The Pasch configuration
(Encyclopaedia of Mathematics entry)

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March 2001

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book).
**Pasch configuration** - The Pasch configuration or quadrilateral is a collection of four triples isomorphic to \( \{a, b, c\} \), \( \{a, y, z\} \), \( \{x, b, z\} \), and \( \{x, y, c\} \). They have been studied extensively in the context of Steiner triple systems.

A Steiner triple system of order \( v \), \( \text{STS}(v) \), is an ordered pair \( (V, B) \) where \( V \) is a set of cardinality \( v \), called elements or points, and \( B \) is a collection of triples, also called lines or blocks, which collectively have the property that every pair of distinct elements of \( V \) occur in precisely one triple. \( \text{STS}(v) \) exist if and only if \( v \equiv 1 \) or 3 (mod 6), [10]. To within isomorphism, the Steiner triple systems of orders 7 and 9 are unique but for all greater orders, the structure is not unique. A \( (p, l) \)-configuration in a Steiner triple system is a collection of \( l \) lines whose union contains precisely \( p \) points. A configuration whose number of occurrences in an \( \text{STS}(v) \) depends only upon the order \( v \) and not on the structure of the \( \text{STS}(v) \) is called constant and otherwise variable. There are two configurations with \( l=2 \) and five with \( l=3 \), all of which are constant. There are 16 configurations with \( l=4 \) of which the Pasch configuration is the unique (6,4)-configuration and the one containing the least number of points. Five of the 4-line configurations are constant but the Pasch configuration is variable. It was shown in [5] that the number of occurrences of all the other variable 4-line configurations can be expressed in terms of the order \( v \), and the number \( c \) of Pasch configurations in the \( \text{STS}(v) \).

The above gives motivation to the problem of constructing \( \text{STS}(v) \) containing no Pasch configurations, so-called anti-Pasch or quadrilateral free Steiner triple systems. A solution for \( v \equiv 3 \) (mod 6) was first given by Brouwer ([1], see also [9]) and it was a long-standing conjecture that anti-Pasch \( \text{STS}(v) \) also exist for all \( v \equiv 1 \) (mod 6), \( v \neq 7 \) or 13. This was settled in the affirmative in two papers, [11] and [8], published in 2000. The proof resolves the first case of a conjecture by Erdős, [3], that for every \( m \geq 4 \) there is an integer \( v_m \) so that for every \( v \geq v_m \), \( v \equiv 1 \) or 3 (mod 6), there is an \( \text{STS}(v) \) avoiding \((l + 2, l)\)-configurations for \( 4 \leq l \leq m \). Anti-Pasch \( \text{STS}(v) \) have application to erasure-correcting codes, [2]. The theoretical maximum number of Pasch configurations in an \( \text{STS}(v) \) is \( v(v - 1)(v - 3)/24 \) but this is achieved only in the point-line designs obtained from the projective spaces \( PG(n, 2) \), [12].

The Pasch configuration is an example of a trade. A pair of distinct collections of blocks \( (T_1, T_2) \) is said to be mutually \( t \)-balanced if each \( t \)-element subset of the base set \( V \) is contained in precisely the same number of blocks of \( T_1 \) as of \( T_2 \). Each collection \( T_1, T_2 \) is then referred to as a trade. The Pasch
configuration is the smallest trade that can occur in a Steiner triple system. If $T_1$ is the collection \{a, b, c\}, \{a, y, z\}, \{x, b, z\} and \{x, y, c\} then, by replacing each triple with its complement, a collection $T_2$, \{x, y, z\}, \{x, b, c\}, \{a, y, c\} and \{a, b, z\} is obtained which contains precisely the same pairs as $T_1$. This transformation is known as a Pasch switch and when applied to a Steiner triple system yields another, usually non-isomorphic, Steiner triple system. There are 80 non-isomorphic $STS(15)$s of which precisely one is anti-Pasch. It was shown in [4] that all of the remaining 79 systems can be obtained from one another by successive Pasch switches. Other relevant papers in this area are [6] and [7].

The number of Pasch configurations and their distribution within a Steiner triple system is an invariant and provides a simple and useful test to help in determining whether two systems are isomorphic.

References


