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Nonorientable biembeddings of Steiner triple systems

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Abstract

Constructions due to Ringel show that there exists a nonorientable face 2-colourable triangular embedding of the complete graph on n vertices (equivalently a nonorientable biembedding of two Steiner triple systems of order n) for all $n \equiv 3 \pmod{6}$ with $n \geq 9$. We prove the corresponding existence theorem for $n \equiv 1 \pmod{6}$ with $n \geq 13$.

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Topological embedding, complete graph, Steiner triple system.

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1 Introduction

Several recent papers [3, 6, 7, 8] have dealt with the relationship between triangular embeddings of complete graphs and biembeddings of Steiner triple systems. A Steiner triple system of order n , briefly STS(n), is an ordered pair (V, \mathcal{B}) , where V is an n -element set (the *points*) and \mathcal{B} is a set of 3-element subsets of V (the *triples*), such that every 2-element subset of V appears in precisely one triple. A necessary and sufficient condition for the existence of an STS(n) is that $n \equiv 1$ or $3 \pmod{6}$; such values of n are called *admissible*. Given a face 2-colourable triangular embedding of K_n , the complete graph of order n , in an orientable or in a nonorientable surface, the triangular faces in each of the two colour classes form the triples of two STS(n)s. Conversely, given two STS(n)s, say S_1 and S_2 , one may ask whether there is a face 2-colourable triangular embedding of K_n in which the colour classes correspond to isomorphic copies of S_1 and S_2 . Such an embedding is called a *biembedding* of S_1 and S_2 and is described as orientable or nonorientable according as the nature of the surface.

In the nonorientable case, Euler's formula admits the possibility of a face 2-colourable triangular embedding of K_n for all $n \equiv 1$ or $3 \pmod{6}$, $n > 3$. Although there is no such embedding in the case $n = 7$ [5], the unique STS(9) gives rise to a nonorientable biembedding and each of the three pairs obtained from the two nonisomorphic STS(13)s has a nonorientable biembedding [9]. It is also now known that each pair of the 80 nonisomorphic STS(15)s has a nonorientable biembedding [2]. These results lead us to make the following conjecture.

Conjecture 1 *For every admissible $n \geq 9$, every two STS(n)s have a nonorientable biembedding.*

The situation regarding orientable biembeddings is more dubious. Euler's formula shows that such biembeddings can only exist if $n \equiv 3$ or $7 \pmod{12}$. Constructions given by Ringel [15] for $n \equiv 3 \pmod{12}$, and by Youngs [16] for $n \equiv 7 \pmod{12}$ prove that there is at least one orientable biembedding for each such value of n . However, it is known that there are pairs of STS(15)s which admit no orientable biembedding [1].

It is shown in [3] that if there is a nonorientable face 2-colourable triangular embedding of K_n , then there are at least $2^{(n-1)(n-3)/6}$ nonisomorphic

nonorientable face 2-colourable triangular embeddings of K_{3n-2} such that the faces of each of the embeddings of K_{3n-2} can be properly coloured in black and white so that all the embeddings of K_{3n-2} have the same black faces. In the language of STSs, this result can be restated as follows: if two STS(n)s have a nonorientable biembedding, then there is an STS($3n-2$) having a nonorientable biembedding with at least $2^{(n-1)(n-3)/6}$ nonisomorphic STS($3n-2$)s. Hence, constructing nonorientable face 2-colourable triangular embedding of K_n gives us classes of pairs of STS($3n-2$)s that have a nonorientable biembedding. It is worth noticing here that computational evidence suggests that nonorientable face 2-colourable triangulations of K_n are plentiful for all admissible $n \geq 13$. Indeed, it is now known that there are 38 608 nonisomorphic embeddings of this type for $n = 13$ [9].

For $n \equiv 9 \pmod{12}$, Ringel (see page 157 of [15]) constructs such an embedding and he subsequently indicates (on page 158) that a similar construction is possible for $n \equiv 3 \pmod{12}$ ($n \geq 15$); these constructions involve bipartite cascades. An alternative design-theoretical construction valid for all $n \equiv 3 \pmod{6}$ with $n \geq 9$ is given by Ducrocq and Sterboul [4]. However, the triangular embeddings of K_n constructed by Ringel in [15] for $n \equiv 1, 7 \pmod{12}$ ($n \geq 13$) are not face 2-colourable and there appear to be no general constructions of such embeddings given elsewhere in the literature. The other known [12, 13, 14] nonorientable triangular embeddings of K_n for $n \equiv 1 \pmod{12}$ are also not face 2-colourable. A gap therefore exists in the spectrum of known nonorientable face 2-colourable triangular embeddings of K_n , namely $n \equiv 1, 7 \pmod{12}$, $n > 13$.

In [3, 6, 8] recursive constructions are given for both orientable and nonorientable face 2-colourable triangular embeddings of K_n . These may be employed to deal with some of the gaps in the spectrum, either directly or as part of an inductive argument. Thus, for $n \equiv 1 \pmod{12}$ we may consider three subcases:

$$(a) \quad n = 36s + 1 = 3(12s + 1 - 1) + 1, \quad s \geq 1,$$

$$(b) \quad n = 36s + 13 = 6(6s + 3 - 1) + 1, \quad s \geq 1,$$

$$(c) \quad n = 36s + 25 = 3(12s + 9 - 1) + 1.$$

Case (c) follows immediately from Theorem 3 of [6] and the $n \equiv 9 \pmod{12}$ case. Case (b) follows from a nonorientable version of Construction 4 of [8] using a face 2-colourable triangular embedding of the complete tripartite graph $K_{6,6,6}$ having a parallel class in one of the colour classes (such an

embedding is given in [10]), a nonorientable face 2-colourable triangular embedding of K_{13} (see [9]), and the $n \equiv 3 \pmod{12}$ case. The K_{13} embedding also deals with the omitted value $s = 0$ from case (b). Finally, case (a) then follows using Theorem 3 of [6] and an inductive argument. For $n \equiv 7 \pmod{12}$ a similar approach suggests the subcases:

$$(d) \quad n = 36s + 7 = 3(12s + 3 - 1) + 1, \quad s \geq 1,$$

$$(e) \quad n = 36s + 19 = 3(12s + 7 - 1) + 1, \quad s \geq 1,$$

$$(f) \quad n = 36s + 31.$$

Case (d) follows immediately from Theorem 3 of [6] and the $n \equiv 3 \pmod{12}$ case. Case (e) might then follow similarly, using induction, if a method for dealing with case (f) were known. Note that constructing current graphs to prove existence for particular values of n is relatively easy, but discerning a general pattern has proved more difficult.

It is appropriate to note at this point an error which appears in the text preceding Corollary 4 of [3], where the authors state that "... there is a face 2-colourable triangular embedding of K_n in a nonorientable surface for every $n \equiv 1$ or $3 \pmod{6}$ with $n \neq 7 \dots$ ". As noted above, this does not seem to have yet been established. The context in which this statement appears is the use of recursive constructions to produce $2^{O(n^2)}$ nonisomorphic face 2-colourable triangular embeddings of K_n for values of n lying in certain residue classes. A consequence of this error is that, in the nonorientable case, the spectrum of residue classes for which the estimate is correctly established is not as wide as that claimed.

The purpose of the current paper is to close the gap in the known spectrum of nonorientable face 2-colourable triangular embeddings of K_n by establishing that such embeddings exist for all admissible $n \geq 9$. This also ensures that the result prematurely claimed in Corollary 4 of [3] is established, namely, that for every $n \equiv 1$ or $3 \pmod{6}$ with $n \geq 19$, there are at least $2^{(n-1)(n-3)/6}$ nonisomorphic nonorientable face 2-colourable triangular embeddings of K_{3n-2} .

In Section 2, for every $n \equiv 1 \pmod{6}$, $n > 13$, we give a current graph with broken arcs (a cascade) that generates a nonorientable face 2-colourable triangular embedding of K_n . The cascade yields the embedding directly, not as part of an inductive argument. We assume that the reader is familiar with cascades, derived graphs and derived embeddings; details are given in the books by Ringel [15], and by Gross & Tucker [11].

2 The $n \equiv 1 \pmod{6}$ case

To construct a nonorientable face 2-colourable triangular embedding of K_{6s+1} we use a cascade with current group \mathbf{Z}_{6s+1} . The cascade is depicted in a figure as a trivalent digraph with the rotations of the vertices being indicated. The black vertices denote clockwise rotation and the white vertices counter-clockwise rotation. The arcs are of two types: broken and unbroken. The following properties (P1)-(P5) hold:

- (P1) The digraph has $3s$ arcs and an element from each of the $3s$ pairs $\{1, 6s\}, \{2, 6s-1\}, \dots, \{3s, 3s+1\}$ of inverse currents appears as the current on some arc.
- (P2) At every vertex, Kirchhoff's current law (KCL) holds: the sum of currents on the arcs directed into the vertex equals the sum of currents on the arcs directed out of the vertex.
- (P3) The rotations and the arc types yield, up to reversal, exactly one circuit.
- (P4) The digraph is bipartite.
- (P5) The arc with current 1 is traversed by the circuit twice in opposite directions, and there is an arc traversed by the circuit twice in the same direction.

The cascade generates the derived embedding of the derived graph K_{6s+1} . The vertex set of the derived graph is the set $\{0, 1, 2, \dots, 6s\}$ of all elements of \mathbf{Z}_{6s+1} , and the edge set of the derived graph consists of all edges $(x, x+\delta)$, where $x \in \mathbf{Z}_{6s+1}$, $\delta \in \mathbf{Z}_{6s+1} \setminus \{0\}$.

Now we show that a cascade with properties (P1)-(P5) generates a nonorientable face 2-colourable triangular embedding of the derived graph K_{6s+1} . Firstly, note that properties (P1)-(P3) imply that the derived embedding is triangular.

The face set of the derived embedding consists of the faces induced by the vertices of the cascade. The faces induced by a vertex of the cascade are determined by Theorem 4.4.1 of [11], which extends to the nonorientable case as well. Every vertex of the cascade induces $6s+1$ triangular faces, and if F and F' are faces induced by vertices v and v' , respectively, then the faces share a common edge if and only if v and v' are adjacent vertices. Hence, (P4) implies that the derived embedding is face 2-colourable.

It is known [11, page 110] that the derived embedding is nonorientable if the derived graph has a cycle with an odd number of twisted (i.e. type 1) edges. An edge $(x, x + \eta)$ of the derived graph is twisted if in the cascade the arc with current η or $-\eta$ is traversed by the circuit twice in the same direction [11, page 199]. If (P5) holds and there is an edge with current δ such that the edge is traversed by the circuit twice in the same direction, then all the edges $(x, x + \delta)$ (respectively, $(x, x \pm 1)$), $x = 0, 1, \dots, 6s$ of the derived graph are twisted (respectively, not twisted). As a result, the cycle

$$(0, \delta), (\delta, \delta - 1), (\delta - 1, \delta - 2), \dots, (1, 0)$$

of the derived graph has exactly one twisted edge, namely $(0, \delta)$, hence the derived embedding is nonorientable. (Note that for $s \equiv 1 \pmod{2}$ a triangular embedding must be nonorientable.)

The cascades will contain fragments of the form indicated in Fig.1(a) which has two vertical arcs connected by dots; the arcs carry currents d and k , respectively, where $d \geq k$, and the arcs are directed in the same direction if and only if $d - k \equiv 0 \pmod{2}$. The fragment indicated in Fig.1(a) is an abbreviated representation of the ladder-like fragment with exactly $d - k + 1$ vertical rungs shown in Fig.1(b). If we consider these $d - k + 1$ rungs from left to right, then the rungs are directed in alternating fashion up and down, and carry the currents $d, d - 1, d - 2, \dots, k + 1, k$, respectively. The horizontal arcs of the ladder-like fragment are directed from left to right and are assigned currents so that KCL holds at every vertex. All vertices on the top (respectively, bottom) line of the ladder-like fragment have clockwise (respectively, counterclockwise) rotation.

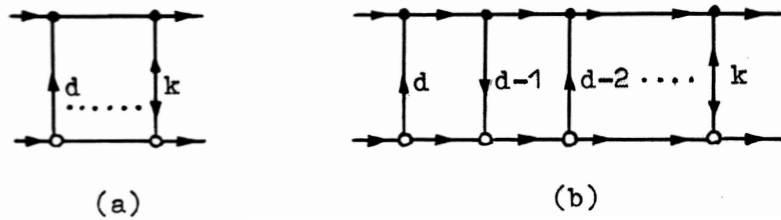


Figure 1: The representation of a ladder-like fragment of a cascade.

Figures 2-5 then give the cascades depending on the residue class of s modulo 4.

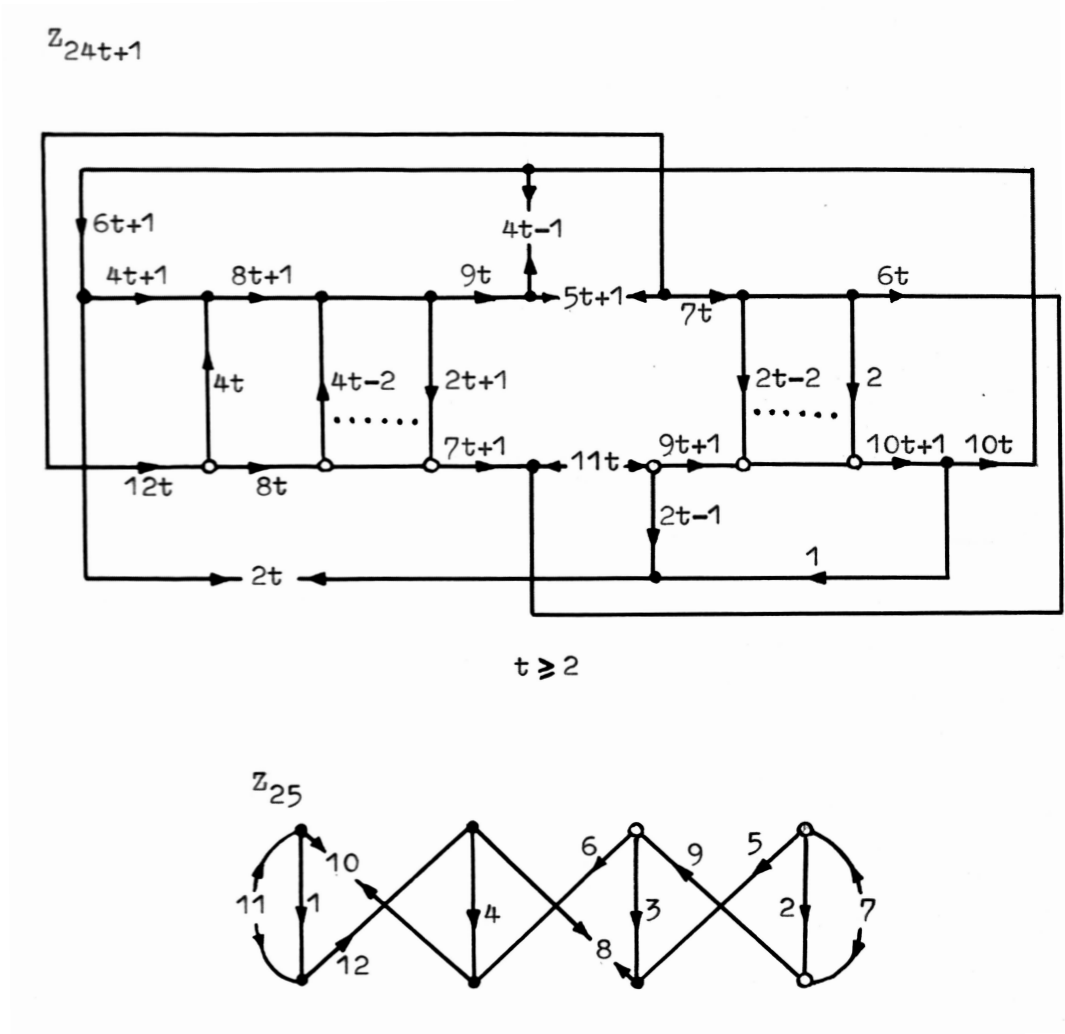


Figure 2: The cascade for the case $s = 4t$.

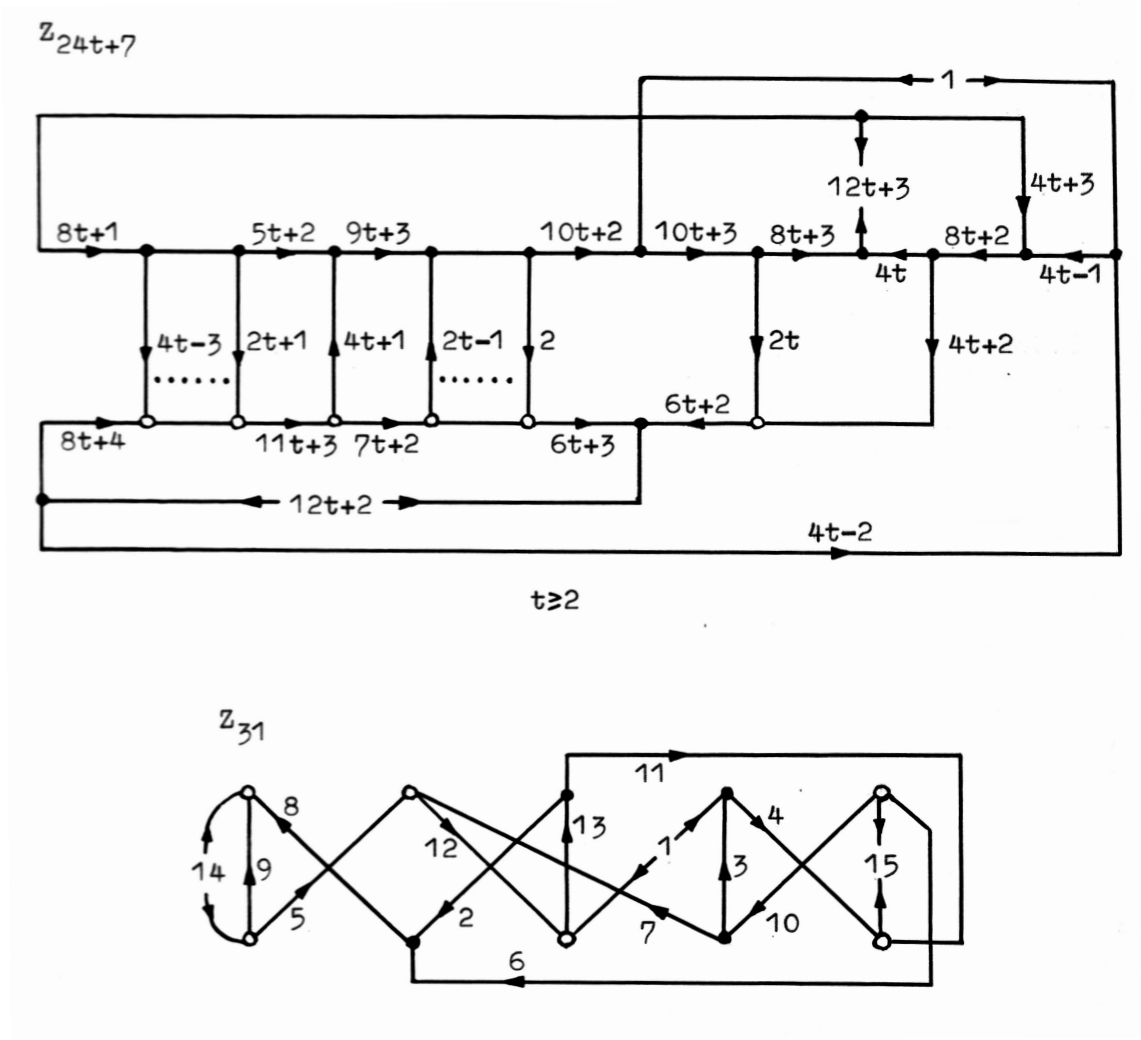


Figure 3: The cascade for the case $s = 4t + 1$.

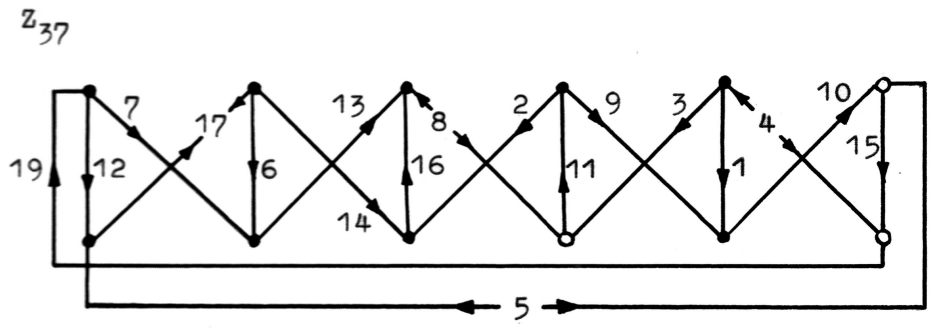
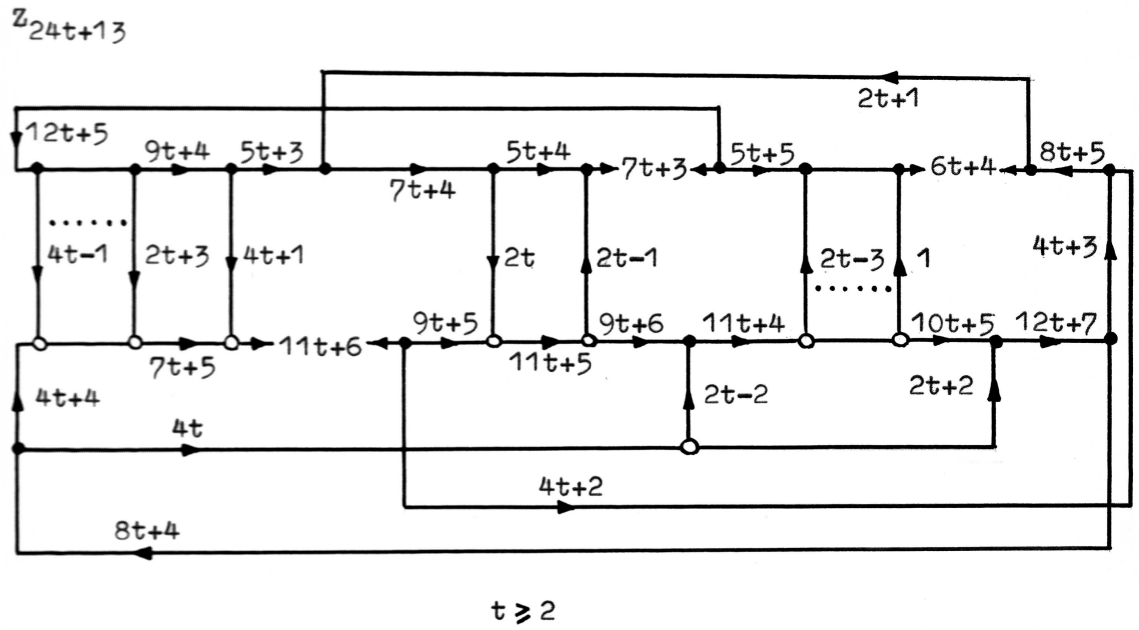
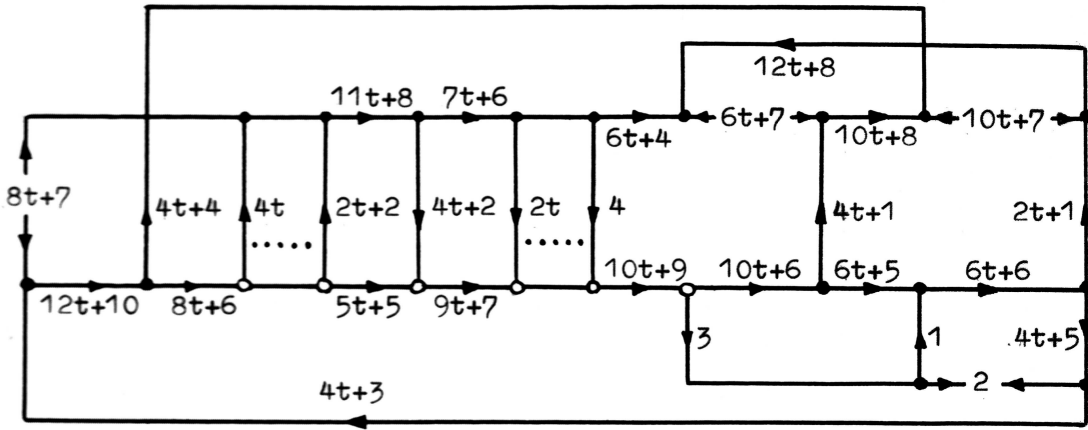


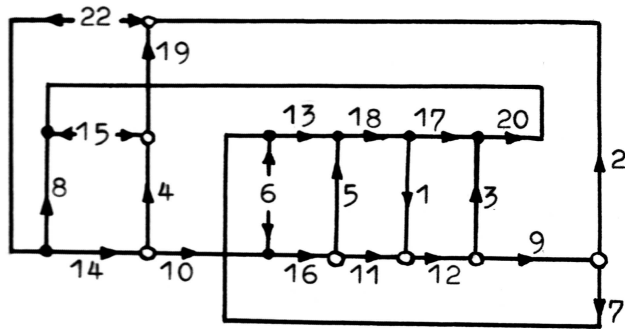
Figure 4: The cascade for the case $s = 4t + 2$.

z_{24t+19}



$t \geq 2$

z_{43}



z_{19}

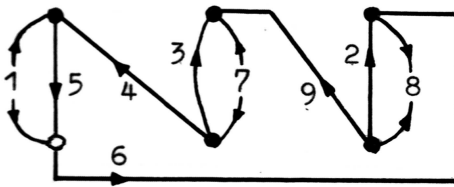


Figure 5: The cascade for the case $s = 4t + 3$.

For each cascade, taking into account the abbreviated representation established in Fig.1, the reader can check the validity of (P1)-(P5).

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