Bi-embeddings of Steiner triple systems of order 15

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Running head:
Bi-embeddings of STS(15)s

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Abstract

It was shown by Gerhard Ringel that one of the three non-isomorphic Steiner triple systems of order 15 having an automorphism of order 5 may be bi-embedded as the faces of a face 2-colourable triangular embedding of K_{15} in a suitable orientable surface. Ringel's bi-embedding was obtained from an appropriate current graph. In a recent paper, the present authors showed that a second STS(15) of this type may also be bi-embedded. In the present paper we show that this second bi-embedding may also be obtained from a current graph. Furthermore, we exhibit a third current graph which yields a bi-embedding of the third STS(15) of this type.

1 Introduction

For fuller details of terminology we refer the reader to [7]. In particular we assume that the reader is familiar with the description of topological embeddings by means of rotation schemes. A detailed explanation of the connection between embeddings of complete graphs and Steiner triple systems is given in [4] and [5].

Given a face 2-colourable triangular embedding of the complete graph K_n in a topological surface (orientable or non-orientable), the faces in each colour class, say black and white, each separately form a Steiner triple system of order n, STS(n). We then say that the two STS(n)s are *bi-embedded*. Ringel [7], in the course of proving the Heawood conjecture, describes *inter alia* how current graphs may be used to obtain a face 2-colourable triangular embedding of K_n in an orientable surface when $n \equiv 3 \pmod{12}$. The method used by Ringel gives an index 3 solution, meaning that the embedding and, consequently, the resulting STS(n)s have an automorphism of order $n/3$. Furthermore, the STS(n)s formed respectively by the black and the white faces of his embedding are isomorphic. These STS(n)s were identified in [5] as being those arising from the Bose construction of Steiner triple systems [3].

Our interest is in the embedding of specific Steiner triple systems rather than the complete graphs from which they are obtained: a study which we call "Topological Design Theory" to distinguish it from the more traditional "Topological Graph Theory". A number of fundamental questions are listed in [5]. One of these is whether every STS(n) with $n \equiv 3$ or $7 \pmod{12}$ can be embedded in an orientable surface. The answer to this question seems to be far beyond current methods, but it is possible to make some progress when $n = 15$ which is the first non-trivial case.

There are 80 non-isomorphic STS(15)s. Of these, precisely three have automorphisms of order 5. They are numbered 1, 76 and 80 in the standard listing given in [6]. System #80 is the Bose system and system #1 is the point-line design corresponding to the projective geometry PG(3,2). It was shown in [1] that, up to isomorphism, there is precisely one orientable face 2-colourable triangular embedding of K_{15} in which both the black and the white systems are isomorphic to system #1. Prior to this present paper, systems #1 and #80 were the only two STS(15)s which were known to be embeddable in an orientable surface.

The purpose of this present paper is to show that:

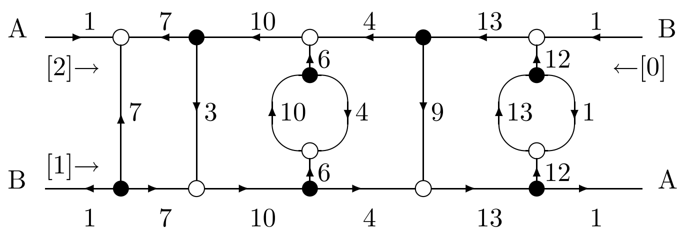
(i) the unique orientable bi-embedding of system #1 with itself given in [1] may be obtained from a current graph similar to that of Ringel, and

(ii) a further similar current graph gives an orientable bi-embedding of system #76 with itself.

There are, therefore, at least three non-isomorphic face 2-colourable triangular embeddings of K_{15} in an orientable surface. Moreover, all three STS(15)s having an automorphism of order 5 can be bi-embedded. At this point it may be appropriate to record that in [2] it is proved that there are at least $2^{n^2/54-O(n)}$ non-isomorphic face 2-colourable triangular embeddings of K_n in an orientable surface for $n \equiv 7$ or 19 (mod 36). However, the methods employed in [2] do not apply to the other residue classes.

2 The embeddings of K_{15}

The current graph used by Ringel [7] to embed K_{15} is shown in Figure 1.



(The two ends labelled A are identified, likewise the two ends labelled B.)

Figure 1.

The key features of this graph are as follows (see also [7] pp 149-151).

(a) Each vertex has valence 3.

(b) A rotation is specified at each vertex, clockwise at solid (●) vertices and anticlockwise at hollow (○) vertices. This rotation induces three directed circuits labelled [0], [1] and [2].

(c) Each edge is directed and labelled with an element of the group Z_{15}. The *log* of each circuit is the cyclically ordered list of elements of Z_{15} encountered in traversing the circuit in the direction indicated, with the convention that if a directed edge labelled x is encountered with the opposite direction to the circuit then its label is replaced by $-x$ in the log. Each element of Z_{15}, apart from 0, then appears precisely once in the log of each circuit.

(d) Kirchoff's current law in Z_{15} holds at each vertex.

(e) If a directed edge with label x is in the circuit $[a]$, and the same edge but with opposite orientation (and consequently, effectively with label $-x$) is in the circuit $[b]$, then $x \equiv b - a \pmod 3$.

(f) The graph is bipartite.

The graph in Figure 1 produces the following logs:

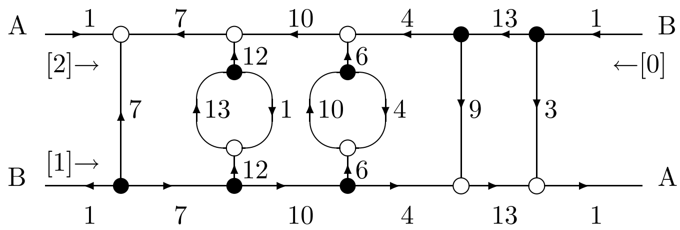
[0]:	1	13	9	11	5	12	7	14	2	6	4	10	3	8
[1]:	14	7	8	5	9	4	10	6	11	2	3	1	13	12
[2]:	1	8	7	10	6	11	5	9	4	13	12	14	2	3

From these we may now obtain a rotation scheme which specifies an embedding of K_{15} in a suitable surface. The vertices of the embedding are labelled with the elements of Z_{15} and the rotation at $i \in Z_{15}$ is determined by taking $a \in \{0, 1, 2\}$ such that $i \equiv a \pmod 3$, and then by adding $i \pmod{15}$ to each element of the log of $[a]$. For example, the rotation at vertex 7 is obtained by adding 7 (mod 15) to the log of [1], i.e. it comprises (6 14 0 12 1 11 2 13 3 9 10 8 5 4).

Ringel proves that properties (a) to (e) ensure that the result of this construction is a rotation scheme representing a triangular embedding of K_{15} in an orientable surface. Property (f) ensures that the embedding is face 2-colourable. Further details are given in [7], [4] and [5].

It is clear from the method of construction that the embedding has an automorphism of order 5, namely $z \to z + 3 \pmod{15}$. Moreover, the two STS(15)s formed respectively by each of the two face colour classes necessarily share this automorphism. It is also clear from the symmetry of the current graph that they are isomorphic, a specific isomorphism being provided by the mapping $z \to -z \pmod{15}$. As we remarked earlier, both systems are in fact isomorphic to system #80, the system produced by the Bose construction.

Figure 2 gives a current graph which generates a face 2-colourable triangular embedding of K_{15} in an orientable surface in which the black and the white faces give STS(15)s isomorphic to system #1.



(The two ends labelled A are identified, likewise the two ends labelled B.)

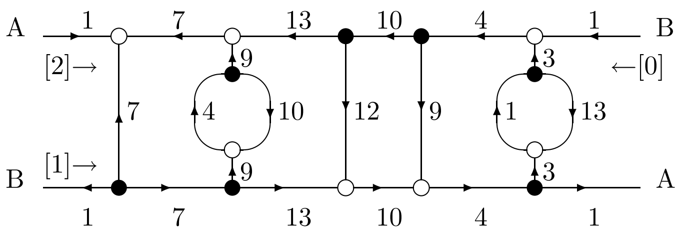
Figure 2.

The graph in Figure 2 produces the following logs:

[0]:	1	3	2	6	4	10	7	14	12	13	9	11	5	8
[1]:	14	7	8	3	1	13	12	5	9	4	10	6	11	2
[2]:	1	8	7	12	14	2	3	10	6	11	5	9	4	13

Note that this current graph also obeys properties (a) to (f). To see that the resulting embedding is isomorphic to the bi-embedding of system #1 given in [1] we compare the rotation schemes. The permutation (0 7 4 13 3 8 2)(1 12)(5 14 11)(9 10) applied to the rotation scheme derived from Figure 2 gives the rotation scheme of the "Class #1" embedding of [1].

Figure 3 gives a current graph which generates a face 2-colourable triangular embedding of K_{15} in an orientable surface in which the black and the white faces give STS(15)s isomorphic to system #76.



(The two ends labelled A are identified, likewise the two ends labelled B.)

Figure 3.

The graph in Figure 3 produces the following logs:

[0]:	1	4	9	5	3	13	7	14	11	6	10	12	2	8
[1]:	14	7	8	6	10	4	9	2	5	11	12	13	1	3
[2]:	1	8	7	9	5	11	6	13	10	4	3	2	14	12

Note that this current graph also obeys properties (a) to (f). The two resulting STS(15)s are again isomorphic and the mapping $z \to -z \pmod{15}$ provides a specific isomorphism. To see that they are isomorphic to system #76, note firstly that one of the systems obtained from Figure 3 has the blocks:

$\{0,1,4\}$	$\{0,2,8\}$	$\{0,3,13\}$	$\{0,5,9\}$	$\{0,6,11\}$	$\{0,7,14\}$
$\{0,10,12\}$	$\{1,2,14\}$	$\{1,3,6\}$	$\{1,5,10\}$	$\{1,7,11\}$	$\{1,8,9\}$
$\{1,12,13\}$	$\{2,3,10\}$	$\{2,4,5\}$	$\{2,6,12\}$	$\{2,7,13\}$	$\{2,9,11\}$
$\{3,4,7\}$	$\{3,5,11\}$	$\{3,8,12\}$	$\{3,9,14\}$	$\{4,6,9\}$	$\{4,8,13\}$
$\{4,10,14\}$	$\{4,11,12\}$	$\{5,6,13\}$	$\{5,7,8\}$	$\{5,12,14\}$	$\{6,7,10\}$
$\{6,8,14\}$	$\{7,9,12\}$	$\{8,10,11\}$	$\{9,10,13\}$	$\{11,13,14\}$	

The mapping which takes the points 0 to 14 above respectively to the points 3, 6, 7, 14, 11, 12, 8, 2, 9, 15, 10, 4, 13, 5, 1 gives an isomorphism of this STS(15) with system #76 as given in [6].

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