A census of minimal pair-coverings with restricted largest block length

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Abstract

The quantity $g^{(k)}(v)$ was introduced in [6] as the minimum number of blocks necessary in a pairwise balanced design on v elements, subject to the condition that the longest block has length k. Recently, we have needed to use all possibilities for such minimal covering designs, and we record all non-isomorphic solutions to the problem for $v \leq 13$.

1 Introduction

Let V be a set of cardinality v. The quantity $g^{(k)}(v)$ was introduced in [6] as the minimum number of blocks necessary in a pairwise balanced design on v elements (PBD(v)), subject to the condition that the longest block has cardinality k. Values of $g^{(k)}(v)$ for $2 \leq k \leq v \leq 13$ are given in [6] and for $2 \leq k \leq v$, $14 \leq v \leq 21$, with two omissions, in [5]. Although these values of $g^{(k)}(v)$ are known and much effort has been put into their determination, a study of the designs which attain these bounds and the number of non-isomorphic solutions for each case has received less attention. Recently, we have needed to use this information and so have provided it. In this paper we record the results, concentrating on the cases $2 \leq k \leq v \leq 13$, although some of the theorems are more general.

An aim of this paper is to be comprehensive. Hence the results range from the elementary to the complex. We also either list all the relevant designs or describe how they may be constructed. Occasionally this involves rehashing well-known information, but the extra space needed is insignificant compared with the advantage of not needing to refer to other papers. We will take as our representation of V the set $\{n : 1 \leq n \leq v, n \in \mathrm{Z}\}$. The elements 10, 11, 12, and 13 will also be called A, B, C, and D, respectively, where appropriate. Often set brackets and commas will be omitted from listings of designs where no confusion arises. The number of pairwise non-isomorphic designs corresponding to each $g^{(k)}(v)$ will be denoted by $N^{(k)}(v)$. We will not re-prove the values of $g^{(k)}(v)$. These are already given in [6]. They are quoted in the statement of all the theorems and used to determine the required designs and values of $N^{(k)}(v)$. Other standard terminology used throughout the paper is as follows. In any of the designs, the number of blocks of cardinality n, $n \geq 2$, will be denoted by b_n. Further, the number of blocks of cardinality n, that is, the frequency, in which a given element occurs will be denoted by f_n. In the proofs of the theorems, the arguments often proceed 'without loss of generality'. This is to be understood throughout the paper and avoids the necessity of a formal statement on every occasion. Similarly, and an example of this occurs immediately below, in places where it is stated that the number of designs has a certain value, this is 'up to isomorphism'. These words too will be omitted but are to be understood.

Finally, one piece of mathematics which recurs throughout the paper concerns sets of one-factors of the complete graph K_6. It is very well known (for example, see [8]) that there is a unique one-factorization of K_6. It is perhaps most appropriately described on the vertex set $\mathrm{Z}_5 \cup \{\infty\}$. The one-factors are the edge sets $\{(\infty, n), (n+1, n+4), (n+2, n+3)\}$, $0 \leq n \leq 4$. It is easily seen that the automorphism group acts doubly transitively on the one-factors. There is a unique set of four one-factors, obtained by removing any one-factor from the one-factorization. There are two sets of three one-factors. One of these is obtained by removing any two one-factors (a 6-cycle) from the one-factorization and the other consists of the one-factors $\{(\infty,0),(1,2),(3,4)\}$, $\{(0,1),(2,3),(4,\infty)\}$, $\{(\infty,2),(0,3),(1,4)\}$. It does not extend to a one-factorization. We will refer to these two sets of three one-factors as the extendable set and non-extendable set, respectively.

2 Designs with 'long' blocks

We begin with some easy general results.

Theorem 2.1 *$g^{(v)}(v)=1$, $N^{(v)}(v)=1$, $v\geq 2$.*

Proof The unique solution is $\{1,2,\ldots,v\}$. □

Theorem 2.2 *$g^{(v-1)}(v)=v$, $N^{(v-1)}(v)=1$, $v\geq 3$.*

Proof The unique solution is $\{1,2,\ldots,v-1\}$ and all pairs $\{i,v\}$, $1\leq i\leq v-1$. □

Theorem 2.3 *$g^{(v-2)}(v)=2v-4$, $N^{(v-2)}(v)=1$, $v\geq 5$.*

Proof The unique solution is $\{1,2,\ldots,v-2\}$, $\{1,v-1,v\}$, and all pairs $\{i,v-1\}$ and $\{i,v\}$, $2\leq i\leq v-2$. □

Before proceeding, we need some further results. In [9], Woodall showed that, in effect, $g^{(k)}(v)\geq 1+(v-k)(3k-v+1)/2$. It was later shown (Theorem 3.3 of [6]) that if $v\equiv 1 \pmod 4$ this bound is achieved for $k>(v-1)/2$ and otherwise for $k\geq(v-1)/2$. Moreover (Theorem 3.4 of [6]), the only designs which achieve these bounds are those which use pairs and triples as blocks other than the 'long' block. Having stated these results we can give the next theorem.

Theorem 2.4 *$g^{(v-3)}(v)=3v-11$, $N^{(v-3)}(v)=1$, $v\geq 6$.*

Proof The unique solution is $\{1,2,\ldots,v-3\}$, $\{1,v-2,v-1\}$, $\{1,v\}$, $\{2,v-2,v\}$, $\{2,v-1\}$, $\{3,v-1,v\}$, $\{3,v-2\}$ and all pairs $\{i,v-2\}$, $\{i,v-1\}$, and $\{i,v\}$, $4\leq i\leq v-3$. □

The next two theorems are also general in nature but require more detailed analysis of the structure of the solutions. This is particularly true of Theorem 2.6.

Theorem 2.5 *$g^{(v-4)}(v) = 4v-21$, $v \geq 7$. $N^{(3)}(7) = 1$, $N^{(4)}(8) = 2$, $N^{(5)}(9) = 3$, $N^{(v-4)}(v) = 4$, $v \geq 10$.*

Proof Let the 'long' block be $\{1, 2, \ldots, v-4\}$. Let the elements $v-3, v-2, v-1, v$ be called X, Y, Z, W.

One solution is obtained by taking the remaining blocks to be {1, X, Y}, {1, Z, W}, {2, X, Z}, {2, Y, W}, {3, X, W}, {3, Y, Z}, and all pairs $\{i, \mathrm{X}\}$, $\{i, \mathrm{Y}\}$, $\{i, \mathrm{Z}\}$, and $\{i, \mathrm{W}\}$, $4 \leq i \leq v-4$; design 2.5.1.

If $v \geq 8$, a second solution is obtained by replacing, in design 2.5.1, {3, Y, Z}, {4, Y}, {4, Z}, by {4, Y, Z}, {3, Y}, {3, Z}; design 2.5.2.

If $v \geq 9$, a third solution is obtained by replacing, in design 2.5.2, {2, Y, W}, {5, Y}, {5, W}, by {5, Y, W}, {2, Y}, {2, W}; design 2.5.3.

If $v \geq 10$, a fourth solution is obtained by replacing, in design 2.5.3, {1, Z, W}, {6, Z}, {6, W}, by {6, Z, W}, {1, Z}, {1, W}; design 2.5.4.

Observe that, in general, the number of non-isomorphic solutions is equal to and corresponds directly with the number of non-isomorphic proper edge-colourings of the complete graph K_4. Let the vertices of K_4 be X, Y, Z, W. Then there are four proper edge-colourings as follows.

#	# colour classes	Colour classes
1	3	{XY, ZW}, {XZ, YW}, {XW, YZ}
2	4	{XY, ZW}, {XZ, YW}, {XW}, {YZ}
3	5	{XY, ZW}, {XZ}, {YW}, {XW}, {YZ}
4	6	{XY}, {ZW}, {XZ}, {YW}, {XW}, {YZ}

To obtain solutions to our problem, triples are formed by assigning each colour class to an element of the 'long' block and completing the design with pairs. The only restriction is when the number of colour classes exceeds the number of elements in the 'long' block. □

Theorem 2.6 *$g^{(v-5)}(v) = 5v-34$, $v \geq 10$. $N^{(5)}(10) = 1$, $N^{(6)}(11) = 3$, $N^{(7)}(12) = 7$, $N^{(8)}(13) = 9$, $N^{(9)}(14) = 10$, $N^{(v-5)}(v) = 11$, $v \geq 15$.*

Proof Let the 'long' block be $\{1, 2, \ldots, v-5\}$. Let the elements $v-4, v-3, v-2, v-1, v$ be called X, Y, Z, W, V. Using the approach introduced in Theorem 2.5, we see that the proper edge-colourings of the complete graph K_5 on vertex set {X, Y, Z, W, V} are as follows.

#	# colour classes	Colour classes
1	5	{YZ, WV}, {XW, ZV}, {XV, YW}, {XZ, YV}, {XY, ZW}
2	6	{YZ, WV}, {XW, ZV}, {XV, YW}, {XZ, YV}, {XY}, {ZW}
3	6	{YZ, WV}, {XW, ZV}, {XV, YW}, {YV, ZW}, {XY}, {XZ}
4	7	{YZ, WV}, {YW, ZV}, {YV, ZW}, {XY}, {XZ}, {XW}, {XV}
5	7	{XV, ZW}, {YW, ZV}, {YZ, WV}, {XY}, {XZ}, {XW}, {YV}
6	7	{XV, YZ}, {YV, ZW}, {XW, ZV}, {XY}, {XZ}, {YW}, {WV}
7	7	{XW, YV}, {YW, ZV}, {ZW, XV}, {XY}, {XZ}, {YZ}, {WV}
8	8	{XY, ZW}, {XZ, YW}, {XW}, {XV}, {YZ}, {YV}, {ZV}, {WV}
9	8	{XY, ZW}, {XV, YZ}, {XZ}, {XW}, {YW}, {YV}, {ZV}, {WV}
10	9	{XY, ZW}, {XZ}, {XW}, {XV}, {YZ}, {YW}, {YV}, {ZV}, {WV}
11	10	{XY}, {XZ}, {XW}, {XV}, {YZ}, {YW}, {YV}, {ZW}, {ZV}, {WV}

The above results can easily be verified by the reader. They are completely straightforward, and only the case of 7 colour classes is not immediately evident. In order to evaluate the ultimate value of $N^{(v-6)}(v)$, the equivalent problem is to determine the number of non-isomorphic proper edge-colourings of the complete graph K_6. This appears not to be in the published literature and the reason may be that it seems an extremely tedious task. Our cursory investigations indicate that there are several hundred solutions. Consequently, for the last theorem in this section, we have restricted our attention to the values we need, that is, $v = 11$, 12, and 13.

Theorem 2.7 $g^{(v-6)}(v) = 6v - 50$, $v \geq 11$.
$N^{(5)}(11) = 1$, $N^{(6)}(12) = 6$, $N^{(7)}(13) = 34$.

Proof Let the 'long' block be $\{1, 2, \ldots, v-6\}$. Let the elements $v-5, v-4, v-3, v-2, v-1, v$ be called $\mathrm{X, Y, Z, W, V, U}$. Using the approach introduced in Theorem 2.5, we see that the value of $N^{(v-6)}(v)$, $v = 11, 12, 13$, can be found by determining the number of proper edge-colourings, with 5, 6, or 7 colours, of the complete graph K_6 on vertex set $\{\mathrm{X, Y, Z, W, V, U}\}$. The notation $3^{x_3}2^{x_2}1^{x_1}$ will be used to indicate that an edge-colouring has x_n colour classes of cardinality n for $n = 1, 2, 3$. There are respectively 1, 2, and 4 such patterns corresponding to 5, 6, and 7 colours. These are 3^5, $3^4 2^1 1^1$, $3^3 2^3$, $3^4 1^3$, $3^3 2^2 1^2$, $3^2 2^4 1^1$, and $3^1 2^6$.

There is just one proper edge-colouring of K_6 with the pattern 3^5; it is the unique one-factorization of K_6. Hence $N^{(5)}(11) = 1$.

Consider the pattern $3^4 2^1 1^1$. There is a unique set of four disjoint one-factors of K_6: $\{\mathrm{XZ, WY, VU}\}$, $\{\mathrm{XW, VZ, UY}\}$, $\{\mathrm{XV, UW, YZ}\}$, $\{\mathrm{XU, YV, ZW}\}$. The remaining edges of K_6 can be partitioned into the pattern $2^1 1^1$ in three ways:

$$\{\mathrm{XY, ZU}\}, \{\mathrm{WV}\}; \{\mathrm{XY, WV}\}, \{\mathrm{ZU}\}; \{\mathrm{WV, ZU}\}, \{\mathrm{XY}\}.$$

However the permutations (ZVUW) and (XVYW) map the first of these to the second and third, respectively, and fix the set of four one-factors. Hence there is just one edge-colouring of K_6 with the pattern $3^4 2^1 1^1$, namely the above set of four one-factors together with any one of the three patterns $2^1 1^1$ above.

Now consider the pattern $3^3 2^3$. There are two sets of three pairwise disjoint one-factors of K_6. Consider first the non-extendable set of one-factors: $\{\mathrm{XY, ZW, VU}\}$, $\{\mathrm{YZ, WV, XU}\}$, $\{\mathrm{XW, YV, ZU}\}$. The six remaining edges are XZ, ZV, VX, YW, WU, and UY. The first three of these edges, and the last three, form 3-cycles and so must all lie in different colour classes. This gives at most six possibilities for the remaining three colour classes: $\{\mathrm{XZ}, \cdot\}, \{\mathrm{ZV}, \cdot\}, \{\mathrm{VX}, \cdot\}$, where the three dots are replaced in turn by the six arrangements of YW, WU, UY. However, the permutation (YWU) permutes these six in two cycles of three and fixes the above set of three one-factors, leaving at most two possibilities for an edge-colouring of K_6 with pattern

$3^3 2^3$, namely the above set of three one-factors together with either of the following:

$$\{\mathrm{XZ}, \mathrm{YW}\}, \{\mathrm{ZV}, \mathrm{WU}\}, \{\mathrm{VX}, \mathrm{UY}\};$$
$$\{\mathrm{XZ}, \mathrm{YW}\}, \{\mathrm{ZV}, \mathrm{UY}\}, \{\mathrm{VX}, \mathrm{WU}\}.$$

To see that these two edge-colourings are non-isomorphic, consider, for each of them, the three subgraphs induced in the edge-coloured graph made up of the colour classes of cardinality 3 by the vertices of the three colour classes of cardinality 2. For both colourings, all three subgraphs have pattern $2^1 1^2$. In the first colouring, the three subgraphs have two edges of the same colour from the same colour class of cardinality 3, while in the second colouring, the three subgraphs have two edges of the same colour from different colour classes of cardinality 3.

The extendable set of three one-factors of K_6 is $\{\mathrm{XY}, \mathrm{ZW}, \mathrm{VU}\}$, $\{\mathrm{YZ}, \mathrm{WV}, \mathrm{UX}\}$, $\{\mathrm{XW}, \mathrm{YU}, \mathrm{ZV}\}$. The six remaining edges are XZ, ZU, UW, WY, YV and VX, which form a 6-cycle. There are two possibilities for the sequence of three colours round the edges of the 6-cycle, namely $(c_1 c_2 c_3 c_1 c_2 c_3)$ and $(c_1 c_2 c_1 c_3 c_2 c_3)$. Since the permutation (XZUWYV) cyclically permutes the vertices of the 6-cycle, and fixes the above set of three one-factors, all partitions corresponding to a particular colour sequence give isomorphic colourings of K_6. This leaves two non-isomorphic colourings of K_6, corresponding to the two colour sequences, namely the above set of three one-factors together with either of the following:

$$\{\mathrm{XZ}, \mathrm{WY}\}, \{\mathrm{ZU}, \mathrm{YV}\}, \{\mathrm{UW}, \mathrm{VX}\};$$
$$\{\mathrm{XZ}, \mathrm{YV}\}, \{\mathrm{ZU}, \mathrm{WY}\}, \{\mathrm{UW}, \mathrm{VX}\}.$$

Thus there are, in total, four proper edge-colourings of $\mathcal{K}_6$ with the pattern $3^3 2^3$. This completes the determination of $N^{(6)}(12)$: we have $N^{(6)}(12) = 1 + 1 + 4 = 6$.

Now consider the pattern $3^4 1^3$. There is a unique set of four pairwise disjoint one-factors of K_6; since there is just one way to partition the remaining three edges into the pattern 1^3, there is just one proper edge-colouring of K_6 with the pattern $3^4 1^3$:

{XZ, WY, VU}, {XW, VZ, UY}, {XV, UW, YZ},
{XU, YV, ZW}, {XY}, {ZU}, {WV}.

Now consider the pattern $3^3 2^2 1^2$. There are two sets of pairwise disjoint one-factors of K_6. Consider first the non-extendable set: $\{\mathrm{XY}, \mathrm{ZW}, \mathrm{VU}\}$, $\{\mathrm{YZ}, \mathrm{WV}, \mathrm{XU}\}$, $\{\mathrm{XW}, \mathrm{YV}, \mathrm{ZU}\}$. The six remaining edges are XZ, ZV, VX, YW, WU and UY. The first three of these edges, and the last three, form 3-cycles and so must all lie in different colour classes. Both the permutations (XZV) and (YWU) fix the above set of three one-factors, leaving at most two possibilities for an edge-colouring of K_6 with pattern $3^3 2^2 1^2$, namely the above set of three one-factors together with either of the following:

$$\{\mathrm{XZ}, \mathrm{YW}\}, \{\mathrm{ZV}, \mathrm{WU}\}, \{\mathrm{VX}\}, \{\mathrm{UY}\};$$
$$\{\mathrm{XZ}, \mathrm{YW}\}, \{\mathrm{ZV}, \mathrm{UY}\}, \{\mathrm{VX}\}, \{\mathrm{WU}\}.$$

To see that these two edge-colourings are non-isomorphic, consider, for each of them, the two subgraphs induced in the edge-coloured graph made up of the colour classes of cardinality 3 by the vertices of the two colour classes of cardinality 2. For both colourings, both subgraphs have pattern $2^1 1^2$. In the first colouring, the two subgraphs have two edges of the same colour from the same colour class of cardinality 3, while in the second colouring, the two subgraphs have two edges of the same colour from different colour classes of cardinality 3.

The extendable set of three one-factors of K_6 is $\{\mathrm{XY}, \mathrm{ZW}, \mathrm{VU}\}$, $\{\mathrm{YZ}, \mathrm{WV}, \mathrm{UX}\}$, $\{\mathrm{XW}, \mathrm{YU}, \mathrm{ZV}\}$. The six remaining edges are XZ, ZU, UW, WY, YV and VX, which form a 6-cycle. There are four possibilities for the sequence of four colours round the edges of the 6-cycle: $(c_1c_2c_1c_2c_3c_4)$, $(c_1c_2c_1c_3c_2c_4)$, $(c_1c_2c_3c_1c_2c_4)$, and $(c_1c_2c_3c_2c_1c_4)$. Since the permutation (XZUWYV) cyclically permutes the vertices of the 6-cycle, and fixes the above set of three one-factors, all partitions corresponding to a particular colour sequence give isomorphic colourings of K_6. This leaves four non-isomorphic colourings of K_6, corresponding to the four colour sequences, namely the above set of three one-factors together with each of the following:

$$\{\mathrm{XZ}, \mathrm{UW}\}, \{\mathrm{ZU}, \mathrm{WY}\}, \{\mathrm{YV}\}, \{\mathrm{VX}\};$$
$$\{\mathrm{XZ}, \mathrm{UW}\}, \{\mathrm{ZU}, \mathrm{YV}\}, \{\mathrm{WY}\}, \{\mathrm{VX}\};$$
$$\{\mathrm{XZ}, \mathrm{WY}\}, \{\mathrm{ZU}, \mathrm{YV}\}, \{\mathrm{UW}\}, \{\mathrm{VX}\};$$
$$\{\mathrm{XZ}, \mathrm{YV}\}, \{\mathrm{ZU}, \mathrm{WY}\}, \{\mathrm{UW}\}, \{\mathrm{VX}\}.$$

Thus there are, in total, six proper edge-colourings of K_6 with the pattern $3^3 2^2 1^2$.

Now consider the pattern $3^2 2^4 1^1$. There is a unique set of two pairwise disjoint one-factors of K_6: $\{XY, ZW, VU\}$, $\{YZ, WV, UX\}$; these form the colour classes of cardinality three. It is easy to check that each colour class of cardinality two must take the form of an image, under some power of the permutation (XYZWVU), of one of the following: $\{XV, YW\}$, $\{XZ, YU\}$, $\{XV, ZU\}$, $\{XW, YV\}$. We shall say that all such images of the first, second, third, and fourth of these are of types a, b, c, and d, respectively. Using the fact that the only permutations of X, Y, Z, W, V, U which fix the set of two one-factors are those in the group $\langle(XYZWVU), (YU)(ZV)\rangle$, we find that there are eleven possibilities for a proper edge-colouring of K_6 with the pattern $3^2 2^4 1^1$. The colour classes of cardinalities two and one are given in the following table. The left-hand column describes the types of the four classes of cardinality two. These results are easily checked by the reader.

a,a,a,d	$\{XZ, WU\}, \{XV, YW\}, \{YU, ZV\}, \{XW, YV\}, \{ZU\}$
a,a,c,d	$\{XV, YW\}, \{XZ, WU\}, \{XW, YU\}, \{YV, ZU\}, \{ZV\}$
a,b,c,c	$\{XV, YW\}, \{ZV, WU\}, \{XZ, YV\}, \{XW, YU\}, \{ZU\}$
a,b,b,d	$\{XV, YW\}, \{XZ, YU\}, \{ZV, WU\}, \{XW, YV\}, \{ZU\}$
	$\{YU, ZV\}, \{XZ, YW\}, \{XV, WU\}, \{XW, YV\}, \{ZU\}$
a,b,c,d	$\{XV, YW\}, \{XZ, YU\}, \{YV, WU\}, \{XW, ZU\}, \{ZV\}$
b,b,b,d	$\{XZ, YW\}, \{XV, YU\}, \{ZV, WU\}, \{XW, YV\}, \{ZU\}$
b,b,c,c	$\{XZ, YW\}, \{XV, YU\}, \{XW, ZV\}, \{YV, WU\}, \{ZU\}$
b,b,c,d	$\{YW, XZ\}, \{WU, XV\}, \{XW, YU\}, \{YV, ZU\}, \{ZV\}$
	$\{XV, YU\}, \{XZ, YW\}, \{YV, WU\}, \{XW, ZU\}, \{ZV\}$
b,c,c,c	$\{XZ, YW\}, \{XW, YU\}, \{YV, WU\}, \{XV, ZU\}, \{ZV\}$

Finally, consider the pattern $3^1 2^6$. The colour class of cardinality three is the one-factor $\{XY, ZW, VU\}$. The remaining edges may be partitioned into the following three 4-cycles: $\{XZ, ZY, YW, WX\}$, $\{XV, VY, YU, UX\}$, $\{ZV, VW, WU, UZ\}$; these are not, of course, colour classes, but we use them to facilitate the enumeration. We may consider the three 4-cycles to be equivalent, since the powers of the permutation (XUW)(YVZ) fix the colour class of cardinality three and cyclically permute the 4-cycles. Using the fact that the only permutations of X, Y, Z, U, V, W which fix the one-factor are those in the group $\langle(XY),(ZW),(VU),(XW)(YZ),(XU)(YV),(WU)(ZV)\rangle$, we find that there are ten possibilities for a proper edge-colouring of K_6 with the pattern $3^1 2^6$. The colour classes of cardinality two are given in the following table. The left-hand column describes the number of different colours in each of the three 4-cycles. All the cases are straightforward to check, with the exception of the case 4/4/4, which is more involved. It is trivial to check that the cases 2/2/3, 2/2/4 and 2/3/4 are impossible.

2/2/2	$\{XZ,YW\},\{XW,YZ\},\{XV,YU\},\{XU,YV\},\{ZV,WU\},\{ZU,WV\}$
2/3/3	$\{XZ,YW\},\{XW,YU\},\{XV,YZ\},\{XU,YV\},\{ZV,WU\},\{ZU,WV\}$
2/4/4	$\{XZ,WV\},\{XW,ZV\},\{XV,YU\},\{XU,YV\},\{YZ,WU\},\{YW,ZU\}$
	$\{XZ,WV\},\{XW,ZU\},\{XV,YU\},\{XU,YV\},\{YZ,WU\},\{YW,ZV\}$
3/3/3	$\{XZ,YW\},\{XW,ZV\},\{XV,YU\},\{XU,YZ\},\{YV,WU\},\{ZU,WV\}$
3/3/4	$\{XZ,YV\},\{XW,YZ\},\{XV,ZU\},\{XU,YW\},\{YU,WV\},\{ZV,WU\}$
3/4/4	$\{XZ,WU\},\{XW,ZU\},\{XV,YW\},\{XU,YV\},\{YZ,WV\},\{YU,ZV\}$
	$\{XZ,WV\},\{XW,ZU\},\{XV,YW\},\{XU,YV\},\{YZ,WU\},\{YU,ZV\}$
4/4/4	$\{XZ,WV\},\{XW,ZU\},\{XV,YW\},\{XU,YZ\},\{YV,WU\},\{YU,ZV\}$
	$\{XZ,WV\},\{XW,YV\},\{XV,WU\},\{XU,YZ\},\{YW,ZU\},\{YU,ZV\}$

This completes the determination of $N^{(7)}(13)$: we have $N^{(7)}(13) = 1+1+4+1+6+11+10 = 34$. □

3 The cases $k = 2$ and $k = 3$

The first case is trivial; the second involves the theory of triple systems.

Theorem 3.1 *$g^{(2)}(v) = v(v-1)/2$, $N^{(2)}(v) = 1$, $v \geq 2$.*

Proof The unique solution is all pairs. □

Theorem 3.2

$$g^{(3)}(v) = \begin{cases} v(v-1)/6, & v \equiv 1 \text{ or } 3 \pmod 6, v \geq 3; \\ v(v+1)/6, & v \equiv 0 \text{ or } 2 \pmod 6, v \geq 6; \\ (v^2+v+4)/6, & v \equiv 4 \pmod 6, v \geq 4; \\ (v^2-v+16)/6, & v \equiv 5 \pmod 6, v \geq 5. \end{cases}$$

$$N^{(3)}(v) = \begin{cases} 1, & 3 \leq v \leq 9; \\ 2, & v = 10, 11 \text{ or } 13; \\ 5, & v = 12. \end{cases}$$

Proof

(i) For $v \equiv 1$ or 3 (mod 6), $v \geq 3$, solutions are Steiner triple systems on v elements (STS(v)). These contain $v(v-1)/6$ triples. For $v = 3$, 7, and 9, the solutions are unique; there are two solutions for $v = 13$. These are all given below.

STS(3):	123.
STS(7):	123, 145, 167, 246, 257, 347, 356.
STS(9):	123, 456, 789, 147, 258, 369, 159, 267, 348, 168, 249, 357.
STS(13) #1:	123, 145, 167, 189, 1AB, 1CD, 246, 257, 28A, 29C, 2BD, 348, 35C, 36A, 37B, 39D, 479, 4AD, 4BC, 56D, 58B, 59A, 68C, 69B, 78D, 7AC.
STS(13) #2:	In STS(13)# 1, replace 36A, 39D, 56D, 59A, by 36D, 39A, 56A, 59D.

STS(13)#1 has a cyclic automorphism; an alternative presentation is as the set of blocks generated by the orbit starters $\{0,1,4\}$ and $\{0,2,7\}$ under the action of the mapping $z \mapsto z+1$ (mod 13). In this representation, STS(13)#2 is obtained by replacing the blocks $\{0,1,4\}, \{0,2,7\}, \{2,4,9\}$, $\{7,9,1\}$ by the blocks $\{0,1,7\}$, $\{0,2,4\}$, $\{2,7,9\}$, $\{4,9,1\}$.

(ii) For $v \equiv 0$ or 2 (mod 6), $v \geq 6$, solutions are obtained from an STS(v+1) by deleting any element. Hence $g^{(3)}(v) = v(v+1)/6$.

The point transitivity of the unique STS(7) and STS(9) implies that there is just one solution for $v = 6$ or 8. Similarly, since STS(13)#1 has a cyclic automorphism, deleting any element yields the same solution for $v = 12$. However, the automorphism partitioning of STS(13)#2 is $(1, 2, 5, 6, 8, \mathrm{D})(3, 9, \mathrm{A})$ $(4, \mathrm{B}, \mathrm{C})(7)$ and four further solutions are obtained by deleting, in turn, one element from each cycle. In fact all five solutions are non-isomorphic. Hence $N^{(3)}(12) = 5$. Further information concerning the designs in parts (i) and (ii) of this proof may be found in [3] on which much of the above is based.

(iii) For $v \equiv 4 \pmod 6$, $v \geq 4$, solutions are obtained by taking the maximum number of triples which may be formed; the design being completed with the missing pairs. Such a system is called a maximum partial Steiner triple system (MSTS(v)) and contains $(v^2 - 2v - 2)/6$ triples [2]. With the $(v + 2)/2$ missing pairs this gives $g^{(3)}(v) = (v^2 + v + 4)/6$.

For $v = 4$, the unique solution is 123, 14, 24, 34.

For $v = 10$, it is well known that there are two solutions. We briefly indicate how these may be derived. Firstly, recall the definitions of b_n and f_n. Now note that $b_2 + b_3 = 19$ and $b_2 + 3b_3 = 45$ giving $b_2 = 6$, $b_3 = 13$. Since $f_2 + 2f_3 = 9$, f_2 is odd. It follows that, for nine of the elements $f_2 = 1$, $f_3 = 4$; whilst for the tenth element, $f_2 = 3$, $f_3 = 3$. Hence any solution contains the pairs A1, A2, A3, 45, 67, 89, and the triples A47, A69, A58. Now consider the elements 1, 2, and 3 and how these are distributed in the remaining ten triples. There are two possibilities. The first possibility is that 123 is a triple and that there are three triples containing each of 1, 2, and 3. Now consider the one-factorization of the complete graph K_6 with vertex-set $\{4, 5, 6, 7, 8, 9\}$. Two of the one-factors can be taken to be $45, 67, 89$ (pairs) and $47, 69, 58$ (triples with A). Assign the three further one-factors respectively to the triples containing 1, 2, and 3. This gives as the completion 123, 148, 156, 179, 249, 257, 268, 346, 359, 378.

The second possibility is that there are three triples, each containing one of 12, 13, and 23, and two triples containing each of

1, 2, and 3. There remains a tenth triple and it is easily verified that the completion is 468, 128, 134, 236, 156, 179, 257, 249, 359, 378.

(iv) For $v \equiv 5 \pmod 6$, $v \geq 5$, solutions are obtained again by taking an MSTS(v) and completing with the missing pairs. In this case the number of triples is $(v^2 - v - 8)/6$, and there are 4 missing pairs which are isomorphic to ab, bc, cd, da [2]. Hence $g^{(3)}(v) = (v^2 - v + 16)/6$.

For $v = 5$, the unique solution is 123, 145, 24, 25, 34, 35.

For $v = 11$, there are two solutions whose derivation we sketch. The four pairs are 12, 23, 34, 14. Two of the triples contain 13 and 24, respectively; also there must be three triples containing each of 1, 2, 3, and 4. Since there is a total of 17 triples, this leaves three triples to be formed from the elements 5, 6, 7, 8, 9, A, and B. It is easily seen that, in order for the design to be completed, each of these elements must occur in one of the three triples. There are two possibilities.

The first of these is 567, 589, 6AB, and the system is completed by 135, 246, 168, 17A, 19B, 25A, 279, 28B, 369, 37B, 38A, 45B, 478, 49A.

The second possibility is 567, 589, 5AB, and the system is completed by 135, 245, 168, 17A, 19B, 26A, 279, 28B, 36B, 378, 39A, 469, 47B, 48A. Note that the pairs occurring with the elements 1, 2, 3, 4, and 5 are again one-factors of the one-factorization of K_6 with vertex-set $\{6, 7, 8, 9, \mathrm{A}, \mathrm{B}\}$.

It is clear from their construction that the two designs for $v = 10$ in part (iii) are non-isomorphic as are the two designs for $v = 11$ in part (iv). Any MSTS(v) with $v = 10$ or $v = 11$ is isomorphic to one of these designs. They are all listed in [2], from which further information may be obtained. □

4 The remaining cases with $k = 4$

The combined results of Sections 2 and 3 leave only the determination of $N^{(4)}(v)$, $9 \leq v \leq 13$, $N^{(5)}(12)$, $N^{(5)}(13)$ and $N^{(6)}(13)$. We next deal with the cases with $k = 4$, beginning with the largest value, $v = 13$.

Theorem 4.1 $g^{(4)}(v) = 13$, $N^{(4)}(v) = 1$, $11 \leq v \leq 13$.

Proof

(i) For $v = 13$, the unique solution is the projective plane PG(2, 3). The lines of the plane are 123A, 456A, 789A, 147B, 258B, 369B, 159C, 267C, 348C, 168D, 249D, 357D, ABCD. An alternative presentation of this design is as the set of blocks generated by the orbit starter $\{0, 1, 4, 6\}$ under the action of the mapping $z \mapsto z + 1 \pmod{13}$.

(ii) PG(2,3) has a cyclic automorphism. The unique solution for $v = 12$ is obtained by deleting a single point, say D, from the lines listed in part (i).

(iii) Similarly for $v = 11$, a solution is obtained by deleting two points, say C and D, from the lines listed in part (i), thus obtaining a linear space of order 11. All linear spaces of order 11 are classified in [4] where the uniqueness of one having 13 lines is confirmed. □

Theorem 4.2 $g^{(4)}(v) = 12$, $N^{(4)}(v) = 1$, $9 \leq v \leq 10$.

Proof

(i) Let $v = 10$. It is known that $g^{(4)}(10) = 12$, [6]. So $b_2+b_3+b_4 = 12$ and $b_2 + 3b_3 + 6b_4 = 45$; there are two possible solutions. The first of these is $b_2 = 3$, $b_3 = 4$, $b_4 = 5$. Consider the five blocks of cardinality 4. It is easily verified that no element can occur in three or more of them and hence every element must occur twice. The blocks are 1234, 1567, 2589, 368A, 479A. Further $f_2 + 2f_3 = 3$; consequently, either $f_2 = 1$, $f_3 = 1$, or $f_2 = 3$, $f_3 = 0$. Since $b_3 = 4$, it follows that at least one element has $f_3 > 1$; hence there is no solution corresponding to this possibility.

The second possibility is $b_2 = 0$, $b_3 = 9$, $b_4 = 3$. Further, $2f_3 + 3f_4 = 9$; consequently either $f_3 = 0$, $f_4 = 3$, or $f_3 = 3$,

$f_4 = 1$. This can occur only if one element, say A, occurs in all three blocks of cardinality 4 and all other elements occur once. The blocks of cardinality 4 are 123A, 456A, 789A. If we now delete element A, we have left twelve triples on 9 elements; this is the unique STS(9). So the system is completed by 147, 258, 369, 159, 267, 348, 168, 249, 357. Observe that the solution may be regarded as adjoining an extra point to each block of one parallel class of the unique STS(9), or by deleting three collinear points, and the line containing them, from PG(2,3) given in part (i) of Theorem 4.1.

(ii) Let $v = 9$. It is known [6] that $g^{(4)}(9) = 12$. So $b_2+b_3+b_4 = 12$ and $b_2 + 3b_3 + 6b_4 = 36$; again there are two possible solutions. The first of these is $b_2 = 6$, $b_3 = 2$, $b_4 = 4$. It is easily verified that it is not possible to construct four blocks of cardinality 4 on 9 elements. So we are left with only the second possibility, $b_2 = 3$, $b_3 = 7$, $b_4 = 2$. Now consider the two blocks of cardinality 4. If they are disjoint, that is, 1234 and 5678, then it is possible to form at most four triples, all containing the element 9. So the two blocks of cardinality 4 must be 1234 and 1567. Now the elements 2, 3, 4, 5, 6, and 7 must occur with five other elements and so each must occur in a pair. These can be taken to be 27, 35, 46. The triple 189 is then forced and the system is completed by 258, 368, 478, 269, 379, 459. Thus this design, which was given in [6], is unique. □

5 The case $k = 5$, $v = 12$

We prove that the design given in [6] is unique.

Theorem 5.1 $g^{(5)}(12) = 18$, $N^{(5)}(12) = 1$.

Proof Recall that the number of blocks of cardinality n, $n \geq 2$, is denoted by b_n. The proof proceeds by various stages. The first of these is to prove that $b_5 = 1$. Assume otherwise. There are two possibilities to consider. The first is that there are two disjoint blocks of cardinality 5, 12345 and 6789A. Then the pairs ij, $1 \leq i \leq 5$, $6 \leq j \leq 10$ must occur in separate blocks; thus $g^{(5)}(12) \geq 2+25 = 27$, a contradiction. The second possibility is that there are two blocks of cardinality 5 intersecting in a point, 12345 and 16789. In this case, the pairs ij, $2 \le i \le 5$, $6 \le j \le 9$ together with the pair 1A must occur in separate blocks; thus $g^{(5)}(12) \ge 2 + 16 + 1 = 19$, another contradiction. So $b_5 = 1$. Let $B_1 = 12345$.

The next stage is to determine the values of b_2, b_3, and b_4. First count blocks and then pairs covered to give $b_2+b_3+b_4 = 18-1 = 17$ and $b_2 + 3b_3 + 6b_4 = 66 - 10 = 56$. There are three solutions: $(b_2, b_3, b_4) = (8, 2, 7)$ or $(5, 7, 5)$ or $(2, 12, 3)$. Now consider the 35 pairs ij, $i \in B_1$, $j \in B_1' = \{6, 7, 8, 9, \mathrm{A}, \mathrm{B}, \mathrm{C}\}$. Because each block of cardinality 2, 3, or 4 can contain only one value of $i \in B_1$, the maximum number of such pairs contained in blocks of this cardinality is 1, 2, or 3, respectively. Hence, if $b_2 = 8$, $b_3 = 2$, $b_4 = 7$, only 8+4+21=33 pairs can be covered. If $b_2 = 5$, $b_3 = 7$, $b_4 = 5$, only 5+14+15=34 pairs can be covered. If $b_2 = 2$, $b_3 = 12$, $b_4 = 3$, then 2+24+9=35 pairs can be covered. So this last solution is the only possibility, and every block must contain an element of B_1, that is, must intersect the block of cardinality 5.

Now consider the possibilities for the three blocks of cardinality 4. Denote these three blocks by B_2, B_3, B_4. Let $j \in B_1'$. Then $j \in B_2 \cup B_3 \cup B_4$; otherwise there would be six pairs of elements of B_1' with j and only five elements of B_1 with which to assemble them in triples. This leaves just two possibilities: the first of which is $678 \subset B_2$, $69\mathrm{A} \subset B_3$, $7\mathrm{BC} \subset B_4$; the second possibility is $678 \subset B_2$, $69\mathrm{A} \subset B_3$, $6\mathrm{BC} \subset B_4$.

For the first possibility, we may assume that $B_2 = 1678$. Then 289, 38A, 48B, 58C are triples; then $B_3 = 469\mathrm{A}$ and $B_4 = 27\mathrm{BC}$. This forces the triples 1AC, 36C, but the pair 9C cannot now be extended.

For the second possibility, let $B_2 = 1678$, $B_3 = 269\mathrm{A}$, $B_4 = 36\mathrm{BC}$. This immediately forces the pairs 46, 56. The system is completed by writing down a one-factorization of the complete graph K_6 with vertex-set $\{7, 8, 9, \mathrm{A}, \mathrm{B}, \mathrm{C}\}$ in which the edges $\{78, 9\mathrm{A}, \mathrm{BC}\}$ belong to different one-factors. It is straightforward to verify that, up to isomorphism, this can be done uniquely. One such completion is by the triples 19B, 1AC, 27B, 28C, 379, 38A, 47A, 48B, 49C, 57C, 589, 5AB. $\square$

6 The case $k = 5$, $v = 13$.

This is a particularly interesting case in that we obtain non-isomorphic solutions having different block structures.

Theorem 6.1 *$g^{(5)}(13) = 19$, $N^{(5)}(13) = 3$.*

Proof The proof initially follows the same general strategy as the previous theorem. First, as in the last section, it is not possible to have two disjoint blocks of cardinality 5. Next, assume that the system contains two blocks of cardinality 5 that intersect in a point, 12345 and 16789. Then the pairs ij, $2 \le i \le 5$, $6 \le j \le 9$ occur in separate blocks, and this produces $2 + 16 = 18$ blocks. None of the pairs 1A, 1B, 1C, 1D can occur in any of these blocks, and so the nineteenth block must be 1ABCD. The system is completed by assigning one of the elements A, B, C, D to each of the pairs ij, $2 \le i \le 5$, $6 \le j \le 9$, to form a transversal design on three groups $(\{2,3,4,5\}, \{6,7,8,9\}, \{\mathrm{A,B,C,D}\})$ of cardinality 4 (TD (3,4)). It is well known that this is equivalent to both a Latin square of side 4 and a one-factorization of the complete bipartite graph $K_{4,4}$. It is also well known (for example, see [1]) that there are two non-isomorphic solutions, and the completions are listed below.

- #1 26A, 27B, 28C, 29D, 36B, 37A, 38D, 39C, 46C, 47D, 48A, 49B, 56D, 57C, 58B, 59A
- #2 In #1 above, replace 48A, 49B, 58B, 59A by 48B, 49A, 58A, 59B.

It remains to consider the case where the system contains only one block of cardinality 5, $B_1 = 12345$. Then $b_2 + b_3 + b_4 = 19 - 1 = 18$ and $b_2 + 3b_3 + 6b_4 = 78 - 10 = 68$. There are three solutions: $(b_2, b_3, b_4) = (8, 0, 10)$ or $(5, 5, 8)$ or $(2, 10, 6)$. By considering the 40 pairs ij, $i \in B_1$, $j \in B_1' = \{6, 7, 8, 9, \mathrm{A, B, C, D}\}$ and the maximum numbers of these which can occur in blocks of cardinality 2, 3 or 4, the first two possibilities can be eliminated. So $b_2 = 2$, $b_3 = 10$, $b_4 = 6$, and every block must contain an element of B_1.

Now, for an element $i \in B_1$, consider the number of blocks of cardinality 2, 3, or 4 in which i occurs. Using the notation introduced earlier, $f_2 + 2f_3 + 3f_4 = 8$. Since $f_n \le b_n$, there are five solutions:

$(f_2, f_3, f_4) = (0,4,0)$ or $(0,1,2)$ or $(1,2,1)$ or $(2,3,0)$ or $(2,0,2)$. Let the number of elements $i \in B_1$ whose frequency distribution is respectively each of these solutions be p_n, $1 \le n \le 5$. Then $p_3 + 2p_4 + 2p_5 = 2$, $4p_1 + p_2 + 2p_3 + 3p_4 = 10$ and $2p_2 + p_3 + 2p_5 = 6$. There are three solutions: $(p_1, p_2, p_3, p_4, p_5) = (2,2,0,0,1)$ or $(1,3,0,1,0)$ or $(1,2,2,0,0)$.

For the first solution the structure of the 18 blocks is as follows, where $\cdot$ signifies an element $j \in B_1'$.

$$\begin{array}{llll}
5\cdot & 1\cdot\cdot & 2\cdot\cdot & 3\cdot\cdot\cdot \\
5\cdot & 1\cdot\cdot & 2\cdot\cdot & 3\cdot\cdot\cdot \\
 & 1\cdot\cdot & 2\cdot\cdot & 4\cdot\cdot\cdot \\
 & 1\cdot\cdot & 2\cdot\cdot & 4\cdot\cdot\cdot \\
 & 3\cdot\cdot & 4\cdot\cdot & 5\cdot\cdot\cdot \\
 & & & 5\cdot\cdot\cdot
\end{array}$$

Let the triples containing the element 1 be 167, 189, 1AB, 1CD and the blocks containing the element 3 be 368, 37AC, 39BD. This leaves only eight possible triples from B_1' with which to complete the quadruples and these form two mutually exclusive collections of 4 triples, namely, (i) 69A, 6BC, 78B, 8AD, and (ii) 69C, 6AD, 78D, 8BC. Since these two collections are isomorphic under the permutation (AC)(BD), which also fixes the blocks of the system constructed so far, it suffices to consider just the first of these collections of 4 triples. This gives quadruples 469A, 478B, and thus a triple 4CD; this is a contradiction. So the first solution does not lead to a completed system. We proceed similarly in the case of the second solution. The structure of the 18 blocks is as follows.

$$\begin{array}{llll}
2\cdot & 1\cdot\cdot & 2\cdot\cdot & 3\cdot\cdot\cdot \\
2\cdot & 1\cdot\cdot & 2\cdot\cdot & 3\cdot\cdot\cdot \\
 & 1\cdot\cdot & 2\cdot\cdot & 4\cdot\cdot\cdot \\
 & 1\cdot\cdot & 3\cdot\cdot & 4\cdot\cdot\cdot \\
 & & 4\cdot\cdot & 5\cdot\cdot\cdot \\
 & & 5\cdot\cdot & 5\cdot\cdot\cdot
\end{array}$$

Exactly the same argument as before eliminates this case.

The third solution does lead to a system. Again, consider the structure of the 18 blocks.

$2\cdot$	$1\cdot\cdot$	$2\cdot\cdot$	$3\cdot\cdot\cdot$
$5\cdot$	$1\cdot\cdot$	$2\cdot\cdot$	$3\cdot\cdot\cdot$
	$1\cdot\cdot$	$5\cdot\cdot$	$4\cdot\cdot\cdot$
	$1\cdot\cdot$	$5\cdot\cdot$	$4\cdot\cdot\cdot$
	$3\cdot\cdot$		$2\cdot\cdot\cdot$
	$4\cdot\cdot$		$5\cdot\cdot\cdot$

Let the triples containing the element 1 be 167, 189, 1AB, 1CD, and the blocks containing the element 3 be 368, 37AC, 39BD. The design can now be uniquely completed with the quadruples 46BC, 48AD, 269A, 578B: the triples 479, 27D, 28C, 56D, 59C, and the pairs 2B, 5A are forced. To our knowledge, this design has not appeared previously in the published literature. □

7 The case $k = 6, v = 13$

This final case is surprisingly complex.

Theorem 7.1 $g^{(6)}(13) = 24$, $N^{(6)}(13) = 9$.

Proof The system contains a block of cardinality 6, $B_1 = 123456$. From [7],

$$\sum_{A(1)} \binom{k_i - 2}{2} + \sum_{A(0)} \binom{k_i}{2} = |A(1)| - (v - k)(3k - v + 1)/2$$

where $A(n)$, $n = 0, 1$, is the set of blocks, X, such that $|X \cap B_1| = n$ and k_i are the other block lengths. The maximum value of the right-hand side of the equation is $23 - 21 = 2$. It therefore follows that there are no further blocks of cardinality 6, no blocks of cardinality 5, and at most two blocks of cardinality 4, which must intersect B_1. Count blocks and pairs covered to give $b_2 + b_3 + b_4 = 24 - 1 = 23$ and $b_2 + 3b_3 + 6b_4 = 78 - 15 = 63$. There are two solutions: $(b_2, b_3, b_4) = (6, 15, 2)$ or $(3, 20, 0)$. But the latter possibility can be eliminated. Since there are no quadruples, and for each $i \in B_1$, the number of pairs ij, $j \in B_1' = \{7, 8, 9, \mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}\}$ is 7, there must be at least one such pair which can not be covered by the triples. Since $b_3 = 3$, there are not enough pairs to cover all the uncovered pairs for $i \in B_1$. Therefore $b_2 = 6$, $b_3 = 15$, $b_4 = 2$. Moreover, as in the previous two theorems, every block must contain an element of B_1.

Now consider the possibilities for the two blocks of cardinality 4. There are three of them; (i) $B_2 = 1789$, $B_3 = 1\text{ABC}$, (ii) $B_2 = 1789$, $B_3 = 2\text{ABC}$, (iii) $B_2 = 1789$, $B_3 = 27\text{AB}$. The first of these can be easily eliminated. The pairs jD, $7 \leq j \leq 12$, must be assigned the elements 1, 2, 3, 4, 5, and 6 in some order. But this gives a repeated pair containing the element 1. The argument for possibility (ii) is similar. Assign triples 1AD, 27D, 3BD, 48D, 5CD, 69D. There then must be pairs 1B, 1C, 28, 29. This leaves only two pairs but there must be at least one containing each of the elements 3, 4, 5, and 6. Again there is a contradiction and so only possibility (iii) remains: $B_2 = 1789$, $B_3 = 27\text{AB}$.

It is now necessary to determine how the system may be completed. A number of preliminary comments are in order. First we note that, if any two solutions are isomorphic, then the block B_1 is stabilized and the two blocks of cardinality 4 must either be stabilized or map to each other. On the set $\{1, 2, 7, 8, 9, \text{A}, \text{B}\}$, there are only six such mappings: the identity $I = P_0$, the transpositions $(89) = P_1$, $(\text{AB}) = P_2$, and the permutation $(89)(\text{AB}) = P_3$, stabilize both B_2 and B_3; the permutations $(12)(8\text{A})(9\text{B}) = P_4$ and $(12)(8\text{B})(9\text{A}) = P_5$ map each block to the other. Hence any isomorphism between two solutions must be an extension of one of these permutations to the set $\{n : 1 \leq n \leq 13, n \in \text{Z}\}$.

Secondly, we list the pairs of elements of B_1' which remain to be covered and partition them into four sublists L1 to L4.

(L1) CD
(L2) 7C, 8C, 9C, AC, BC
(L3) 7D, 8D, 9D, AD, BD
(L4) 8A, 8B, 9A, 9B

There are four cases to consider.

(I) There are triples 1CD, 28C, 29D, which force pairs 1A, 1B.

(II) There are triples 3CD, 1AC, 1BD, 28C, 29D and *precisely one* triple containing the element 3 and a pair from sublist L4.

(III) There are triples 3CD, 1AC, 1BD, 28C, 29D, 38A, 39B.

(IV) There are triples 3CD, 1AC, 1BD, 28C, 29D, 38B, 39A.

In cases I, II and IV the pair 37 is also forced. This exhausts all possibilities. Also, a solution obtained from any one of the cases can not be isomorphic to a solution obtained from any other case. Clearly this is true of case I. In case II the element 3 occurs three times in the pairs, whereas it occurs only once in cases III and IV. To see that cases III and IV yield non-isomorphic solutions, observe that there is no extension of the permutations P_n, $0 \leq n \leq 5$, which interchanges the triples of the two cases. All solutions modulo permutation of the elements 3, 4, 5, 6, in case I and the elements 4, 5, 6, in the other cases are given and isomorphs are then rejected.

(I) There are six solutions.

(a) 38A, 3BD, 37D, 48B, 4AD, 49C, 59A, 5BD, 57C, 69B, 6AC, 68D, 39, 47, 58, 67.

(b) 38A, 3BC, 37C, 48B, 4AD, 49C, 59A, 5BC, 57D, 69B, 6AC, 68D, 39, 47, 58, 67.

(c) 38A, 3BD, 37C, 48B, 4AD, 49C, 59A, 5BC, 58D, 69B, 6AC, 67D, 39, 47, 57, 68.

(d) 38A, 3BD, 39C, 48B, 4AD, 47C, 59A, 5BC, 57D, 69B, 6AC, 68D, 37, 49, 58, 67.

(e) 38A, 3BD, 39C, 48B, 4AD, 47C, 59A, 5BC, 58D, 69B, 6AC, 67D, 37, 49, 57, 68.

(f) 38A, 3BD, 39C, 48B, 4AC, 47D, 59A, 5BC, 58D, 69B, 6AD, 67C, 37, 49, 57, 68.

To determine which of the above solutions are isomorphic it is necessary to consider whether any extensions of the permutations P_n, $0 \leq n \leq 5$, are isomorphisms or automorphisms of the systems. The same procedure is used in possibilities II, III and IV below. For each possibility the extensions are given together with their actions on the solutions. Details, although tedious, are easily checked by the reader.

P_0: stabilizes all solutions.
P_1(CD)(35)(46): stabilizes a, b, e, f; interchanges c and d.
P_2(34)(56): interchanges a & f, b & e, c & d.
P_3(CD)(36)(45): stabilizes c, d; interchanges a & f, b & e.

Hence there are three non-isomorphic solutions corresponding to this possibility.

(II) There are six solutions.

(a) 39B, 47C, 4AD, 48B, 59C, 57D, 58A, 6BC, 68D, 69A, 37, 38, 3A, 49, 5B, 67.

(b) 38B, 47C, 4AD, 49B, 59C, 57D, 58A, 6BC, 68D, 69A, 37, 39, 3A, 48, 5B, 67.

(c) 38A, 47C, 4AD, 49B, 59C, 57D, 58B, 6BC, 68D, 69A, 37, 39, 3B, 48, 5A, 67.

(d) 39B, 47C, 48D, 49A, 59C, 5AD, 58B, 6BC, 67D, 68A, 37, 38, 3A, 4B, 57, 69.

(e) 39A, 47C, 48D, 49B, 59C, 5AD, 58B, 6BC, 67D, 68A, 37, 38, 3B, 4A, 57, 69.

(f) 38A, 47C, 48D, 49B, 59C, 5AD, 58B, 6BC, 67D, 69A, 37, 39, 3B, 4A, 57, 68.

P_0: stabilizes all solutions.
P_3(CD)(465): maps a to f, b to e, c to d.
P_4(56): interchanges a & d, b & e, c & f.
P_5(CD)(45): stabilizes b; interchanges a & c.
P_5(CD)(46): stabilizes e; interchanges d & f.

Hence there are two non-isomorphic solutions corresponding to this possibility.

(III) There are six solutions.

(a) 47C, 48D, 49A, 59C, 57D, 58B, 6BC, 6AD, 37, 4B, 5A, 67, 68, 69.

(b) 47C, 4AD, 48B, 59C, 57D, 6BC, 68D, 69A, 37, 49, 58, 5A, 5B, 67.

(c) 47C, 4AD, 59C, 57D, 58B, 6BC, 68D, 69A, 37, 48, 49, 4B, 5A, 67.

(d) 47C, 48D, 49A, 59C, 5AD, 58B, 6BC, 67D, 37, 4B, 57, 68, 69, 6A.

(e) 47C, 48D, 59C, 5AD, 58B, 6BC, 67D, 69A, 37, 49, 4A, 4B, 57, 68.

(f) 47C, 4AD, 48B, 59C, 58D, 6BC, 67D, 69A, 37, 49, 57, 5A, 5B, 68.

P_0: stabilizes all solutions.
P_3(CD)(45): stabilizes a.
P_3(CD)(46): stabilizes f.
P_3(CD)(465): maps b to e, c to d.
P_4(56): interchanges a & f, b & d, c & e.
P_5(CD)(465): maps a to f.
P_5(CD)(45): interchanges b & c.
P_5(CD)(46): interchanges d & e.

Hence there are two non-isomorphic solutions corresponding to this possibility.

(IV) There are four solutions.

(a) 47C, 48D, 49B, 59C, 57D, 58A, 6BC, 6AD, 37, 4A, 5B, 67, 68, 69.

(b) 47C, 4AD, 49B, 59C, 57D, 58A, 6BC, 68D, 37, 48, 5B, 67, 69, 6A.

(c) 47C, 48D, 49B, 59C, 5AD, 6BC, 67D, 68A, 37, 4A, 57, 58, 5B, 69.

(d) 47C, 4AD, 49B, 59C, 58D, 6BC, 67D, 68A, 37, 48, 57, 5A, 5B, 69.

P_0: stabilizes all solutions.
P_3(CD)(45): stabilizes a.
P_3(CD)(46): stabilizes d.
P_3(CD)(465): maps b to c.
P_4: interchanges a & d, b & c.
P_5(CD)(45): stabilizes b.
P_5(CD)(46): stabilizes c.
P_5(CD)(465): maps a to d.

Hence there are two non-isomorphic solutions corresponding to this possibility.

To summarize, there are, in total, precisely nine non-isomorphic solutions: Ia, Ib, Ic, IIa, IIb, IIIa, IIIb, IVa, IVb. □

8 Conclusion

We record the values of $N^{(k)}(v)$, and, for completeness, of $g^{(k)}(v)$, $2 \leq k \leq v \leq 13$ in the tables below. The latter values are taken from [6].

k/v	2	3	4	5	6	7	8	9	10	11	12	13
2	1	1	1	1	1	1	1	1	1	1	1	1
3		1	1	1	1	1	1	1	2	2	5	2
4			1	1	1	1	2	1	1	1	1	1
5				1	1	1	1	3	1	1	1	3
6					1	1	1	1	4	3	6	9
7						1	1	1	1	4	7	34
8							1	1	1	1	4	9
9								1	1	1	1	4
10									1	1	1	1
11										1	1	1
12											1	1
13												1

Table of values of $N^{(k)}(v)$, $v \leq 13$

k/v	2	3	4	5	6	7	8	9	10	11	12	13
2	1	3	6	10	15	21	28	36	45	55	66	78
3		1	4	6	7	7	12	12	19	21	26	26
4			1	5	8	10	11	12	12	13	13	13
5				1	6	10	13	15	16	16	18	19
6					1	7	12	16	19	21	22	24
7						1	8	14	19	23	26	28
8							1	9	16	22	27	31
9								1	10	18	25	31
10									1	11	20	28
11										1	12	22
12											1	13
13												1

Table of values of $g^{(k)}(v)$, $v \leq 13$

It would be good to know the values of $N^{(v-6)}(v)$ for $v \geq 14$, particularly as the ultimate value is the number of non-isomorphic edge-colourings of the complete graph K_6. It may also be possible to extend the above results to $v = 15$.

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