On the bi-embeddability of certain Steiner triple systems of order 15

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September 2001

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Bi-embeddings of STS(15)s

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AMS classifications:
05B07, 05C10.

Abstract

There are 80 non-isomorphic Steiner triple systems of order 15. A standard listing of these is given in [8]. We prove that systems #1 and #2 have no bi-embedding together in an orientable surface. This is the first known example of a pair of Steiner triple systems of order n, satisfying the admissibility condition $n \equiv 3$ or $7 \pmod{12}$, which admits no orientable bi-embedding. We also show that the same pair has five non-isomorphic bi-embeddings in a non-orientable surface.

1 Introduction

The background to this paper lies in the result of Ringel and Youngs [9, 11] that, for all $n \equiv 0$, 3, 4 or 7 (mod 12), there exists a triangulation of the complete graph K_n in an orientable surface of appropriate genus. Here we give a brief summary of those aspects required for the purpose of the present paper; further details may be found in [1, 3, 5, 6] as well as in Ringel's book [9].

For $n \equiv 3$ or 7 (mod 12) results of Ringel and Youngs establish the existence of a triangulation of K_n, in an orientable surface, and having the additional property that the faces may be properly 2-coloured. The triangular faces in each of the two colour classes of such a triangulation necessarily form a Steiner triple system of order n (STS(n)); that is a set of triples from a point set of cardinality n such that every pair of points (corresponding to the edges of K_n) lies in a unique triple (the face, in the relevant colour class, which contains that edge). In such a triangulation we will say that the two STS(n)s, are embedded together in the surface.

Given a pair of STS(n)s, say A and B, one may ask whether there exists an embedding of A together with B. The answer to this question will sometimes be no, for example if A and B have a triple in common. However, the question may be refined to ask if A and an isomorphic copy of B can be embedded together. With this question in mind, we define a *bi-embedding* of A and B to be an embedding of A with an isomorphic copy of B.

It is not known whether every STS(n) with $n \equiv 3$ or 7 (mod 12) has a bi-embedding with some other STS(n) in an orientable surface. An affirmative answer would entail the existence of $n^{O(n^2)}$ non-isomorphic face 2-colourable triangulations of K_n in an orientable surface, since there are $n^{n^2/6-o(n^2)}$ non-isomorphic STS(n)s [10]. However, the best existing lower bounds for the numbers of such triangulations all have the form $2^{O(n^2)}$ [3, 7]. The lowest non-trivial specific value of n for which one might investigate the question is $n = 15$.

There are 80 non-isomorphic STS(15)s and it is known that at least three of these have bi-embeddings in an orientable surface. In each of these three cases the bi-embedding is of a system with an isomorphic copy of itself. Using the standard listing of the STS(15)s given in [8], the three systems involved are #1 (which is the point-line design of the projective geometry PG(3,2)), #76 and #80. The bi-embedding of system #80 was given by Ringel [9], that of #1 was given in [1], and that of #76 together with current graphs which generate all three bi-embeddings was given in [2]. It seems to be a difficult problem to determine whether or not all the remaining 77 STS(15)s admit a bi-embedding in an orientable surface. However, a more tractable question is whether particular pairs of STS(15)s may be bi-embedded together.

The rich structure of PG(3,2) facilitated computer analysis which resulted in the construction of the unique (up to isomorphism) bi-embedding of system #1 with itself in an orientable surface. None of the other STS(15)s possesses a comparable degree of symmetry. However, system #2 may be obtained from #1 by means of a Pasch-switch. A *Pasch configuration* also known as a *quadrilateral* in a Steiner triple system is a set of four triples whose union has cardinality six. Such a configuration is isomorphic to $\{\{a,b,c\},\{a,y,z\},\{x,b,z\},\{x,y,c\}\}$. A *Pasch-switch* is the operation of replacing this set of four triples by $\{\{x,y,z\},\{x,b,c\},\{a,y,c\},\{a,b,z\}\}$ which covers the same pairs. System #1 has 105 Pasch configurations and it was shown in [4] that switching any one of these results in a copy of system #2. Using this fact, we show (Theorem 3.1) that there is no bi-embedding of system #1 with system #2 in an orientable surface. This is the first example of a pair of STS(n)s (with $n \equiv 3$ or 7 (mod 12)) which cannot be bi-embedded in an orientable surface.

Ringel and Youngs' work also dealt with triangulations of K_n in non-orientable surfaces and one may ask questions, similar to those given above, for bi-embeddings of STS(n)s in non-orientable surfaces. Here the necessary conditions are $n \equiv 1$ or 3 (mod 6). In the course of the investigation we prove that there are precisely five non-isomorphic bi-embeddings of system #1 with system #2 in a non-orientable surface.

2 Method

In a bi-embedding of two STS(15)s there will be 15 vertices, 105 edges and 70 triangular faces. The genus of the surface may be determined from Euler's formula. In the orientable case the surface is S_{11}, the sphere with 11 handles, and in the non-orientable case it is $\bar{S}_{22}$, the sphere with 22 crosscaps. We will refer to the colour classes for the faces as black and white.

A triangulation of K_n may be described by means of a *rotation scheme*. This comprises a set of circularly ordered lists, one for each vertex of K_n. The list corresponding to the vertex x, the *rotation at* x, gives the remaining $n-1$ vertices in the order in which they appear around x in the given embedding.

If the embedding is in an orientable surface then a consistent orientation, say clockwise, may be selected for the entire rotation scheme. As an example, Table 1 gives a rotation scheme for an embedding of K_7 in a torus. In fact this embedding is unique up to isomorphism. The vertices of K_7 are taken to be the points of Z_7.

0:	1	3	2	6	4	5
1:	2	4	3	0	5	6
2:	3	5	4	1	6	0
3:	4	6	5	2	0	1
4:	5	0	6	3	1	2
5:	6	1	0	4	2	3
6:	0	2	1	5	3	4

Table 1. Rotation scheme for embedding K_7.

Given a triangular embedding of K_n, by considering each pair of adjacent triangular faces, $\langle i, j, k\rangle$ and $\langle i, k, l\rangle$, it is easy to see that the rotation scheme must satisfy the following:

Rule R. If the rotation at i contains $\ldots jkl \ldots$ then the rotation at k contains either $\ldots lij \ldots$ or $\ldots jil \ldots$.

The converse is also true (see for example [9], p76), namely a rotation scheme on n points (with the rotation at each point x containing all the $n-1$ points apart from x) which satisfies Rule R represents a triangular embedding of K_n in some surface. The surface may or may not be orientable. It will be orientable if it is possible to orient the rotations at the vertices consistently, i.e. to satisfy the following:

Rule R*. If the rotation at i contains $\ldots jkl \ldots$ then the rotation at k contains $\ldots lij \ldots$.

The necessity of Rule R* may be seen in a similar fashion to that of Rule R. A proof of its sufficiency is given in [9].

We take the vertices of K_{15} to be the elements of Z_{15}. Without loss of generality, the rotation at 0 can be taken as:

$$0: \quad 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12 \ 13 \ 14$$

Since a face 2-colourable embedding is sought, it can be assumed that, for $i = 1, 2, \ldots, 7$, the triangles $\langle 0, 2i-1, 2i \rangle$ are coloured black and the triangles $\langle 0, 2i, 2i+1 \rangle$ (with "15" replaced by "1") are coloured white. We look for bi-embeddings where the white and black systems are isomorphic to systems #1 and #2 respectively.

It was shown in [1] that there are precisely 480 differently labelled copies of system #1 on the point set Z_{15} and containing the seven white triples $\{0, 2i, 2i+1\}$; it was also explained there how these 480 copies may be obtained. In that paper we sought bi-embeddings of system #1 with itself. There are also 480 differently labelled copies of system #1 on the point set Z_{15} and containing the seven black triples $\{0, 2i-1, 2i\}$, and these may be obtained from the 480 white systems by applying the permutation (0)(14 13 12 . . . 1). In the case being considered in the current paper, we seek bi-embeddings of system #1 with system #2. The strategy employed is to take the 480 black systems just identified and to apply permutations and Pasch-switches which yield all the differently labelled copies of system #2 on the point set Z_{15} and which contain the seven black triples $\{0, 2i-1, 2i\}$.

Given a realisation of system #2 there are precisely $15 \times (14 \times 12 \times 10 \times \ldots \times 2)$ ways of mapping the blocks through a single point onto the seven black triples $\{0, 2i-1, 2i\}$. However, system #2 has an automorphism group of order 192 [8]. Consequently the number of differently labelled copies of system #2 on the point set Z_{15} and containing the seven specified triples is $15.2^7.7!/192 = 105 \times 480$. All such systems may be obtained in one of two ways from the 480 copies of system #1 containing the same black triples.

The first of these is by switching any Pasch configuration which does not involve the seven specified triples. There are $7 \times 6 = 42$ Pasch configurations in system #1 which involve triples through a specified point, and consequently there are $105 - 42 = 63$ which do not. Thus we obtain 63×480 copies of system #2 containing the seven specified triples. We show below that these are distinct, and we refer to them as Type I copies.

The second possibility is that a copy of system #2 containing the seven specified triples results from a Pasch switch on a copy of system #1 which does not contain all the specified triples. The Pasch configuration involved in such a switch which lies in system #2 must contain two of the specified triples, say $\{0, 2i-1, 2i\}$ and $\{0, 2j-1, 2j\}$ $(i \neq j)$ together with a pair of other triples which may either be $\{\{x, 2i-1, 2j-1\}, \{x, 2i, 2j\}\}$ or $\{\{x, 2i-1, 2j\}, \{x, 2i, 2j-1\}\}$. The corresponding copy of system #1 will contain five of the specified triples together with either $\{\{0, 2i-1, 2j-$

$1\}, \{0, 2i, 2j\}, \{x, 2i-1, 2i\}, \{x, 2j-1, 2j\}\}$ or $\{\{0, 2i-1, 2j\}, \{0, 2i, 2j-1\}, \{x, 2i-1, 2i\}, \{x, 2j-1, 2j\}\}$. If we apply the permutation $(2i-1\ \ 2j)$ (alternatively $(2i\ \ 2j-1)$) to the former case or $(2i-1\ \ 2j-1)$ (alternatively $(2i\ \ 2j)$) to the latter case we obtain a copy of system #1 containing all seven of the specified triples. The process is reversible; we may start with any of the 480 copies of system #1 containing all seven of the specified triples, apply an appropriate permutation, carry out the corresponding Pasch-switch, and obtain a copy of system #2 containing the seven specified triples. There are $2 \times 7 \times 6 = 84$ transformations to consider, leading to 84×480 copies of system #2 containing the seven specified triples. We will refer to these as Type II copies. We show below that these are distinct from the Type I copies and that there are precisely 42×480 distinct Type II copies, each copy being generated precisely twice by the procedure described above. Thus we are able to construct all $(63+42) \times 480 = 105 \times 480$ distinct copies of system #2 containing the seven specified black triples.

Lemma 2.1 *(a) The 63×480 Type I copies are all distinct.*

(b) The Type I copies are all distinct from the Type II copies.

(c) The Type II copies form 42×480 distinct pairs of identical systems.

Proof. Parts (a) and (b) follow immediately from the fact that in a copy of system #2, there is precisely one Pasch configuration which may be switched to give a copy of system #1 [4]. To establish part (c), note firstly that a copy of system #1 containing five of the specified triples together with $\{0, 2i-1, 2j-1\}$ and $\{0, 2i, 2j\}$ may be obtained from a copy containing all seven of the specified triples by means of either the permutation $(2i-1\ \ 2j)$ or the permutation $(2i\ \ 2j-1)$. A similar duplication occurs in respect of a copy of system #1 containing five of the specified triples together with $\{0, 2i-1, 2j\}$ and $\{0, 2i, 2j-1\}$. Thus there are at most 42×480 distinct Type II systems. If two Type II systems are identical then they arise from identical copies of system #1 as described and only two transpositions of the forms described are capable of producing such a copy of system #1 from a copy containing all seven of the specified triples. □

Putting together a white system #1 and a black system #2, the assumed rotation at 0 together with the lists of black and white triples determines a potential rotation scheme. As a consequence, there are $480 \times (105 \times 480)$

potential bi-embeddings of system #1 with system #2. Each of these was examined to check firstly that the potential rotation at each vertex indeed comprises a single 14-cycle and, in such cases, secondly that the whole scheme satisfies Rule R. The rotation schemes so identified were then further tested against Rule R* to determine those which are orientable.

The procedure just described leads to the conclusion that there is no bi-embedding of system #1 with system #2 in an orientable surface. However non-orientable bi-embeddings were obtained. Isomorphisms between these bi-embeddings may be determined in the manner given below. The same approach can also be used to determine the automorphism groups. Since the black and the white systems are themselves non-isomorphic, mappings which reverse the colours cannot form isomorphisms between (or automorphisms of) the bi-embeddings obtained.

Consider two rotation schemes, R_1 and R_2, defined on the points of Z_{15} and representing bi-embeddings of system #1 (white) and system #2 (black). To determine those mappings (if any) $\phi : Z_{15} \to Z_{15}$ which take R_1 to R_2 we only need consider $15 \times 14 = 210$ possibilities. For suppose two points x and y are fixed in R_1, then once their images $\phi(x)$ and $\phi(y)$ are chosen in R_2, the circularly ordered rotations at x in R_1 and at $\phi(x)$ in R_2 must correspond. Since y corresponds to $\phi(y)$, the images of the remaining points are determined up to a reversal of one of these rotations. However, only one of the two orientations is possible because colour reversals are not allowed. Thus there are 210 possible mappings which might provide an isomorphism and, similarly, there are 210 possible mappings of an embedding which might provide an automorphism.

3 Results

From the $480 \times (105 \times 480)$ possibilities described above, 1050 bi-embeddings of system #1 with system #2 were identified and these fall into just five isomorphism classes. A representative of each class is given in Table 2. None of these bi-embeddings can be oriented to satisfy Rule R* and so there is no orientable bi-embedding of these systems. Each isomorphism class contains 210 bi-embeddings satisfying Rule R. Consequently all the bi-embeddings have only the trivial automorphism. These computational results have been verified by two independently written computer programs. We summarise the results as follows.

Theorem 3.1 *Up to isomorphism, there are five non-orientable bi-embeddings of system #1 with system #2. There is no orientable bi-embedding of these systems.* □

Class #1 Representative														
0:	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1:	2	0	14	3	5	7	10	11	6	13	9	12	8	4
2:	0	1	4	14	5	8	6	7	9	11	12	10	13	3
3:	4	0	2	13	11	8	7	12	10	6	9	5	1	14
4:	0	3	14	2	1	8	11	7	13	12	6	9	10	5
5:	6	0	4	10	8	2	14	12	7	1	3	9	11	13
6:	0	5	13	1	11	14	10	3	9	4	12	8	2	7
7:	8	0	6	2	9	14	11	4	13	10	1	5	12	3
8:	0	7	3	11	4	1	12	6	2	5	10	14	13	9
9:	10	0	8	13	1	12	14	7	2	11	5	3	6	4
10:	0	9	4	5	8	14	6	3	12	2	13	7	1	11
11:	12	0	10	1	6	14	7	4	8	3	13	5	9	2
12:	0	11	2	10	3	7	5	14	9	1	8	6	4	13
13:	14	0	12	4	7	10	2	3	11	5	6	1	9	8
14:	0	13	8	10	6	11	7	9	12	5	2	4	3	1

(Continued on the next page)

Class #2 Representative														
0:	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1:	2	0	14	3	5	8	12	10	7	6	11	9	13	4
2:	0	1	4	9	6	8	7	5	14	12	10	11	13	3
3:	4	0	2	13	10	6	8	11	12	7	9	5	1	14
4:	0	3	14	6	13	1	2	9	11	7	12	8	10	5
5:	6	0	4	10	9	3	1	8	11	14	2	7	13	12
6:	0	5	12	9	2	8	3	10	14	4	13	11	1	7
7:	8	0	6	1	10	14	11	4	12	3	9	13	5	2
8:	0	7	2	6	3	11	5	1	12	4	10	13	14	9
9:	10	0	8	14	12	6	2	4	11	1	13	7	3	5
10:	0	9	5	4	8	13	3	6	14	7	1	12	2	11
11:	12	0	10	2	13	6	1	9	4	7	14	5	8	3
12:	0	11	3	7	4	8	1	10	2	14	9	6	5	13
13:	14	0	12	5	7	9	1	4	6	11	2	3	10	8
14:	0	13	8	9	12	2	5	11	7	10	6	4	3	1

Class #3 Representative														
0:	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1:	2	0	14	3	5	8	12	10	7	6	11	9	13	4
2:	0	1	4	14	5	7	8	6	9	12	10	11	13	3
3:	4	0	2	13	10	8	6	11	12	7	9	5	1	14
4:	0	3	14	2	1	13	6	9	11	7	12	8	10	5
5:	6	0	4	10	9	3	1	8	11	14	2	7	13	12
6:	0	5	12	14	10	13	4	9	2	8	3	11	1	7
7:	8	0	6	1	10	14	11	4	12	3	9	13	5	2
8:	0	7	2	6	3	10	4	12	1	5	11	13	14	9
9:	10	0	8	14	12	2	6	4	11	1	13	7	3	5
10:	0	9	5	4	8	3	13	6	14	7	1	12	2	11
11:	12	0	10	2	13	8	5	14	7	4	9	1	6	3
12:	0	11	3	7	4	8	1	10	2	9	14	6	5	13
13:	14	0	12	5	7	9	1	4	6	10	3	2	11	8
14:	0	13	8	9	12	6	10	7	11	5	2	4	3	1

(Continued on the next page)

Class #4 Representative

0:	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1:	2	0	14	4	13	3	8	6	10	11	7	5	12	9
2:	0	1	9	11	12	10	13	4	6	7	5	8	14	3
3:	4	0	2	14	9	8	1	13	11	5	6	12	10	7
4:	0	3	7	9	10	8	11	6	2	13	1	14	12	5
5:	6	0	4	12	1	7	2	8	10	14	13	9	11	3
6:	0	5	3	12	8	1	10	13	9	14	11	4	2	7
7:	8	0	6	2	5	1	11	14	10	3	4	9	12	13
8:	0	7	13	11	4	10	5	2	14	12	6	1	3	9
9:	10	0	8	3	14	6	13	5	11	2	1	12	7	4
10:	0	9	4	8	5	14	7	3	12	2	13	6	1	11
11:	12	0	10	1	7	14	6	4	8	13	3	5	9	2
12:	0	11	2	10	3	6	8	14	4	5	1	9	7	13
13:	14	0	12	7	8	11	3	1	4	2	10	6	9	5
14:	0	13	5	10	7	11	6	9	3	2	8	12	4	1

Class #5 Representative

0:	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1:	2	0	14	3	9	7	13	4	11	6	12	5	10	8
2:	0	1	8	9	14	4	7	10	12	6	5	11	13	3
3:	4	0	2	13	10	5	7	12	11	8	14	1	9	6
4:	0	3	6	10	14	2	7	11	1	13	8	12	9	5
5:	6	0	4	9	13	7	3	10	1	12	8	14	11	2
6:	0	5	2	12	1	11	9	3	4	10	8	13	14	7
7:	8	0	6	14	12	3	5	13	1	9	10	2	4	11
8:	0	7	11	3	14	5	12	4	13	6	10	1	2	9
9:	10	0	8	2	14	12	4	5	13	11	6	3	1	7
10:	0	9	7	2	12	13	3	5	1	8	6	4	14	11
11:	12	0	10	14	5	2	13	9	6	1	4	7	8	3
12:	0	11	3	7	14	9	4	8	5	1	6	2	10	13
13:	14	0	12	10	3	2	11	9	5	7	1	4	8	6
14:	0	13	6	7	12	9	2	4	10	11	5	8	3	1

Table 2. Isomorphism class representatives.

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