of differently labelled 2to-embeddings of $K_{n,n,n}$ to be made for a variety of residue classes for n.

Our final comment in this Section concerns the number of non-isomorphic 2to-embeddings of $K_{n,n,n}$. An isomorphism class can contain at most $6n!$ different realisations of such an embedding on a fixed point set with a fixed tripartition. Thus, for example, in the case $n \equiv 0 \pmod 3$, the number of non-isomorphic embeddings is at least $2^{n^2/9}/6n!$ and estimating the factorial gives $2^{n^2/9-O(n\log n)}$.

3 Triangulations of complete graphs (I)

We start this Section by recalling from our Introduction that two of the principal ingredients for our next Construction, namely 2to-embeddings of K_n and K_{2m+1}, are known to exist for $n \equiv 3$ or 7 (mod 12) and $m \equiv 1$ or 3 (mod 6).

Construction 4 *Suppose that $n \equiv 3$ or 7 (mod 12) and that $m \equiv 1$ or 3 (mod 6). Then, from a 2to-embedding of K_n we may construct a 2to-embedding of $K_{m(n-1)+1}$.*

Proof. Suppose initially that m and $n-1$ are coprime. We shall point out where this assumption is used and, subsequently, how it may be dropped.

Let η be a 2to-embedding of K_n with faces properly coloured black and white, and let a fixed orientation of the surface be chosen (say, clockwise). Fix a vertex z of K_n and remove from η the vertex z, together with all open arcs and open triangular faces originally incident with z. We obtain a face 2-coloured triangular embedding ϕ of $G = K_n \setminus \{z\} \cong K_{n-1}$ in a bordered surface S; note that the boundary of the hole in S (i.e. the border of S) forms a Hamiltonian cycle D in G.

We now take m disjoint copies of the embedding ϕ (including the proper 2-colouring of triangular faces inherited from η). Denote these by $\phi^i : G^i \to S^i$ $(0 \le i \le m-1)$, where the surfaces S^i are mutually disjoint and the natural mapping $f^i : G \to G^i$ which assigns the superscript i to each vertex of G is a colour-preserving and orientation-preserving isomorphism of the embeddings ϕ and ϕ^i.

In the embedding ϕ we have $t = (n-1)(n-3)/6$ white triangular faces; let $\mathcal{T}$ be the set of these faces and let $\mathcal{T}^i = f^i(\mathcal{T})$ be the corresponding set of all white triangular faces in the embedding ϕ^i for $i = 0, 1, \ldots, m-1$. We now focus attention on an individual white triangular face T of ϕ. Denote the vertices of this face by a, b, c, so that the cyclic permutation (abc) corresponds to the clockwise orientation of the boundary cycle C of the face T. Next take a 2to-embedding ψ_T of the complete tripartite graph $K_{m,m,m}$ in a surface S_T disjoint from each S^i and whose three vertex-parts are $\{a_T^i\}, \{b_T^i\}$ and $\{c_T^i\}$. By Construction 1, we may select ψ_T to have a parallel class of black triangular faces $\{a_T^i, b_T^i, c_T^i\}$ and we may choose the orientation of ψ_T to ensure that it induces the cyclic permutations $(a_T^i c_T^i b_T^i)$ of the boundary cycles C_T^i of these faces. Note the difference between the cyclic permutations (abc) on S and $(a_T^i c_T^i b_T^i)$ on S_T.

Now, for each $i \in Z_m$, remove from the embedding ϕ^i the open triangular face $T^i = f^i(T)$, thereby creating in each S^i a new hole with boundary curve $C^i = f^i(C)$ corresponding to the 3-cycle $(a^i b^i c^i)$ in ϕ^i. Similarly remove from ψ_T the open triangular faces $\{a_T^i, b_T^i, c_T^i\}$ for $i \in Z_m$. Then, for $i \in Z_m$, we identify the closed curve C^i in the embedding ϕ^i with the curve C_T^i in the embedding ψ_T in such a way that $a^i \equiv a_T^i$, $b^i \equiv b_T^i$, and $c^i \equiv c_T^i$.

As in Construction 2, we apply the above procedure successively to *each* white triangular face $T \in \mathcal{T}$ (assuming that the corresponding embeddings ψ_T are mutually disjoint), and we thereby obtain from the surfaces S^i a new connected triangulated surface with a boundary. Denote this surface by $\hat{S}$. Roughly speaking, $\hat{S}$ is obtained from the surfaces S^i by adding $|\mathcal{T}|$ "bridges". Clearly, $\hat{S}$ has m holes, and their (disjoint) boundary curves correspond to the Hamiltonian cycles $D^i = f^i(D)$ in the graphs G^i. Also, it is easy to see that the chosen orientations of ϕ^i and ψ_T guarantee that the bordered surface $\hat{S}$ is orientable, inheriting the clockwise orientation from ϕ^i and ψ_T. Note that $\hat{S}$ also inherits the proper 2-colouring of triangular faces from these embeddings. Since we have $t = (n-1)(n-3)/6$ black triangles in S (and hence in each S^i), and for each of the t white triangles T in S we added, in ψ_T, another $(2m^2 - m)$ triangles, the total number of triangular faces on $\hat{S}$

is equal to $mt + (2m^2 - m)t = m^2(n-1)(n-3)/3$. For each collection of $(2m^2 - m)$ triangles added, m^2 are white and $(m^2 - m)$ are black; hence it is easy to check that exactly half of the triangles on $\hat{S}$ are black, as expected.

Let H be the graph that triangulates the bordered surface $\hat{S}$; we need a precise description of H. Let $D = (u_1 u_2 \ldots u_{n-1})$ be our Hamiltonian cycle in $G = K_n \setminus \{z\}$ (thus, $V(G) = \{u_j : 1 \leq j \leq n-1\}$). Since n is odd, every other edge of D is incident to a white triangle on $\hat{S}$; let these edges be u_2u_3, u_4u_5,...,$u_{n-1}u_1$. From the above construction it may be seen that the graph H is obtained as follows. For $1 \leq j \neq j' \leq n-1$, each vertex u_j of G gives rise to m vertices u_j^i $(0 \leq i \leq m-1)$ of H, and each edge $u_j u_{j'}$ of G incident to a white triangle gives rise to m^2 edges $u_j^i u_{j'}^{i'}$ $(i, i' \in Z_m)$ of H. Since each edge of G except the $(n-1)/2$ edges u_1u_2, u_3u_4,..., $u_{n-2}u_{n-1}$ is incident to exactly one white triangle, the total number of edges of the graph H is $m^2(|E(G)| - (n-1)/2) + m(n-1)/2$ $= m(n-1)(m(n-3)+1)/2$. To see its structure, note that for each edge $u_j u_{j'}$ of $G \cong K_{n-1}$ (except when $\{u_j, u_{j'}\} = \{u_l, u_{l+1}\}$, $l = 1, 3, 5, \ldots, n-2$), H contains all edges of the form $u_j^i u_{j'}^{i'}$, $i, i' \in Z_m$. However, if $\{u_j, u_{j'}\} = \{u_l, u_{l+1}\}$ for some $l = 1, 3, \ldots, n-2$ then H contains no edge $u_j^i u_{j'}^{i'}$ with $i \neq i'$, although it does contain the edges $u_j^i u_{j'}^i$. Also, H contains no edge of the form $u_j^i u_j^{i'}$, $i, i' \in Z_m$. We see that, abstractly, H is isomorphic to $K_{m(n-1)}$ minus $(n-1)/2$ pairwise disjoint copies of (K_{2m} minus a 1-factor), one on each of the sets $\{u_l^0, u_l^1, \ldots, u_l^{m-1}, u_{l+1}^0 u_{l+1}^1, \ldots, u_{l+1}^{m-1}\}$ with missing 1-factor $\{\{u_l^i, u_{l+1}^i\} : i = 0, 1, \ldots, m-1\}$, for $l = 1, 3, 5, \ldots, n-2$.

Let $\omega : H \to \hat{S}$ be the embedding just constructed. We recall that the boundary curves of the m holes in $\hat{S}$ are D^i, the images of our Hamiltonian cycle D under the isomorphisms f^i, $i \in Z_m$. In order to complete the construction to obtain a 2to-embedding of $K_{m(n-1)+1}$ we build an auxiliary triangulated bordered surface S^* and paste it to $\hat{S}$ so that the m holes of $\hat{S}$ will be capped. The construction of S^* commences with voltage assignments.

Let μ be the plane embedding of the multigraph M as depicted in Figure 2.

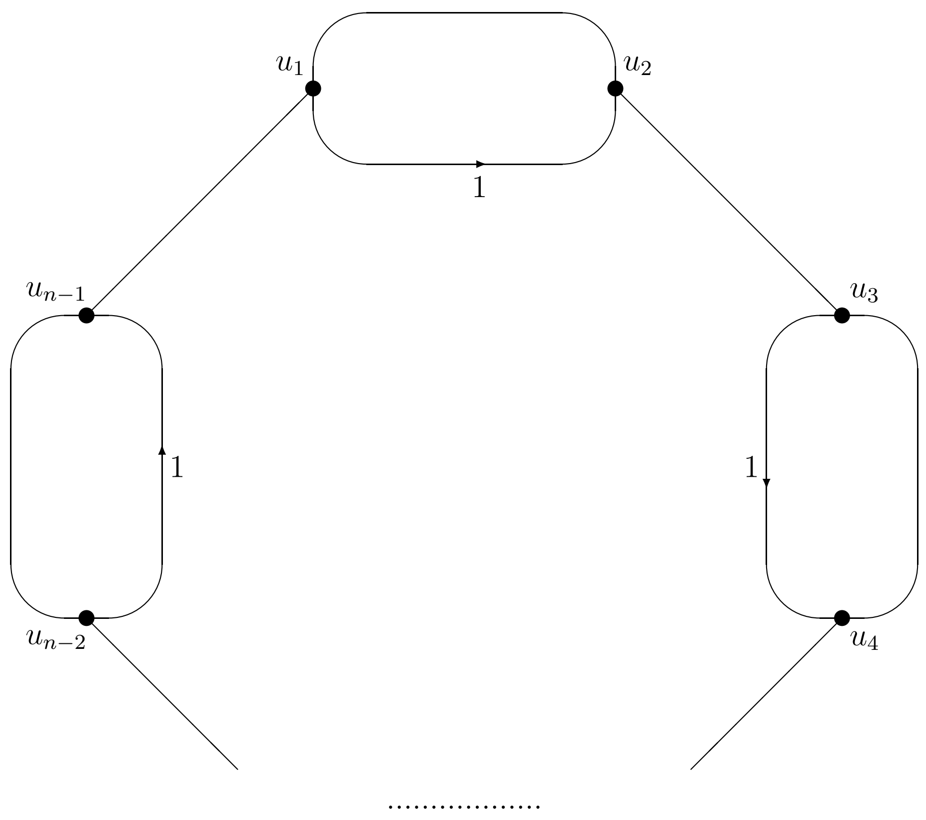

Figure 2. The plane embedding of the graph M.

Figure 2 also shows voltages α on directed edges of M, taken in the group Z_m. Edges with no direction assigned carry the zero voltage.

The lifted graph M^α has the vertex set $\{u^i_j : 1 \leq j \leq n-1, i \in Z_m\}$. (We are deliberately using the same letters for vertices of M^α as for vertices of the graphs G^i, but assume that these graphs are disjoint; such notation will be of advantage later.) The lifted embedding $\mu^\alpha : M^\alpha \to R$ is orientable and has the following face boundaries.

(a) $(n-1)/2$ faces whose boundaries correspond to $(2m)$-cycles of the form $(u^0_{2j-1}u^0_{2j}u^{m-1}_{2j-1}u^{m-1}_{2j} \dots u^1_{2j})$ for $1 \leq j \leq (n-1)/2$.

(b) m faces whose boundaries correspond to $(n-1)$-cycles of the form $(u_{n-1}^i u_{n-2}^i \dots u_1^i)$ for $i \in Z_m$.

(c) One face whose boundary corresponds to a single $(m(n-1))$-cycle $(u_1^0 u_2^1 u_3^1 u_4^2 u_5^2 u_6^3 \dots u_{n-1}^0)$. (**Note:** It is here that use is made of the assumption that m and $n-1$ are coprime; if this were not the case then a multiplicity of faces with shorter boundary cycles would be obtained.)

We now remove all the open faces of type (a) from the surface R, leaving an orientable surface R^1 with $(n-1)/2$ vertex-disjoint boundaries $(u_{2j-1}^0 u_{2j}^0 u_{2j-1}^{m-1} u_{2j}^{m-1} \dots u_{2j}^1)$, $1 \leq j \leq (n-1)/2$. We cap each of these in turn by taking, for each j, a 2to-embedding of K_{2m+1} with colour classes black and white on the points $\{\infty_j, u_{2j}^0 u_{2j-1}^0 u_{2j}^1 u_{2j-1}^1 \dots u_{2j-1}^{m-1}\}$, in which the rotation at ∞_j is the cycle $(u_{2j}^0 u_{2j-1}^0 u_{2j}^1 u_{2j-1}^1 \dots u_{2j-1}^{m-1})$ and in which the face corresponding to the 3-cycle $(\infty_j u_{2j}^0 u_{2j-1}^0)$ is coloured black. Here also for convenience we are using the same letters for the vertices of our K_{2m+1} embeddings as for the vertices of M^α, but we assume that the corresponding surfaces are disjoint. From each embedding of K_{2m+1} we remove the point ∞_j, all open edges incident with ∞_j, and all open triangular faces incident with ∞_j. This results in a face 2-colourable embedding of K_{2m} in an orientable surface R_j with a boundary cycle $(u_{2j}^0 u_{2j-1}^0 u_{2j}^1 u_{2j-1}^1 \dots u_{2j-1}^{m-1})$. We then glue the surface R_j to the surface R^1, identifying points carrying the same labels on each of the two surfaces. This procedure is repeated successively for each of the $(n-1)/2$ faces of R of type (a). Let $\mu' : M' \to R'$ denote the embedding eventually obtained. It is easy to check that $|E(M')| = m^2(n-1)$.

We next remove from R' all the open faces of type (b) leaving an orientable surface S^* with m vertex-disjoint boundaries $(u_{n-1}^i u_{n-2}^i \dots u_1^i)$, $i \in Z_m$.

Let M^* be the graph obtained from M' by adding a new vertex ∞ and joining it to each vertex of M' (and keeping all other edges in M' unchanged). We construct an embedding $\mu^* : M^* \to S^*$ from the embedding of M' in S^* by inserting the vertex ∞ in the centre of the face F bounded by the $(m(n-1))$-gon and joining this point by open arcs (within F) to *every* vertex on the boundary of F (that is, to every vertex of M^α). Instead of the face F we now have $m(n-1)$ new triangular faces on S^*. These new triangular faces have boundary cycles of the forms $(\infty u_j^k u_{j+1}^{k+1})$ (j odd) and $(\infty u_j^k u_{j+1}^k)$ (j even). We colour these new triangular faces as follows.

The edge $u_1^0 u_2^1$ already lies in a black triangular face of μ' because $(\infty_1 u_1^0 u_2^1)$ corresponds to a white triangular face of the K_{2m+1} embedding employed in the construction of μ'. We therefore colour white the face of μ^* which is bounded by the 3-cycle $(\infty u_1^0 u_2^1)$. It is easy to see that, by an extension of this argument we must colour white those alternate triangles with boundary cycles $(\infty u_j^k u_{j+1}^{k+1})$ for j odd. The remaining alternate triangles, those with boundary cycles of the form $(\infty u_j^k u_{j+1}^k)$ for j even, do not share an edge with any existing triangular face of μ' and these are coloured black.

By this process, the triangular faces of μ^* are properly 2-coloured, and the number of such faces is

$$\frac{(n-1)}{2}\frac{2m(2m-2)}{3}+m(n-1)=\frac{m(2m+1)(n-1)}{3},$$

exactly half of which are coloured black.

We are ready for the final step of the construction. Our method of constructing the orientable surface $\hat{S}$ guarantees that a chosen orientation of $\hat{S}$ induces *consistent* orientations of the boundary cycles of the m holes of $\hat{S}$; we may assume that the orientation induces the cyclic ordering of the cycles D^i in the form that was used before, namely, $D^i = f^i(D) = (u_1^i u_2^i \dots u_{n-1}^i)$, $i \in Z_m$. The bordered surface S^* has m holes as well, and again, the method of construction implies that an orientation of S^* can be chosen so that the boundary cycles of the holes are oriented in the form $D^{*i} = (u_{n-1}^i \dots u_2^i u_1^i)$, $i \in Z_m$. It remains to do the obvious – namely, for each i to paste together the boundary cycles D^i and D^{*i} so that corresponding vertices u_j^i get identified. As the result, we obtain an orientable surface $\bar{S}$ and a triangular embedding $\sigma : K \to \bar{S}$ of some graph K. We claim that $K \cong K_{m(n-1)+1}$ and that the triangulation is face 2-colourable.

Obviously, $|V(K)| = m(n-1)+1$. A straightforward edge count shows that

$$\begin{aligned}|E(K)| &= |E(H)|+|E(M^*)|-m|E(D)|\\ &= \frac{m(n-1)(m(n-3)+1)}{2}+(n-1)(m^2+m)-m(n-1)\\ &= \frac{m(n-1)(m(n-1)+1)}{2}=|E(K_{m(n-1)+1})|.\end{aligned}$$

It is easy to check that, except for edges incident with ∞ and edges contained in the m $(n-1)$-cycles D^{*i}, the graph M^* contains *exactly those edges which are missing in* H. This shows that there are no repeated edges or loops in K, and thus $K \cong K_{m(n-1)+1}$. As regards the face 2-colouring, we just have to see what happens along the identified $(n-1)$-cycles D^i and D^{*i} (since both triangulations of $\hat{S}$ and S^* are already known to be face 2-colourable). But according to the construction, if $l = 1, 3, 5, \ldots, n-2$, a triangular face on $\hat{S}$ that contains the edge $u_l^i u_{l+1}^i$ is black, while the face on S^* containing this edge is white because the embeddings of K_{2m+1} employed had the faces with boundary cycles $(\infty_j u_{2j}^i u_{2j-1}^i)$ coloured black.

As an additional check we independently determine the genus of the surface $\bar{S}$. We have

$$\begin{aligned}\gamma(H) &= m\gamma(K_n) + |\mathcal{T}|[\gamma(K_{m,m,m}) + m - 1] - (m-1)\\ &= \frac{m(n-3)(n-4)}{12} + \frac{(n-1)(n-3)}{6}\left[\frac{(m-1)(m-2)}{2} + m - 1\right]\\ &\quad -(m-1).\end{aligned}$$

From Euler's formula we obtain

$$2\gamma(M^\alpha) = \left[2 + \frac{3m(n-1)}{2} - \frac{(n-1)}{2} - m - 1 - m(n-1)\right]$$

and we also have

$$\gamma(M^*) = \gamma(M^\alpha) + \frac{(n-1)}{2}\gamma(K_{2m+1}).$$

Finally, we have $\gamma(\bar{S}) = \gamma(K) = \gamma(H) + \gamma(M^*) + m - 1$ and this reduces to $(m^2n^2 - 2m^2n + m^2 - 5mn + 5m + 6)/12$, which equals $\gamma(K_{m(n-1)+1})$, as expected. This completes the proof of the construction in the case in which m and $n-1$ are coprime.

To deal with the case when m and $n-1$ are not coprime, we return to Figure 2 and make a generalisation to the construction. The voltages shown as 1 may be replaced respectively by voltages $x_1, x_2, \ldots, x_{(n-1)/2}$ provided that

(a) each x_i is coprime with m, and

(b) $\sum_{i=1}^{(n-1)/2} x_i$ is coprime with m.

Condition (a) ensures that μ^α has $(n-1)/2$ faces with boundary cycles of length $2m$ on each of the sets of points $\{u_{2j-1}^0, u_{2j}^0, u_{2j-1}^1, u_{2j}^1, \ldots, u_{2j-1}^{m-1}, u_{2j}^{m-1}\}$, while condition (b) ensures that μ^α has a single face with boundary cycle of length $m(n-1)$. In effect, condition (b) replaces the condition that m and $n-1$ should be coprime. The subsequent steps in the proof then proceed as before with the obvious changes. We leave readers to verify the details for themselves. If we select $x_1 = x_3 = x_5 = \ldots = x_{(n-1)/2} = 1$ and $x_2 = x_4 = \ldots = x_{(n-3)/2} = m-1$, then condition (a) is trivially satisfied and $\sum_{i=1}^{(n-1)/2} x_i = 1 \pmod{m}$, thereby ensuring that (b) is also satisfied. □

It is possible to generalise the previous Construction in a number of ways. Firstly, it is not necessary to use the same $K_{m,m,m}$ embedding ψ to form ψ_T for each of the $(n-1)(n-3)/6$ white triangles $T \in \mathcal{T}$. If we have k differently labelled 2to-embeddings of $K_{m,m,m}$, all of which have a common parallel class of (consistently oriented) black triangular faces, then we have k choices for ψ_T for each $T \in \mathcal{T}$. Similarly to Construction 2, this enables us to produce a large number (here $k^{(n-1)(n-3)/6}$) of differently labelled 2to-embeddings of $K_{m(n-1)+1}$. In fact, this number may be increased by a further factor reflecting the choice of currents x_i assigned in the generalised version of Figure 2 and the available choice of K_{2m+1} embeddings. With these latter variables held fixed, if the k embeddings of $K_{m,m,m}$ all have the same black triangles, then the resulting embeddings of $K_{m(n-1)+1}$ also all have the same black triangles and the same rotation at the point ∞. By reversing the colours, this provides a plentiful supply of embeddings to motivate our second generalisation.

It is not necessary for the embeddings ϕ^i, described in the first section of the proof, to be copies of the same embedding ϕ. All that the Construction requires is that these embeddings have the "same" white triangular faces and the "same" cycle of $n-1$ points around the border, all with the "same" orientations.

We can combine our generalisations as follows.

Construction 5 *Suppose that $n \equiv 3$ or $7 \pmod{12}$ and that $m \equiv 1$ or $3 \pmod{6}$. Suppose also that we have k differently labelled 2to-embeddings of $K_{m,m,m}$, all of which have a common parallel class of (consistently oriented) black triangular faces. Then we may construct $k^{(n-1)(n-3)/6}$ differently labelled 2to-embeddings of $K_{m(n-1)+1}$. Furthermore*

1. *If the k embeddings of $K_{m,m,m}$ all have the same black triangular faces then the resulting embeddings of $K_{m(n-1)+1}$ also all have the same black triangular faces and the same rotation about the point ∞.*

2. *If we have N differently labelled 2to-embeddings of K_n, all having the same white triangular faces and a common rotation about a particular point z, then the number of differently labelled 2to-embeddings of $K_{m(n-1)+1}$ may be increased to $N^m k^{(n-1)(n-3)/6}$.*

Proof. The proof is a generalisation of that of the previous Construction, with additional features as discussed informally above. We leave the reader to complete the details. □

In applying Construction 5 we can make use of our earlier results concerning embeddings of $K_{m,m,m}$. For $m \equiv 3 \pmod 6$ we can take k to be either $(m-1)!$ or $2^{m^2/9}$, and for $m \equiv 1 \pmod 6$ we can take $k = (m-1)!$. The case $m = 3$ is investigated in [1] where it is shown, *inter-alia*, that there are at least $2^{n^2/54-O(n)}$ non-isomorphic 2to-embeddings of K_n for $n \equiv 7$ or 19 (mod 36), and at least $2^{2n^2/81-O(n)}$ for $n \equiv 19$ or 55 (mod 108). The latter estimate is achieved by a second application of the Construction, making use of aspects (1) and (2).

By way of fresh and explicit applications, we here consider the cases $m = 7$ and $m = 9$. For $m = 7$, using the value $k = 6! = 720$, we may construct $720^{(n-1)(n-3)/6}$ differently labelled 2to-embeddings of K_{7n-6} for $n \equiv 3$ or 7 (mod 12). There are therefore at least $720^{(n-1)(n-3)/6}/(7n-6)!$ isomorphism classes. Putting $w = 7n - 6$ and estimating the factorial term, we may express this as follows.

Corollary 5.1 *If $w \equiv 15$ or 43 (mod 84) then there are at least $2^{aw^2-O(w \log w)}$ non-isomorphic 2to-embeddings of K_w, where $a = \log_2(720)/294 \approx 0.032285$.* □

Note that the residue classes 15 and 43, modulo 84, cover additional values to those dealt with in [1]. In the case $m = 9$, using $k = 8! = 40320$ and applying a similar argument gives the following.

Corollary 5.2 *If $w \equiv 19$ or 55 (mod 108) then there are at least $2^{bw^2-O(w \log w)}$ non-isomorphic 2to-embeddings of K_w, where $b = \log_2(40320)/486 \approx 0.031480$.* □

Note that $b > 2/81 \approx 0.024691$, the latter being the corresponding constant for these residue classes which appears in [1]. It is clear that we can obtain many other similar estimates.

4 Triangulations of complete graphs (II)

We start this Section by recalling from our Introduction that one of the principal ingredients for our next Construction, namely a 2to-embedding of K_n having a parallel class of triangular faces in one of the two colour classes, is known to exist for $n \equiv 3 \pmod{12}$. Indeed, we can assume the existence of parallel classes in each of the two colour classes.

Construction 6 *Suppose that $n \equiv 3$ (mod 12) and that $m \equiv 1$ (mod 4). Then, from a 2to-embedding of K_n with a parallel class of triangular faces in one colour class we may construct a 2to-embedding of K_{mn}.*

Proof. Let η be a 2to-embedding of K_n with faces properly coloured black and white, and let a fixed orientation of the surface be chosen (say, clockwise). We may assume that η has a parallel class $\mathcal{P}$ of white triangular faces.

Take m disjoint copies of the embedding η (including the proper 2-colouring of triangular faces). Denote these by $\eta^i : K_m^i \to S^i$ $(0 \le i \le m-1)$, where the surfaces S^i are mutually disjoint and the natural mapping $f^i : K_m \to K_m^i$ which assigns the superscript i to each vertex of K_m is a colour-preserving and orientation-preserving isomorphism of the embeddings η and η^i.

In the embedding η we have $t = n(n-1)/6$ white triangular faces of which $n/3$ lie in the parallel class $\mathcal{P}$ and the remaining $n(n-3)/6$ lie outside $\mathcal{P}$. Let $\mathcal{T}$ be the set of these remaining faces. Denote by $\mathcal{P}^i$ and $\mathcal{T}^i$ the corresponding sets of white triangular faces in the embedding η^i for $i = 0, 1, \ldots, m-1$.

We firstly focus attention on an individual white triangular face $T \in \mathcal{T}$. Denote the vertices of this face by a, b, c, so that the cyclic permutation (abc) corresponds to the clockwise orientation of the boundary cycle C of the face T. Next take a 2to-embedding ψ_T of the complete tripartite graph $K_{m,m,m}$ in a surface S_T disjoint from each S^i and whose three vertex-parts are $\{a_T^i\}, \{b_T^i\}$ and $\{c_T^i\}$. By Construction 1, we may select ψ_T to have a parallel class of black triangular faces $\{a_T^i, b_T^i, c_T^i\}$ and we may choose the orientation of ψ_T to ensure that it induces the cyclic permutations $(a_T^i c_T^i b_T^i)$ of the boundary cycles C_T^i of these faces. Note the difference between the cyclic permutations (abc) on S and $(a_T^i c_T^i b_T^i)$ on S_T.

Now, for each $i \in Z_m$, remove from the embedding η^i the open triangular face $T^i = f^i(T)$, thereby creating in each S^i a hole with boundary curve $C^i = f^i(C)$ corresponding to the 3-cycle $(a^i b^i c^i)$ in η^i. Similarly remove from

ψ_T the open triangular faces $\{a_T^i, b_T^i, c_T^i\}$ for $i \in Z_m$. Then, for $i \in Z_m$, we identify the closed curve C^i in the embedding η^i with the curve C_T^i in the embedding ψ_T in such a way that $a^i \equiv a_T^i$, $b^i \equiv b_T^i$, and $c^i \equiv c_T^i$.

As in Constructions 2 and 4, we apply the above procedure successively to *each* white triangular face $T \in \mathcal{T}$ (assuming that the corresponding embeddings ψ_T are mutually disjoint), and we thereby obtain from the surfaces S^i a new connected triangulated surface $\hat{S}$. Roughly speaking, $\hat{S}$ is obtained from the surfaces S^i by adding $|\mathcal{T}|$ "bridges". It is easy to see that the chosen orientations of η^i and ψ_T guarantee that the surface $\hat{S}$ is orientable, inheriting the clockwise orientation from η^i and ψ_T. Note that $\hat{S}$ also inherits the proper 2-colouring of triangular faces from these embeddings. Since we have $t = n(n-1)/6$ black triangles in S and $n/3$ white triangles in $\mathcal{P}$, and for each of the $t - n/3$ white triangles $T \in \mathcal{T}$ we added, in ψ_T, another $(2m^2 - m)$ triangles, the total number of triangular faces on $\hat{S}$ is equal to $mt + mn/3 + (2m^2 - m)(t - n/3) = mn(mn - 3m + 2)/3$, exactly half of which are black.

Let $\hat{H}$ be the graph that triangulates the surface $\hat{S}$ and $\hat{\eta} : \hat{H} \to \hat{S}$ the corresponding embedding. We need a precise description of $\hat{H}$. Let $\{(u_{3j+1}u_{3j+2}u_{3j+3}) : 0 \le j \le n/3 - 1\}$ be the set of clockwise boundary cycles of the parallel class $\mathcal{P}$ in K_n (thus, $V(K_n) = \{u_j : 1 \le j \le n\}$). Then $V(\hat{H}) = \{u_j^i : 1 \le j \le n, 0 \le i \le m-1\}$. If $u_j u_{j'}$ $(j \ne j')$ is an edge of a triangle in $\mathcal{P}$, then $E(\hat{H})$ contains the edges $u_j^i u_{j'}^i$, but no edges $u_j^i u_{j'}^{i'}$ with $i \ne i'$. On the other hand, if $u_j u_{j'}$ $(j \ne j')$ is an edge of a triangle in $\mathcal{T}$ then $E(\hat{H})$ contains all edges $u_j^i u_{j'}^{i'}$ (both for $i \ne i'$ and $i = i'$). Finally, $E(\hat{H})$ contains no edges $u_j^i u_j^{i'}$. The total number of edges of the graph $\hat{H}$ is $mn(mn - 3m + 2)/2$. Abstractly, $\hat{H}$ is isomorphic to K_{mn} minus $n/3$ pairwise disjoint copies of (K_{3m} minus a parallel class of triangles), one on each of the sets $\{u_{3j+1}^i, u_{3j+2}^i, u_{3j+3}^i : i \in Z_m\}$ with missing parallel class $\{\{u_{3j+1}^i, u_{3j+2}^i, u_{3j+3}^i\} : i \in Z_m\}$, for $j = 0, 1, \ldots, n/3 - 1$.

We now focus attention on an individual white triangular face $T_j \in \mathcal{P}$ with boundary cycle $D_j = (u_{3j+1}u_{3j+2}u_{3j+3})$. For ease of notation we refer to T_j and D_j simply as T and D. Next take a 2to-embedding ϕ_T of the complete graph K_{3m} in a surface S'_T disjoint from $\hat{S}$ and having a parallel class of black triangular faces. We may assume that the vertices of this embedding are labelled $\{v_{3j+1}^i, v_{3j+2}^i, v_{3j+3}^i : i \in Z_m\}$ and that the orientation of ϕ_T induces the cyclic permutations $(v_{3j+1}^i v_{3j+3}^i v_{3j+2}^i)$ of the boundary cycles D_T^i of the parallel class. Note the difference between the cyclic permutations

$(u_{3j+1}u_{3j+2}u_{3j+3})$ on S and $(v^i_{3j+1}v^i_{3j+3}v^i_{3j+2})$ on S'_T.

Now, for each $i \in Z_m$, remove from the embedding $\hat{\eta}$ the open triangular face $T^i = f^i(T)$, thereby creating m holes in $\hat{S}$ with boundary curves $D^i = f^i(D)$ corresponding to the 3-cycles $(u^i_{3j+1}u^i_{3j+2}u^i_{3j+3})$. Similarly remove from ϕ_T the open triangular faces $\{v^i_{3j+1}, v^i_{3j+2}, v^i_{3j+3}\}$ for $i \in Z_m$. Finally, for $i \in Z_m$, we identify the closed curve D^i in the embedding $\hat{\eta}$ with the curve D^i_T in the embedding ϕ_T in such a way that $u^i_{3j+k} \equiv v^i_{3j+k}$ for $k = 1, 2, 3$.

Applying the above procedure successively to *each* white triangular face $T \in \mathcal{P}$ (and assuming that the corresponding embeddings ϕ_T are mutually disjoint), we obtain from the surface $\hat{S}$ a new connected triangulated surface $\bar{S}$. It is easy to see that the chosen orientations of $\hat{\eta}$ and ϕ_T guarantee that the surface $\bar{S}$ is orientable, inheriting the clockwise orientation from $\hat{\eta}$ and ϕ_T. Note that $\bar{S}$ also inherits the proper 2-colouring of triangular faces from these embeddings. The total number of triangular faces on $\bar{S}$ is equal to

$$\frac{mn(mn-3m+2)}{3} + \frac{n}{3} \cdot \frac{3m(3m-1)}{3} - \frac{2mn}{3} = \frac{mn(mn-1)}{3},$$

exactly half of which are black.

Let $\bar{H}$ be the graph that triangulates the surface $\bar{S}$ and $\bar{\eta} : \bar{H} \to \bar{S}$ the corresponding embedding. Each embedding ϕ_T (for $T = T_j$) adds to $E(\hat{H})$ the edges $u^i_k u^{i'}_{k'}$ for $i \neq i'$ and $k, k' \in \{3j+1, 3j+2, 3j+3\}$. Thus $|E(\bar{H})| = |E(\hat{H})| + (n/3) \cdot \binom{m}{2} \cdot 9 = mn(mn-1)/2$. It follows that $\bar{H} \cong K_{mn}$, so that $\bar{\eta}$ is a 2to-embedding of K_{mn}.

As a final check we independently determine the genus of the surface $\bar{S}$. We have

$$\begin{aligned}\gamma(\bar{S}) &= m\gamma(K_n) + \frac{n}{3}[\gamma(K_{3m}) + m - 1] + \frac{n(n-3)}{6}[\gamma(K_{m,m,m}) + m - 1] \\ &\quad -(m-1) \\ &= \frac{(mn-3)(mn-4)}{12} = \gamma(K_{mn}),\end{aligned}$$

as expected. □

We now make some observations and generalisations concerning the preceding Construction. If the embedding of K_n has a parallel class of triangular faces in each of the colour classes then the resulting K_{mn} embedding will have a parallel class in black. If the K_{3m} bridges each have a parallel class in white (as well as in black) then the resulting K_{mn} embedding will have a parallel class in white. These observations show that the Construction may be applied recursively.

It is not necessary to use the same $K_{m,m,m}$ embedding ψ to form ψ_T for each of the $n(n-3)/6$ white triangles $T \in \mathcal{T}$. If we have k differently labelled 2to-embeddings of $K_{m,m,m}$, all of which have a common parallel class of (consistently oriented) black triangular faces, then we have k choices for ψ_T for each $T \in \mathcal{T}$. Similarly to Constructions 2 and 5, this enables us to produce a large number (here $k^{n(n-3)/6}$) of differently labelled 2to-embeddings of K_{mn}. In fact, this number may be increased by a further factor reflecting the available choice of K_{3m} embeddings. With this latter variable held fixed, if the k embeddings of $K_{m,m,m}$ all have the same black triangles, then the resulting embeddings of K_{mn} also all have the same black triangles. By reversing the colours, this provides a plentiful supply of embeddings to motivate the following further generalisation.

It is not necessary for the embeddings η^i, described in the first section of the proof, to be copies of the same embedding η. All that the Construction requires is that these embeddings have the "same" white triangular faces (including a common parallel class), all with the "same" orientations.

We can combine our generalisations as follows.

Construction 7 *Suppose that $n \equiv 3 \pmod{12}$ and that $m \equiv 1 \pmod{4}$. Suppose also that we have k differently labelled 2to-embeddings of $K_{m,m,m}$, all of which have a common parallel class of (consistently oriented) black triangular faces. Then we may construct $k^{n(n-3)/6}$ differently labelled 2to-embeddings of K_{mn} all having a common parallel class in black. Furthermore*

1. *If the k embeddings of $K_{m,m,m}$ all have the same black triangular faces then the resulting embeddings of K_{mn} also all have the same black triangular faces.*

2. *If we have N differently labelled 2to-embeddings of K_n, all having the same white triangular faces and a common parallel class in white, then the number of differently labelled 2to-embeddings of K_{mn} may be increased to $N^m k^{n(n-3)/6}$.*

Proof. The proof is a generalisation of that of the previous Construction, with additional features as discussed informally above. We leave the reader to complete the details. □

In applying Construction 7 we can make use of our earlier results concerning embeddings of $K_{m,m,m}$. For $m \equiv 9 \pmod{12}$ we can take k to be either $(m-1)!$ or $2^{m^2/9}$, and for $m \equiv 1$ or $5 \pmod{12}$ we can take $k = (m-1)!$.

As an explicit application, we here consider the cases $m = 5$ and $m = 9$. For $m = 5$, using the value $k = 4! = 24$, we may construct $24^{n(n-3)/6}$ differently labelled 2to-embeddings of K_{5n} for $n \equiv 3 \pmod{12}$. There are therefore at least $24^{n(n-3)/6}/(5n)!$ isomorphism classes. Putting $w = 5n$ and estimating the factorial term, we may express this as follows.

Corollary 7.1 *If $w \equiv 15 \pmod{60}$ then there are at least $2^{aw^2-O(w\log w)}$ non-isomorphic 2to-embeddings of K_w, where $a = \log_2(24)/150 \approx 0.030566$.*

□

In the case $m = 9$, using $k = 8! = 40320$ and applying a similar argument gives the following.

Corollary 7.2 *If $w \equiv 27 \pmod{108}$ then there are at least $2^{bw^2-O(w\log w)}$ non-isomorphic 2to-embeddings of K_w, where $b = \log_2(40320)/486 \approx 0.031480$.*

□

It is clear that we can obtain many other similar estimates.

5 Concluding Remarks

Our estimates for the number of isomorphism classes for K_n embeddings obtained from the constructions in Sections 3 and 4 have the form $2^{an^2-O(n\log n)}$. In [1] the corresponding estimates have the form $2^{an^2-O(n)}$. By using a similar technique to that described in [1], it should be possible, at least in the case of the constructions of Section 3, to reduce the $O(n\log n)$ term to $O(n)$. Also in [1], it was shown that conclusions could be drawn about the automorphism groups of some of the embeddings, in particular some of the embeddings were shown to have only the trivial automorphism group. Again with particular reference to the constructions of Section 3, it seems feasible that similar results could be obtained by similar methods.

The current results, together with those of [1] establish (for various values of a) the existence of $2^{an^2-o(n^2)}$ non-isomorphic 2to-embeddings of K_n for $n \equiv 7$ or $19 \pmod{36}$, $n \equiv 15 \pmod{60}$, $n \equiv 15$ or $43 \pmod{84}$, $n \equiv 27$

(mod 108), and for many other residue classes. We conjecture that similar results in fact hold for *all* values of $n \equiv 3$ or 7 (mod 12).

In [3] it is remarked that the main construction given there is a "topologised" version of a familiar $n \to 3n-2$ construction for Steiner triple systems. Our Construction 4 may be viewed as a topologised version of an $n \to m(n-1)+1$ construction for Steiner triple systems. There are many recursive constructions for Steiner triple systems and it is possible that some more of these also possess topological counterparts. Further research in this area may facilitate a fuller coverage of the residue classes 3 and 7 modulo 12.

Another important question concerns the true order of growth of the number of isomorphism classes for 2to-embeddings of K_n. The number of differently labelled Steiner triple systems of order n is bounded above by $(e^{-1/2}n)^{n^2/6}$ (see [9]). An easy argument given in [1] then shows that the number of non-isomorphic 2to-embeddings of K_n is bounded above by $n^{n^2/3}$. There is a large gap between our estimates, essentially 2^{an^2}, and this upper bound. It would be of considerable interest to see this gap narrowed.

Finally we observe that the constructions of Sections 3 and 4 may be applied to the non-orientable case. If we start Constructions 4 and 6 with a non-orientable face 2-colourable triangulation of K_n, then these Constructions produce non-orientable face 2-colourable triangulations of $K_{m(n-1)+1}$ and K_{mn} respectively. The results of Constructions 5 and 7 extend in a similar way.

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