Mendelsohn Directed Triple Systems

Mike Grannell, Terry S. Griggs and Kathleen A.S. Quinn
Department of Pure Mathematics, The Open University,
Walton Hall, Milton Keynes MK7 6AA

Abstract

We introduce a class of ordered triple systems which are both Mendelsohn triple systems and directed triple systems. We call these *Mendelsohn directed triple systems* (MDTS(v, λ)), characterise them, and prove that they exist if and only if $\lambda(v-1) \equiv 0 \pmod 3$. This is the same spectrum as that of *regular* directed triple systems, of which they are a special case. We also prove that cyclic MDTS(v, λ) exist if and only if $\lambda(v-1) \equiv 0 \pmod 6$. In so doing we simplify a known proof of the existence of cyclic directed triple systems. Finally, we enumerate some 'small' MDTS.

1 Introduction

The concepts of a *Mendelsohn triple system* and a *directed triple system* are well known. Let V be a set of cardinality v, and $\mathcal{B}$ a collection of ordered triples of distinct elements of V. The pair $(V, \mathcal{B})$ is said to be a *Mendelsohn triple system* MTS(v, λ) or a *directed triple system* DTS(v, λ) if every ordered pair of distinct elements of V is contained in precisely λ ordered triples; the two types of system are distinguished by the definition of containment. In a Mendelsohn triple system containment is *cyclic*: an ordered triple (x, y, z) contains the ordered pairs (x, y), (y, z) and (z, x). However, in a directed triple system containment is *transitive*: an ordered triple (x, y, z) contains the ordered pairs (x, y), (y, z) and (x, z). For more information on these systems, see [3, 5, 10]. In graph-theoretic terms, each of the above types of triple system is a decomposition of the complete digraph on v vertices, with each arc taken λ times, into isomorphic copies of a particular digraph on three vertices. The problem of decomposition into other digraphs on three vertices in considered in [9].

Clearly any cyclic shift of all triples of an MTS(v, λ) will result in an MTS(v, λ). In addition, reversing all the triples of an MTS(v, λ) will produce an MTS(v, λ). Thus the property of being Mendelsohn is invariant under all permutations of the positions of the elements in the triples. This is not true of directed triple systems. The converse of a DTS(v, λ) is also a DTS(v, λ), but if the triples are shifted cyclically, a DTS(v, λ) will not necessarily result.

In this paper we are interested in directed triple systems which have the additional property that any cyclic shift of all triples results in a DTS(v, λ). An example with $v = 4$ and $\lambda = 1$ is given by the following triples: $(0, 2, 1)$, $(2, 0, 3)$, $(1, 3, 0)$, $(3, 1, 2)$. We call such systems *Mendelsohn directed triple systems* and denote them by MDTS(v, λ), because they are both Mendelsohn and directed, and remain so under all permutations of the positions of the elements in the triples. This follows from Lemma 1.1 below. In the proof of this lemma, by the *union* $A \cup B$ of two multisets A and B, we mean that if an element occurs m times in A and n times in B, then it occurs $m + n$ times in $A \cup B$.

Lemma 1.1 *Let* $(V, \mathcal{B})$ *be an* MDTS(v, λ). *Denote by* $S_{a,b}$ *the multiset of ordered pairs* (x, y) *in positions* a *and* b *of the triples of* $\mathcal{B}$. *Then* $S_{1,2} = S_{2,1}$, $S_{2,3} = S_{3,2}$ *and* $S_{3,1} = S_{1,3}$.

Proof Let U_λ denote the multiset whose elements are all the ordered pairs of distinct elements of V, each occurring precisely λ times. Since $(V, \mathcal{B})$ is a DTS(v, λ), we have $S_{1,2} \cup S_{1,3} \cup S_{2,3} = U_\lambda$.

It follows from the definition of an MDTS(v, λ) that $(V, \mathcal{B})$ remains a DTS(v, λ) under all permutations of the positions of the elements in the triples. In particular, $(V, \mathcal{B})$ remains a DTS(v, λ) under a transposition of any two positions. Considering the three such transpositions, (12), (23) and (13), in turn, we obtain

$$\begin{aligned} S_{2,1} \cup S_{2,3} \cup S_{1,3} &= U_\lambda; \\ S_{1,3} \cup S_{1,2} \cup S_{3,2} &= U_\lambda; \\ S_{3,2} \cup S_{3,1} \cup S_{2,1} &= U_\lambda. \end{aligned}$$

Comparing the first of these equations with the equation in the first paragraph of the proof, we obtain $S_{1,2} = S_{2,1}$. Similarly, comparing the second of these equations with the equation in the first paragraph, we obtain $S_{2,3} = S_{3,2}$. Finally, comparing the third equation with the equation in the first paragraph, and using $S_{1,2} = S_{2,1}$ and $S_{2,3} = S_{3,2}$, it follows that $S_{1,3} = S_{3,1}$. □

We shall henceforth say that any collection of ordered triples which obey the three multiset equations in the statement of Lemma 1.1 satisfy the *order conditions*. It is not difficult to see that a DTS(v, λ) which satisfies the order conditions is an MDTS(v,λ). Thus a DTS(v,λ) is an MDTS(v,λ) if and only if it satisfies the order conditions.

Finally in this introductory section, we consider *regular* directed triple systems, since, as will be clear from the definition, every Mendelsohn directed triple system is regular. The concept of a regular DTS$(v,1)$ was introduced by Colbourn and Colbourn [6], and extends to DTS(v,λ) for any λ. For a given directed triple system $(V,\mathcal{B})$, let $c_i(x)$ denote the number of times element x appears in position i of a triple. $(V,\mathcal{B})$ is said to be *regular* if and only if $c_1(x) = c_2(x) = c_3(x)$ for all $x \in V$. Colbourn and Colbourn proved that there exists a regular DTS$(v,1)$ if and only if $v \equiv 1 \pmod 3$. The necessary condition $v \equiv 1 \pmod 3$ for the existence of a regular DTS$(v,1)$ can easily be generalised to values of λ greater than 1, as follows.

Lemma 1.2 *If there exists a regular* DTS(v,λ) *then* $\lambda(v-1) \equiv 0 \pmod 3$.

Proof Suppose that $(V,\mathcal{B})$ is a regular DTS(v,λ). Since each ordered pair of elements is transitively contained in the same number of blocks, it follows that each element appears in the same number of blocks. Now $(V,\mathcal{B})$ has $\lambda v(v-1)/3$ blocks; therefore each element appears in $(\lambda v(v-1)/3)(3/v) = \lambda(v-1)$ blocks. Since $(V,\mathcal{B})$ is regular, this number must be divisible by 3. □

It follows immediately from the order conditions that every MDTS(v,λ) is regular. Hence Lemma 1.2 gives a necessary condition for the existence of an MDTS(v,λ).

In Section 2, we settle the existence question for MDTS(v,λ): we show that there exists an MDTS(v,λ) for every v and λ for which the above necessary condition holds. In doing so we obtain an independent proof and generalisation of Colbourn and Colbourn's result that there exists a regular DTS$(v,1)$ if and only if $v \equiv 1 \pmod 3$. In Section 3, we settle the existence question for cyclic MDTS(v,λ): we show that there exists a cyclic MDTS(v,λ) for every v and λ for which $\lambda(v-1) \equiv 0 \pmod 6$. In doing so we obtain a simplification of a proof by Cho, Han and Kang [4] of necessary and sufficient conditions for the existence of cyclic directed triple systems. Finally, in Section 4, we enumerate the non-isomorphic MDTS$(v,1)$ for $v \leq 10$.

2 Existence

Theorem 2.1 below completely settles the existence question for Mendelsohn directed triple systems. In the proof, we use results on *pairwise balanced* *designs*. A (v, K) *pairwise balanced design*, PBD(v, K), is a pair $(V, \mathcal{B})$ where V is a set of cardinality v, and $\mathcal{B}$ is a collection of subsets of V, called *blocks*, with the property that the size of every block is in the set K and every pair of elements of V is contained in precisely one block. Given a set K of positive integers, the *PBD-closure* of K is the set $B(K) = \{v : \exists \text{ PBD}(v, K)\}$.

Note that if there exists an MDTS(k, λ) for all $k \in K$, then it follows that there exists an MDTS(v, λ) for all $v \in B(K)$. This is because, given a PBD(v, K), for each $k \in K$ we may replace all the blocks of cardinality k by an MDTS(k, λ), to yield an MDTS(v, λ).

Theorem 2.1 *There exists an* MDTS(v, λ) *if and only if* $\lambda(v-1) \equiv 0 \pmod 3$.

Proof Since an MDTS(v, λ) is regular, it follows from Lemma 1.2 that it must satisfy $\lambda(v-1) \equiv 0 \pmod 3$. We now show that this necessary condition for the existence of an MDTS(v, λ) is sufficient.

We begin by considering $\lambda = 1$. We exhibit an MDTS$(v, 1)$ for $v = 4, 7, 10, 19$. Since the PBD-closure of $\{4, 7\}$ is $\{v : v \equiv 1 \pmod 3, v \neq 10, 19\}$ [2, 11], it follows that there exists an MDTS$(v, 1)$ for all $v \equiv 1 \pmod 3$.

We gave an example of an MDTS$(4, 1)$ in Section 1, namely the system given by the triples $(0, 2, 1), (2, 0, 3), (1, 3, 0), (3, 1, 2)$. An MDTS$(7, 1)$ is given by the triples generated by $(0, 1, 3)$ and $(0, 6, 4)$ under the action of the mapping $i \mapsto i + 1 \pmod 7$. An MDTS$(19, 1)$ is given by the triples generated by $(0, 1, 5)$, $(0, 18, 14)$, $(0, 2, 8)$, $(0, 17, 11)$, $(0, 3, 10)$ and $(0, 16, 9)$ under the action of the mapping $i \mapsto i + 1 \pmod{19}$. (In fact, the MDTS$(7, 1)$ and MDTS$(19, 1)$ given here are special cases of Lemma 3.3 in the next section.) An MDTS$(10, 1)$ is given below. Each triple (x, y, z) is written simply as xyz.

021	054	087	347	593	836	274	952	628
203	506	809	158	671	914	385	763	439
130	460	790	269	482	725	196	841	517
312	645	978						

We now consider $\lambda = 3$. We exhibit an MDTS$(v, 3)$ for $v = 3, 4, 5, 6, 8$. Since the PBD-closure of $\{3, 4, 5\}$ is $\{v : v \neq 6, 8\}$ [1, 8], it follows that there exists an MDTS$(v, 3)$ for all v.

An MDTS$(3, 3)$ is given by the six orderings of 0,1,2. An MDTS$(4, 3)$ can be formed by taking three copies of an MDTS$(4, 1)$. An MDTS$(5, 3)$ is given by the triples generated by $(0, 1, 4), (0, 4, 1), (0, 2, 3)$ and $(0, 3, 2)$ under the action of the mapping $i \mapsto i + 1 \pmod 5$. An MDTS$(6, 3)$ is given by the triples generated by $(\infty,0,1),(0,\infty,3),(0,2,\infty),(0,1,2),(0,3,2)$ and $(0,4,3)$ under the action of the permutation $(\infty)(01234)$. An MDTS$(8,3)$ is given by the triples generated by $(\infty,0,1)$, $(0,\infty,3)$, $(0,2,\infty)$, $(0,5,4)$, $(0,1,3)$, $(0,6,4)$, $(0,1,5)$ and $(0,6,2)$ under the action of the permutation $(\infty)(0123456)$.

Since n copies of an MDTS(v,λ) form an MDTS$(v,n\lambda)$, it follows from the above results for $\lambda=1$ and $\lambda=3$ that an MDTS(v,λ) exists for all v and λ satisfying the necessary condition $\lambda(v-1)\equiv 0 \pmod 3$. □

The following corollary to Theorem 2.1 shows that we have an independent proof and generalisation of Colbourn and Colbourn's result [6] that there exists a regular DTS$(v,1)$ if and only if $v\equiv 1 \pmod 3$.

Corollary 2.2 *There exists a regular* DTS(v,λ) *if and only if* $\lambda(v-1)\equiv 0 \pmod 3$.

Proof Lemma 1.2 states that $\lambda(v-1)\equiv 0 \pmod 3$ is a necessary condition for the existence of a regular DTS(v,λ). Since every MDTS(v,λ) is regular, it follows from Theorem 2.1 above that this condition is also sufficient. □

3 Cyclic systems

In this section we completely settle the existence question for cyclic Mendelsohn directed triple systems. An MDTS(v,λ) is *cyclic* if it has an automorphism which permutes its points in a cycle of length v. The following lemma gives a necessary condition for the existence of such MDTS(v,λ).

Lemma 3.1 *If there exists a cyclic* MDTS(v,λ)*, then* $\lambda(v-1)\equiv 0 \pmod 6$.

Proof Suppose that there exists a cyclic MDTS(v,λ). Then $\lambda(v-1)\equiv 0 \pmod 3$, by Theorem 2.1. We show that also $\lambda(v-1)\equiv 0 \pmod 2$.

Suppose to the contrary that $\lambda(v-1)$ is odd. Then the number of orbits of the MDTS, namely $\lambda(v-1)/3$, is odd; denote it by n. Consider any two of the three positions, say i and j. From Lemma 1.1 we know that, for any element $x\neq 0$, the number of orbits in which positions i and j contain the sub-orbit generated by $(0,x)$ is equal to the number of orbits in which positions i and j contain the sub-orbit generated by $(x,0)$. This latter sub-orbit is the negative of the sub-orbit generated by $(0,x)$, since it is also generated by $(0,-x)$. Since the number n of orbits is odd, the multiset of sub-orbits appearing in places i and j must consist of pairs comprising orbits and their negatives, except for an odd number of occurrences of the only self-negative sub-orbit, that generated by $(0, v/2)$.

Now assume that the orbits are generated by the triples $(0, x_1, y_1), (0, x_2, y_2), \ldots, (0, x_n, y_n)$. Then the sub-orbits occurring in positions 1 and 2 are generated by the pairs $(0, x_1), (0, x_2), \ldots, (0, x_n)$, those occurring in positions 1 and 3 are generated by the pairs $(0, y_1), (0, y_2), \ldots, (0, y_n)$, and those occurring in positions 2 and 3 are generated by the pairs $(0, y_1 - x_1), (0, y_2 - x_2), \ldots, (0, y_n - x_n)$. It follows from the discussion in the above paragraph that

$$\begin{aligned} x_1 + x_2 + \cdots + x_n &\equiv v/2 \pmod{v}, \\ y_1 + y_2 + \cdots + y_n &\equiv v/2 \pmod{v}, \\ (y_1 + y_2 + \cdots + y_n) - (x_1 + x_2 + \cdots + x_n) &\equiv v/2 \pmod{v} \end{aligned}$$

which is a contradiction. This completes the proof. □

We now prove that the necessary condition in Lemma 3.1 for the existence of a cyclic MDTS(v, λ) is also sufficient. Since a cyclic MDTS(v, λ_1) and a cyclic MDTS(v, λ_2) together form a cyclic MDTS$(v, \lambda_1 + \lambda_2)$, it is enough to prove that there exists a cyclic MDTS(v, λ) for the following values of v and λ.

(i) $\lambda = 1$ and $v \equiv 1 \pmod 6$;

(ii) $\lambda = 2$ and $v \equiv 4 \pmod 6$;

(iii) $\lambda = 3$ and $v \equiv 3, 5 \pmod 6$;

(iv) $\lambda = 6$ and $v \equiv 0, 2 \pmod 6$.

We cover all of these cases in Lemmas 3.3 to 3.8 below. The proofs of some of these use the following preliminary lemma, which is easily seen to be true.

Lemma 3.2 *Let $x, y \in \mathrm{Z}_v$. Then the ordered triples generated by $(0, x, y)$ and $(0, -x, -y)$ under the action of the mapping $i \mapsto i + 1 \pmod v$ satisfy the order conditions.*

The proof of the next lemma uses a result on *Steiner triple systems*. Recall that a Steiner triple system of order v, STS(v), is a pair $(V, \mathcal{B})$ where V is a set of cardinality v and $\mathcal{B}$ is a collection of unordered triples of elements of V, called *blocks*, with the property that every pair of elements of V is contained in precisely one block.

Lemma 3.3 *If* $v \equiv 1 \pmod 6$ *then there exists a cyclic* MDTS$(v, 1)$.

Proof There exists a cyclic Steiner triple system of order v for all $v \equiv 1 \pmod 6$ [12]. Choose a representation with $V = \mathrm{Z}_v$, invariant under the cyclic group generated by the mapping $i \mapsto i + 1 \pmod v$. Replace each orbit of the Steiner triple system generated by a block $\{0, x, y\}$ by the ordered triples generated by $(0, x, y)$ and $(0, -x, -y)$ under the action of the same group. Since the STS(v) contains no short orbits, it cannot contain a block $(0, x, -x)$ and it then follows from Lemma 3.2 that this gives an MDTS$(v, 1)$. This MDTS$(v, 1)$ is also invariant under a cyclic automorphism. □

Lemma 3.4 *If* $v \equiv 4 \pmod{12}$ *then there exists a cyclic* MDTS$(v, 2)$.

Proof If $v \equiv 4 \pmod{12}$ then there exists a cyclic DTS$(v, 1)$ [6]. By Lemma 3.2, if we replace each base triple $(0, x, y)$ of the system by the two base triples $(0, x, y)$ and $(0, -x, -y)$, we obtain a set of base triples for a cyclic MDTS$(v, 2)$. □

Lemma 3.5 *If* $v \equiv 10 \pmod{12}$ *then there exists a cyclic* MDTS$(v, 2)$.

Proof If $v = 10$ the following ordered triples generate a cyclic MDTS$(10, 2)$.

$$(0,1,4),(0,1,8),(0,2,6),(0,8,5),(0,9,2),(0,9,5)$$

Otherwise, let $v = 12s + 10$, $s \geq 1$. Then the following ordered triples generate a cyclic MDTS$(v, 2)$.

$$\begin{array}{ll}
(0, 2, 6s+6), & (0, 6s+7, 6s+4), \\
(0, 6s+3, 12s+9), & ((0, 12s+8, 1), \\
 & \\
(0, 1, 10s+7), & (0, 12s+9, 2s+3) \\
(0, 2, 6), & (0, 12s+8, 12s+4), \\
(0, 3, 6s+5), & (0, 12s+7, 6s+5), \\
(0, 4s+1, 10s+8), & (0, 8s+9, 2s+2), \\
(0, 6s+1, 10s+5), & (0, 6s+9, 2s+5), \\
 & \\
(0, 2x-1, 10s+6+x), & (0, 12s+11-2x, 2s+4-x), \\
 & \qquad x \in \{3, 4 \ldots, 2s\}, \\
(0, 2x, 6s+4+x), & (0, 12s+10-2x, 6s+6-x), \\
 & \qquad x \in \{4, 5, \ldots, 2s+1\}.
\end{array}$$

□

Lemma 3.6 *If v is odd, then there exists a cyclic* $\mathrm{MDTS}(v,3)$.

Proof If v is odd, then the ordered triples

$$(0,1,v-1),(0,2,v-2),\ldots,(0,v-1,1)$$

generate a cyclic $\mathrm{MDTS}(v,3)$. □

Lemma 3.7 *If $v \equiv 0,8 \pmod{12}$ then there exists a cyclic* $\mathrm{MDTS}(v,6)$.

Proof If $v \equiv 0,8 \pmod{12}$ then there exists a cyclic $\mathrm{DTS}(v,3)$ [4]. By Lemma 3.2, if we replace each base triple $(0,x,y)$ of the system by the two base triples $(0,x,y)$ and $(0,-x,-y)$, we obtain a set of base triples for a cyclic $\mathrm{MDTS}(v,6)$. □

Lemma 3.8 *If $v \equiv 2 \pmod 4$ then there exists a cyclic* $\mathrm{MDTS}(v,6)$.

Proof Let $v = 4s+2$. Then the following ordered triples generate a cyclic $\mathrm{MDTS}(v,6)$.

$$\begin{array}{lll}
(0,2x-1,4s+3-2x), & & x \in \{1,2,\ldots,s,s+2,\ldots,2s+1\} \text{ taken twice}, \\
(0,2x,4s+2-2x), & & x \in \{1,2,\ldots,2s-1\}, \\
(0,2x,2s+7-2x), & & x \in \{4,5,\ldots,s\}, \\
(0,2x,6s-3-2x), & & x \in \{s+1,s+2,\ldots,2s-3\}, \\
(0,2x,2s+1), & & x \in \{1,2,3,2s-2,2s-1,2s\}, \\
(0,4s,1), & (0,5,2), & (0,4s-3,4s+1).
\end{array}$$

□

Thus we have proved the following theorem.

Theorem 3.9 *There exists a cyclic* $\mathrm{MDTS}(v,\lambda)$ *if and only if* $\lambda(v-1) \equiv 0 \pmod 6$.

Proof This follows from Lemmas 3.1 to 3.8. □

Lemmas 3.6 and 3.8 above can be used to considerably simplify Cho, Han and Kang's proof [4] of necessary and sufficient conditions for the existence of cyclic directed triple systems. Specifically, Lemmas 2.6, 2.7, 2.8 and 2.9 of [4], which together prove that there exists a cyclic $\mathrm{DTS}(v,3)$ for $v \equiv 3,5 \pmod 6$, can be replaced by Lemma 3.6 above (which is in fact a generalisation of Lemma 2.9 of [4]). Further, Lemmas 2.14, 2.16, 2.18, 2.20, 2.22 and 2.23 of [4], which together prove that there exists a cyclic $\mathrm{DTS}(v,6)$ for all $v \equiv 2,6 \pmod{12}$, can be replaced by Lemma 3.8 above.

4 Enumeration of MDTS$(v, 1)$

We consider two Mendelsohn directed triple systems to be isomorphic if they are isomorphic as directed triple systems. In this section we enumerate the non-isomorphic MDTS$(v, 1)$ for $v \leq 10$. By Theorem 2.1, MDTS$(v, 1)$ exist precisely when $v \equiv 1 \pmod 3$. It is easy to see that the number of non-isomorphic MDTS$(4, 1)$ is one; we gave such a system in Section 1. We now deal with $v = 7$ and $v = 10$.

We shall call any permutation of the positions of a Mendelsohn directed triple system (or a partial Mendelsohn directed triple system) a *position permutation*. We denote the six position permutations by I, F_1, F_2, F_3, C and C^2; these send each ordered triple (x, y, z) to (x, y, z), (x, z, y), (z, y, x), (y, x, z), (z, x, y) and (y, z, x) respectively (so that F_i fixes position i). We denote the group of position permutations by Π.

In determining whether two given MDTS$(v, 1)$ (or partial MDTS$(v, 1)$ satisfying the order conditions) are isomorphic, it is useful to consider graphs that we shall call the *position graphs* of the systems. Each system has three position graphs, one corresponding to positions 1 and 2, one to positions 1 and 3, and one to positions 2 and 3. Given a system, each of its position graphs has as vertices all the elements of the system. $\{x, y\}$ is an edge of the graph corresponding to positions i and j if and only if the ordered pair (x, y) appears in positions i and j in some triple (which implies that (y, x) appears in these positions also).

We enumerate the MDTS$(7, 1)$ and MDTS$(10, 1)$ by starting with known enumerations of MTS$(7, 1)$ and MTS$(10, 1)$. We shall say that an MTS(v, λ) is *directable* if each triple (x, y, z) of the MTS can be individually cyclically re-ordered as one of (x, y, z), (z, x, y) or (y, z, x) in such a way that the resulting system is an MDTS(v, λ); in this case we say that the MTS *underlies* the MDTS. Clearly every directable MTS$(v, 1)$ is *pure*, that is, its underlying BIBD has no repeated blocks. The *converse* of a MTS is the system obtained by replacing each triple (x, y, z) by (z, y, x). Two MTS are said to be *equivalent* if they are isomorphic or if one is isomorphic to the converse of the other. The inequivalent pure MTS$(v, 1)$ have been enumerated for $v = 7$ and $v = 10$.

The number of inequivalent pure MTS$(7, 1)$ is precisely one [10]. Such a system is given by the triples generated by $(0, 1, 3)$ and $(0, 6, 4)$ under the action of the mapping $i \mapsto i + 1 \pmod 7$. This system is directable: for example, the triples $(0, 1, 3)$ and $(0, 6, 4)$ generate an MDTS$(7, 1)$, as we stated in the proof of Theorem 2.1. In the remainder of this paragraph, we denote this system by $\mathcal{D}$. Further, once the ordering of an initial triple of this MTS is determined, the orderings of all other triples are forced. Hence this MTS and its converse underly precisely six MDTS, namely those given by applying the six position permutations to $\mathcal{D}$. Now $\mathcal{D} \cong C(\mathcal{D})$ (under $i \mapsto 3i$) and therefore also $\mathcal{D} \cong C^2(\mathcal{D})$ (under $i \mapsto 2i$). It further follows that $F_1(\mathcal{D}) \cong F_2(\mathcal{D}) \cong F_3(D)$. However it is straightfoward to check, by working with position graphs, that $\mathcal{D}$ is not isomorphic to $F_1(\mathcal{D})$. Hence the number of non-isomorphic MDTS$(7,1)$ is two.

We now discuss the enumeration of MDTS$(10,1)$. Table 2 in [7] lists all the inequivalent pure MTS$(10,1)$; there are 34. We use the notation of [7].

Given an element x of an MTS$(v,1)$, its *adjacency graph* is the graph whose vertices are the other $v-1$ elements of the system, and where $\{y,z\}$ is an edge if and only if there is a triple of the system containing x, y and z. The adjacency graph of an element of an MTS$(10,1)$ is thus one of four types: a cycle of length 9 (type A), two cycles of lengths 6 and 3 (type B), two cycles of length 5 and 4 (type C), or three cycles each of length 3 (type D). The *T-vector* of an MTS$(10,1)$ is the vector (T_A, T_B, T_C, T_D) where T_A, T_B, T_C and T_D are the numbers of elements of the system with an adjacency graph of type A, B, C and D respectively. Table 1 in [7] lists the T-vectors of the 34 pure MTS$(10,1)$, against their underlying BIBDs (clearly a T-vector is in fact determined by the underlying BIBD of an MTS$(v,1)$).

Now if any element x of an MTS$(v,1)$ has an adjacency graph containing a cycle whose length is not a multiple of 3, then by considering the potential orderings of the triples containing x, it can be seen that the MTS$(v,1)$ is not directable. Hence a necessary condition for an MTS$(10,1)$ to be directable is that $T_C = 0$. It therefore follows from Table 1 in [7] that the only possibilities for directable MTS$(10,1)$ are those numbered (2), (3)A, (3)B, (3)C, (14), (20) and (26). Attempts to direct (20) and (26) quickly confirm that these are not directable: the above necessary condition is not sufficient. The others are directable, and we now discuss them in detail.

Mendelsohn triple system (14) can be directed to give an MDTS$(10,1)$ as follows.

123	214	341	532	462	437
359	285	640	826	784	873
963	495	658	036	504	948
308	697	561	057	819	180
716	279	175	720	091	902

For the remainder of the discussion of this MTS, we denote the above system by $\mathcal{D}$. Further, once the ordering of an initial triple of this MTS is determined, the orderings of all other triples are forced. Hence this MTS and its converse underly precisely six MDTS, namely those given by applying the six position permutations to $\mathcal{D}$. Now $\mathcal{D} \cong F_3(\mathcal{D})$: an isomorphism is $(34)(56)(09)$. It follows that $C(\mathcal{D}) \cong F_2(\mathcal{D})$ and $C^2(\mathcal{D}) \cong F_1(\mathcal{D})$. Further, it is an easy exercise to check, using position graphs, that $\mathcal{D}$, $C(\mathcal{D})$ and $C^2(\mathcal{D})$ are pairwise non-isomorphic. Hence this MTS corresponds to precisely three non-isomorphic MDTS.

Mendelsohn triple system (2) can be directed to give an MDTS$(10, 1)$ as follows.

156	517	671	265	735	768	629
582	370	853	847	486	936	792
063	028	207	398	974	604	810
189	459	540	901	095		
123	214	341	432			

The last four triples of this form an MDTS$(4, 1)$ sub-system, which, for the remainder of the discussion of this MTS, we denote by $\mathcal{D}_1$. We denote the partial system formed by the first 26 triples by $\mathcal{D}_0$. Now $\mathcal{D}_0$ and $\mathcal{D}_1$ can be directed independently: once the ordering of a triple in $\mathcal{D}_0$ is chosen, it determines the ordering of all other triples in $\mathcal{D}_0$, but not that of any triple in $\mathcal{D}_1$; and vice versa. Hence this MTS and its converse underly 36 MDTS, namely $\pi_0(\mathcal{D}_0) \cup \pi_1(\mathcal{D}_1)$, $\pi_0, \pi_1 \in \Pi$. We now determine the number of non-isomorphic systems amongst these.

Now $\mathcal{D}_0 \cong F_3(\mathcal{D}_0)$: an isomorphism is (58)(60)(79). It follows that $C(\mathcal{D}_0) \cong F_2(\mathcal{D}_0)$ and $C^2(\mathcal{D}_0) \cong F_1(\mathcal{D}_0)$. Further, it is an easy exercise to check by working with position graphs that $\mathcal{D}_0$, $C(\mathcal{D}_0)$ and $C^2(\mathcal{D}_0)$ are pairwise non-isomorphic. Thus we need only consider systems of the form $\mathcal{D}_0 \cup \pi(\mathcal{D}_1)$, $C(\mathcal{D}_0) \cup \pi(\mathcal{D}_1)$ and $C^2(\mathcal{D}_0) \cup \pi(\mathcal{D}_1)$, where π is a position permutation.

Consider the six systems of the form $\mathcal{D}_0 \cup \pi(\mathcal{D}_1)$. Two of these systems, say $\mathcal{D}_0 \cup \pi(\mathcal{D}_1)$ and $\mathcal{D}_0 \cup \pi'(\mathcal{D}_1)$, are isomorphic if and only if there exists an automorphism α of $\mathcal{D}_0$ such that $\alpha(\pi(\mathcal{D}_1)) = \pi'(\mathcal{D}_1)$. Now Aut $(\mathcal{D}_0) = \{\iota, (07)(23)(58)(69)\}$. It follows that $\mathcal{D}_0 \cup \mathcal{D}_1$, $\mathcal{D}_0 \cup C(\mathcal{D}_1)$ and $\mathcal{D}_0 \cup C^2(\mathcal{D}_1)$ are pairwise non-isomorphic, while if π is any other position permutation, then $\mathcal{D}_0 \cup \pi(\mathcal{D}_1)$ is isomorphic to one of these three.

Now clearly Aut $(\mathcal{D}_0) =$ Aut $(C(\mathcal{D}_0)) =$ Aut $(C^2(\mathcal{D}_0))$. Hence, similarly to the above, the six systems of the form $C(\mathcal{D}_0) \cup \pi(\mathcal{D}_1)$ yield three non-isomorphic systems. Further, none of these three is isomorphic to any of the first three, since this would imply the existence of an isomorphism from $\mathcal{D}_0$ to $C(\mathcal{D}_0)$. Finally, the six systems of the form $C^2(\mathcal{D}_0) \cup \pi(\mathcal{D}_1)$ yield another three non-isomorphic systems.

In summary, there follows a list of the nine non-isomorphic MDTS$(10, 1)$ corresponding to Mendelsohn triple system (2) in Table 2 of [7]. In this list, the entry π_0/π_1 means $\pi_0(\mathcal{D}_0) \cup \pi_1(\mathcal{D}_1)$.

$$I/I \quad I/C \quad I/C^2 \quad C/I \quad C/C \quad C/C^2 \quad C^2/I \quad C^2/C \quad C^2/C^2$$

Thus these nine non-isomorphic MDTS$(10, 1)$ are obtained by cyclically permuting the two parts of the original ordering of the Mendelsohn triple system in the nine possible ways.

Mendelsohn triple system (3)A can be directed to give an MDTS$(10,1)$ as follows.

156	517	681	265	735	971
863	018	629	502	378	053
792	109	936	820	287	390
123	214	341	432		
458	549	894	985		
467	640	076	704		

The last twelve triples of this form three MDTS$(4,1)$ sub-systems, which, for the remainder of the discussion of this MTS, we denote by $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ respectively. We denote the partial system formed by the first 18 triples by $\mathcal{D}_0$.

Now it happens that $\pi(\mathcal{D}_0) \cong \mathcal{D}_0$ for any position permutation π. For example, sample isomorphisms from $\mathcal{D}_0$ to $\pi(\mathcal{D}_0)$ where π runs over the position permutations F_1, F_2, F_3, C and C^2, are (56)(78)(90), (23)(58)(70), (23)(67)(89), (23)(568097) and (589)(607) respectively. Also, Mendelsohn triple systems (3)B and (3)C can be directed in such a way that they can be obtained from the above directed version of (3)A by applying appropriate position permutations to the last two MTS(4,1) sub-systems.

Hence every MDTS$(10,1)$ whose underlying Mendelsohn triple system is (3)A, (3)B or (3)C or the converse of one of these is isomorphic to a system of the form $\mathcal{D}_0 \cup \pi_1(\mathcal{D}_1) \cup \pi_2(\mathcal{D}_2) \cup \pi_3(\mathcal{D}_3)$ for some position permutations π_1, π_2 and π_3. Let X be the set of all $6^3 = 216$ systems of this form. Two elements of X, say $\mathcal{D}_0 \cup \pi_1(\mathcal{D}_1) \cup \pi_2(\mathcal{D}_2) \cup \pi_3(\mathcal{D}_3)$ and $\mathcal{D}_0 \cup \pi'_1(\mathcal{D}_1) \cup \pi'_2(\mathcal{D}_2) \cup \pi'_3(\mathcal{D}_3)$, are isomorphic if and only if there exists an automorphism α of $\mathcal{D}_0$ such that $\{\alpha(\pi_1(\mathcal{D}_1)), \alpha(\pi_2(\mathcal{D}_2)), \alpha(\pi_3(\mathcal{D}_3))\} = \{\pi'_1(\mathcal{D}_1), \pi'_2(\mathcal{D}_2), \pi'_3(\mathcal{D}_3)\}$. Now Aut $(\mathcal{D}_0) =$

$$
\begin{array}{rrr}
\{\iota, & (123)(589)(670), & (132)(598)(607), \\
(150)(286)(397), & (105)(268)(379), & (169)(275)(308), \\
(196)(257)(380), & (187)(290)(356), & (178)(209)(365), \\
(23)(50)(69)(78), & (15)(29)(38)(67), & (10)(27)(36)(89), \\
(12)(56)(79)(80), & (16)(20)(37)(58), & (18)(25)(39)(60), \\
(13)(75)(68)(90), & (17)(26)(30)(59), & (19)(28)(35)(70)\}.
\end{array}
$$

Consider Aut $(\mathcal{D}_0)$ acting on X. It is an easy exercise to see that the subset of X fixed by a permutation $\alpha \in$ Aut $(\mathcal{D}_0)$ has size 216 if $\alpha = \iota$, 0 if α is one of the permutations of order 2 or if $\alpha = (123)(589)(670)$ or $(132)(598)(607)$, and 6 if α is any of the other six permutations of order 3. Hence by Burnside's theorem, the number of non-isomorphic such MDTS$(10,1)$ is

$(216+6\times 6)/18=14$. It is also an easy exercise to show that the subset of X consisting of all systems of the form $\mathcal{D}_0\cup\mathcal{D}_1\cup\pi_2(\mathcal{D}_2)\cup\pi_3(\mathcal{D}_3)$ can be partitioned into 14 isomorphism classes, and hence representatives of these form a list of all non-isomorphic MDTS$(10,1)$ corresponding to Mendelsohn triple systems 3(A), 3(B) and 3(C). There follows a list of such representatives. In this list the entry π_2/π_3 means $\mathcal{D}_0\cup\mathcal{D}_1\cup\pi_2(\mathcal{D}_2)\cup\pi_3(\mathcal{D}_3)$.

I/I	I/F_1	I/F_2	F_1/I	F_1/F_1	F_1/F_2	F_1/F_3
F_1/C^2	F_2/I	F_2/F_1	F_2/C	F_3/F_3	F_3/C	F_3/C^2

Thus we have found that in total there are 3+9+14=26 non-isomorphic MDTS$(10,1)$.

In summary, we present our enumeration results for MDTS$(v,1)$ along with the known results for MTS$(v,1)$ and DTS$(v,1)$ with the same parameters (see [3, 5, 10]).

v	number of inequivalent MTS$(v,1)$	number of non-isomorphic DTS$(v,1)$	number of non-isomorphic MDTS$(v,1)$
4	1	3	1
7	3	2368	2
10	144		26

References

[1] T. Beth, D. Jungnickel and H. Lenz, *Design Theory*, BI Mannheim, 1985.

[2] A.E. Brouwer, Optimal packings of K_4's into K_n, *J. Combin. Theory A* **26**, (1979), 278–297.

[3] F.E. Bennett and A. Mahmoodi, Directed designs, in *The CRC Handbook of Combinatorial Designs* (ed. C.J. Colbourn and J.H. Dinitz), CRC Press, 1996.

[4] C.J. Cho, Y.-H. Han and S.-H. Kang, Cyclic directed triple systems, *J. Korean Math. Soc.* **23** (1986), No. 2, 117-125.

[5] C.J. Colbourn and A. Rosa, Directed and Mendelsohn triple systems, in *Contemporary Design Theory: A Collection of Surveys* (ed. J.H. Dinitz and D.R. Stinson), John Wiley, 1992.

[6] M.J. Colbourn and C.J. Colbourn, The analysis of directed triple systems by refinement, *Ann. Discrete Math.* **15** (1982), 97–103.

[7] B. Ganter, R. Mathon and A. Rosa, A complete census of $(10, 3, 2)$-block designs and of Mendelsohn triple systems of order ten. I. Mendelsohn triple systems without repeated blocks, *Proc. Seventh Manitoba Conference on Numerical Math. and Computing* (1977), 383-398.

[8] H.-D.O.F. Gronau, R.C. Mullin and Ch. Pietsch, The closure of all subsets of $\{3, 4, \ldots, 10\}$ which include 3, *Ars Combin.* **41** (1995), 129–161.

[9] A. Hartman and E. Mendelsohn, The last of the triple systems, *Ars Combin.* **22** (1986), 25-41.

[10] E. Mendelsohn, Mendelsohn designs, in *The CRC Handbook of Combinatorial Designs* (ed. C.J. Colbourn and J.H. Dinitz), CRC Press, 1996.

[11] R.C. Mullin, A.C.H. Ling, R.J.R. Abel and F.E. Bennett, On the closure of subsets of $\{4, 5, \ldots, 9\}$ which contain $\{4\}$, *Ars Combin.* **45** (1997), 33–76.

[12] R. Peltesohn, Eine Losung der beiden Heffterschen Differenzenprobleme, *Compositio Math.* **6** (1939), 251-257.