On the number of designs with affine parameters

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Abstract

A construction is described that yields improved lower bounds for the numbers of 2-designs and resolvable 2-designs with the parameters of $AG_d(n, q)$.

AMS classification: 05B05, 51E05.

Keywords: 2-Design; Affine Geometry; Flats; Enumeration.

1 Introduction

Various lower bounds for the numbers of 2-designs with the parameters of $PG_d(n,q)$ and $AG_d(n,q)$ have been established in the papers [1, 4, 5, 6, 7, 8, 9]. In our recent paper [3], we described two constructions that, for sufficiently large n, facilitate an improved lower bound in the case of $PG_d(n,q)$, and a lower bound for the number of resolved 2-designs with the parameters of $AG_d(n,q)$. The purpose of the present paper is to extend the second construction to produce 2-designs with the parameters of $AG_d(n,q)$ that are not necessarily resolvable. As a result we obtain an improved lower bound for the number of 2-designs with these parameters.

Throughout the paper, q is taken to be a prime power. For $n \geq d \geq 1$, $\left[{n \atop d}\right]_q$ denotes the product

$$\prod_{i=1}^{d} \frac{q^{n-d+i}-1}{q^i-1}.$$

For $n > 0$, $\left[{n \atop 0}\right]_q$ is defined to take the value 1.

A *2-design* $2-(v,k,\lambda)$ is an ordered pair $(V,\mathcal{B})$ where V is a v-set (the *points*) and $\mathcal{B}$ is a collection of k-subsets of V (the *blocks*) such that every unordered pair of distinct points occurs in precisely λ blocks. A 2-design may have repeated blocks; it is said to be *simple* if it has no repeated blocks. It is said to be *resolvable* if the blocks may be partitioned (resolved) into *parallel classes* each of which consists of a set of disjoint blocks that collectively contain all the points of the design. Note that a resolvable 2-design may have more than one resolution into parallel classes. A resolvable 2-design *with a given resolution* is called a *resolved* 2-design. We will say that a resolved 2-design is *semi-simple* if no two of the parallel classes have all their blocks identical, that is to say no parallel classes are repeated. Plainly, a simple resolved 2-design is semi-simple.

A *transversal design* $TD(k,n)$ is an ordered triple $(V,\mathcal{G},\mathcal{B})$, where V is a kn-set (the *points*), $\mathcal{G}$ is a partition of V into k disjoint n-subsets (the *groups*), and $\mathcal{B}$ is a collection of k-subsets of V (the *blocks*) such that every unordered pair of points from different groups occurs in precisely one block, and no block contains more than one point from any group. Such a design is said to be *resolvable* if the blocks may be partitioned (resolved) into n *parallel classes* each of which consists of a set of n disjoint blocks that collectively contain all the points of the design. A resolvable $TD(k,n)$ may be denoted by $rTD(k,n)$. Note that such a design may have more than one resolution into parallel classes. An $rTD(k,m)$ may be constructed from a set of $k-1$ MOLS of side m whenever such a set exists, which it does when $k=q$ (a prime power) and $m=q^r$ with $r \geq 1$.

The set of d-flats of the affine geometry $AG(n,q)$ is denoted by $AG_d(n,q)$ and forms the block set of a $2-(q^n, q^d, [{n-1 \atop d-1}]_q)$ design. A design having these parameters will be called an $A(d,n,q)$ design. Thus an $A(d,n,q)$ design has q^n points, block length q^d, every pair of points appears $[{n-1 \atop d-1}]_q$ times, every point appears $[{n \atop d}]_q$ times, and the number of blocks is $q^{n-d}[{n \atop d}]_q$. A resolvable $A(d,n,q)$ design may be denoted by $rA(d,n,q)$, and a simple resolvable $A(d,n,q)$ design may be denoted by $SrA(d,n,q)$. A resolved $A(d,n,q)$ design, that is a resolvable design with a given resolution, may be called an $RA(d,n,q)$ design. If the design is semi-simple it may be called an $S^*RA(d,n,q)$ design; if it is also simple it may be called an $SRA(d,n,q)$ design. The reader's attention is drawn to the use of the letters r and R; lower case r denotes a design that is known to be resolvable, while upper case R denotes a resolvable design *with a given resolution.* The number of parallel classes in a resolved $A(d,n,q)$ design is $[{n \atop d}]_q$ and each parallel class has q^{n-d} blocks.

Two (unresolved) designs with the same parameters, on the same set of points and, in the case of transversal designs, with the same groups, are *distinct* if they have different blocks. When speaking of the *number of designs* with a given set of parameters, we mean the number of distinct designs. If some of these designs happen to be resolvable, each such design is counted once only; there is no multiple counting for different resolutions of the same design. The same applies to the number of resolvable designs; each is counted once only. Thus the number of $A(d,n,q)$ designs is greater than or equal to the number of $rA(d,n,q)$ designs, and generally the inequality is strict because not all designs are resolvable.

Two resolved designs are *distinct* if they have different blocks, or if they have identical blocks but different parallel classes. When discussing resolvable designs it is necessary to distinguish the number of designs from the number of resolutions of these designs. The latter quantity is called the *number of resolved designs.* We will try to be careful about this point. Thus the number of $RA(d,n,q)$ designs is generally greater than the number of $rA(d,n,q)$ designs because a resolvable design may have many distinct resolutions which are counted according to multiplicity in the former but once only in the latter.

Two designs (or two resolved designs) are *isomorphic* if there is a permutation of the points that maps the blocks of one design to those of the other (and, in the case of resolved designs, preserves the parallel classes). If a given 2-design D is on v points then the largest possible isomorphism class of designs isomorphic to D on the same point set has cardinality $v!$, and this occurs when D has the trivial automorphism group. Thus the number of isomorphism classes in such cases (colloquially, the number of nonisomorphic designs) may be estimated by dividing the number of designs by $v!$. The same argument applies to isomorphism classes of resolved 2-designs.

Our previous paper [3] contained estimates for the number of resolved $A(d, n, q)$ designs. Although our primary focus in the current paper is on the number of $A(d, n, q)$ designs which may or may not be resolvable, in order to complete the comparison with the results of [1], we also provide estimates for the number of resolvable $A(d, n, q)$ designs.

2 Affine Construction

Construction 2.1 (A recursive construction for an $A(d, n, q)$ design)

We first describe the construction and then verify that it produces an $A(d, n, q)$ design. After this we give an example by constructing an $A(2, 3, 3)$ design. The reader may find it helpful to look at this example while studying the general version of the construction.

The $A(d, n, q)$ design produced by this construction is not necessarily resolvable. It will be realized on the point set $V = \bigcup_{i=1}^{q} V_i$, where $V_i = \{1_i, 2_i, \ldots, q_i^{n-1}\}$. Here $n > d > 1$ and the ingredients required are as follows:

(a) an $RA(d-1, n-1, q)$, $\mathcal{D}_i$, on each point set V_i for $i = 1, 2, \ldots, q-1$,

(b) an $A(d-1, n-1, q)$, $\mathcal{D}_q$, on the point set V_q

(c) an $A(d, n-1, q)$, $\mathcal{E}_i$, on each point set V_i for $i = 1, 2, \ldots, q$,

(d) $\left[{n-1 \atop d-1}\right]_q$ transversal designs $TD(q, q^{n-d})$.

Where several designs with the same parameter set are used, it does not matter whether or not these designs are isomorphic. In connection with item (d) we will specify the point sets and groups of the designs below. Each such design has q groups, each of size q^{n-d} and a total of $q^{2(n-d)}$ blocks.

Suppose that for $i = 1, 2, \ldots, q-1$, $\{\mathcal{P}_{j,i} : 1 \leq j \leq [{n-1 \atop d-1}]_q\}$ is the set of parallel classes of the design $\mathcal{D}_i$. The order in which these parallel classes of each $\mathcal{D}_i$ are listed is arbitrary. Let $\{\mathcal{P}_{j,q} : 1 \leq j \leq [{n-1 \atop d-1}]_q\}$ be *any* partition of the blocks of $\mathcal{D}_q$ into classes of size q^{n-d} (note that $\mathcal{D}_q$ is not required to be resolvable). Then for $i = 1, 2, \ldots, q$ let the set of blocks forming the class $\mathcal{P}_{j,i}$ be $\{B_{k,j,i} : 1 \leq k \leq q^{n-d}\}$. For each value of j, $1 \leq j \leq [{n-1 \atop d-1}]_q$, take a $TD(q, q^{n-d})$, $\mathcal{T}_j$, having point set $\{B_{k,j,i} : 1 \leq i \leq q, 1 \leq k \leq q^{n-d}\}$, and groups $\mathcal{P}_{j,i}$ for $i = 1, 2, \ldots, q$. For each block of this design, $\{B_{k_1,j,1}, B_{k_2,j,2}, \ldots, B_{k_q,j,q}\}$, form the block $B^* = \bigcup_{i=1}^{q} B_{k_i,j,i}$. For each fixed j there will be $q^{2(n-d)}$ blocks of the form B^*, each of length $q \cdot q^{d-1} = q^d$ on the point set V. Figure 1 illustrates this part of the construction. Let $\mathcal{B}_a$ be the collection of all the blocks of the form B^* taken over all values of j from $j = 1$ to $j = [{n-1 \atop d-1}]_q$. Then $\mathcal{B}_a$ comprises $q^{2(n-d)}[{n-1 \atop d-1}]_q$ blocks of length q^d on the point set V.

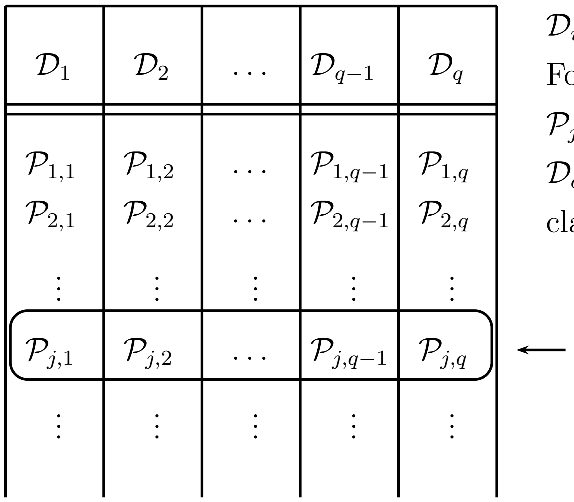
Let $\mathcal{B}_b$ be the collection of all the blocks from all the designs $\mathcal{E}_i$, $1 \leq i \leq q$. Then $\mathcal{B}_b$ comprises $q \cdot q^{n-1-d}[{n-1 \atop d}]_q = q^{n-d}[{n-1 \atop d}]_q$ blocks of length q^d on the point set V. Then put $\mathcal{B} = \mathcal{B}_a \cup \mathcal{B}_b$. We will prove that $\mathcal{B}$ forms an $A(d, n, q)$ design.

By construction, $\mathcal{B}$ contains only blocks of length q^d on the point set V of cardinality q^n. Allowing possible multiplicity, we have

$$|\mathcal{B}| = |\mathcal{B}_a| + |\mathcal{B}_b| = q^{n-d}\left\{ q^{n-d}\begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_q + \begin{bmatrix} n-1 \\ d \end{bmatrix}_q \right\} = q^{n-d}\begin{bmatrix} n \\ d \end{bmatrix}_q .$$

Hence $|\mathcal{B}|$ is the number of blocks in an $A(d, n, q)$ design.

To prove that the blocks of $\mathcal{B}$ form an $A(d, n, q)$ design on the point set V, it remains only to show that every pair of distinct points $x, y \in V$ appears $[{n-1 \atop d-1}]_q$ times. So take any two distinct points $x_i \in V_i$ and $y_h \in V_h$. Suppose initially that $i \neq h$. We may then assume that $i < h$, so that $i \neq q$. Then the point x_i appears in precisely one block of the j^{th} parallel class $\mathcal{P}_{j,i}$ of $\mathcal{D}_i$, say $B_{k_x,j,i}$. If $h < q$ then y_h appears in precisely one block of the j^{th} parallel class $\mathcal{P}_{j,h}$ of $\mathcal{D}_h$, say $B_{k_y,j,h}$. The blocks $B_{k_x,j,i}$ and $B_{k_y,j,h}$ then appear together in precisely one block of the transversal design $\mathcal{T}_j$. Hence the points x_i and y_h appear together in precisely one block B^* for each j in the range $1 \leq j \leq [{n-1 \atop d-1}]_q$ and consequently the pair $x_i y_h$, $i < h < q$, appears in precisely $[{n-1 \atop d-1}]_q$ blocks of $\mathcal{B}_a$. Now consider the case $h = q$. Amongst all the blocks of $\mathcal{D}_q$, the point y_q appears $[{n-1 \atop d-1}]_q$ times. Denote by p_j the

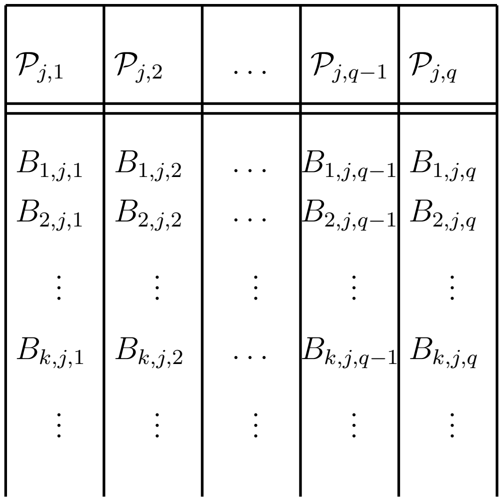


$\mathcal{D}_1$	$\mathcal{D}_2$	$\dots$	$\mathcal{D}_{q-1}$	$\mathcal{D}_q$
$\mathcal{P}_{1,1}$	$\mathcal{P}_{1,2}$	$\dots$	$\mathcal{P}_{1,q-1}$	$\mathcal{P}_{1,q}$
$\mathcal{P}_{2,1}$	$\mathcal{P}_{2,2}$	$\dots$	$\mathcal{P}_{2,q-1}$	$\mathcal{P}_{2,q}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathcal{P}_{j,1}$	$\mathcal{P}_{j,2}$	$\dots$	$\mathcal{P}_{j,q-1}$	$\mathcal{P}_{j,q}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

$\mathcal{D}_i$ is on the point set V_i.
For $i \neq q$, $\mathcal{D}_i$ has parallel classes $\mathcal{P}_{j,i}$ for $1 \leq j \leq \left[{n-1 \atop d-1}\right]_q$.
$\mathcal{D}_q$ is partitioned into $\left[{n-1 \atop d-1}\right]_q$ classes $\mathcal{P}_{j,q}$ each of size q^{n-d}.

The blocks of the classes $\mathcal{P}_{j,i}$ for $1 \leq i \leq q$ are the points of the transversal design $\mathcal{T}_j$, and the classes themselves form the groups of this design.

Figure 1(a). The partitioning of the designs $\mathcal{D}_i$.



$\mathcal{P}_{j,1}$	$\mathcal{P}_{j,2}$	$\dots$	$\mathcal{P}_{j,q-1}$	$\mathcal{P}_{j,q}$
$B_{1,j,1}$	$B_{1,j,2}$	$\dots$	$B_{1,j,q-1}$	$B_{1,j,q}$
$B_{2,j,1}$	$B_{2,j,2}$	$\dots$	$B_{2,j,q-1}$	$B_{2,j,q}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$B_{k,j,1}$	$B_{k,j,2}$	$\dots$	$B_{k,j,q-1}$	$B_{k,j,q}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

$\mathcal{P}_{j,i}$ is on the point set V_i and has blocks $B_{k,j,i}$ for $1 \leq k \leq q^{n-d}$.

The classes $\mathcal{P}_{j,i}$, $1 \leq i \leq q$ form the groups of $\mathcal{T}_j$, and the blocks $B_{k,j,i}$ for $1 \leq k \leq q^{n-d}$ and $1 \leq i \leq q$ form the points of $\mathcal{T}_j$.

Figure 1(b). The transversal design $\mathcal{T}_j$.

number of times it appears in the blocks of $\mathcal{P}_{j,q}$. Then $\sum_{j=1}^{\left[{n-1 \atop d-1}\right]_q} p_j = \left[{n-1 \atop d-1}\right]_q$. Furthermore, the points x_i and y_q appear together p_j times in the blocks B^* generated by the transversal design $\mathcal{T}_j$. Hence the pair x_iy_q $(i < q)$ appears in precisely $\sum_{j=1}^{\left[{n-1 \atop d-1}\right]_q} p_j = \left[{n-1 \atop d-1}\right]_q$ blocks of $\mathcal{B}_a$.

Finally consider the case $i = h$ by taking two points x_i and y_i where $x \neq y$. The pair x_iy_i does not appear in any $\mathcal{D}_j$ with $j \neq i$, and it appears in precisely $\left[{n-2 \atop d-2}\right]_q$ blocks of $\mathcal{D}_i$. Each point of a $TD(q, q^{n-d})$ appears in q^{n-d} blocks of the transversal design. Consequently each of the $\left[{n-2 \atop d-2}\right]_q$ blocks containing the pair x_iy_i appears as a sub-block of q^{n-d} blocks of $\mathcal{B}_a$. (Here and subsequently we use the term "sub-block" to describe a subset of a block which is itself a block of another design.) Hence the pair x_iy_i appears in precisely $q^{n-d}\left[{n-2 \atop d-2}\right]_q$ blocks of $\mathcal{B}_a$. Furthermore, the pair x_iy_i appears in precisely $\left[{n-2 \atop d-1}\right]_q$ blocks of $\mathcal{E}_i$ and so it appears in precisely $\left[{n-2 \atop d-1}\right]_q$ blocks of $\mathcal{B}_b$. But

$$q^{n-d}\begin{bmatrix} n-2 \\ d-2 \end{bmatrix}_q + \begin{bmatrix} n-2 \\ d-1 \end{bmatrix}_q = \begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_q,$$

and so the pair x_iy_i appears in $\left[{n-1 \atop d-1}\right]_q$ blocks of $\mathcal{B}$.

It follows that the blocks of $\mathcal{B}$ form an $A(d, n, q)$ design on the point set V.

□

Example 2.1 (An $A(2, 3, 3)$ design)

An $A(2, 3, 3)$ is a $2 - (27, 9, 4)$ design and it has 39 blocks. The ingredients required by the construction are as follows, where $V_i = \{1_i, 2_i, \ldots, 9_i\}$:

(a) an $RA(1, 2, 3)$, $\mathcal{D}_i$, on each point set V_i for $i = 1, 2$,

(b) an $A(1, 2, 3)$, $\mathcal{D}_3$, on the point set V_3,

(c) an $A(2, 2, 3)$, $\mathcal{E}_i$, on each point set V_i for $i = 1, 2, 3$,

(d) four transversal designs $TD(3, 3)$.

Items (a) are resolved $2 - (9, 3, 1)$ designs and item (b) is an (unresolved) $2 - (9, 3, 1)$ design. Up to isomorphism there is only one design with these parameters which is resolvable with a unique resolution. For $\mathcal{D}_1$ we list the parallel classes in one order and for $\mathcal{D}_2$ we choose a different order. For $\mathcal{D}_3$ we choose an arbitrary partition of the blocks into classes of size three to emphasize that, in general, $\mathcal{D}_q$ need not be resolvable. Each $\mathcal{E}_i$ consists of the single block V_i. The tables below list all these blocks and the partitions into classes $\mathcal{P}_{j,i}$. Blocks are listed without set braces $\{\}$ and commas.

$\mathcal{D}_1$		$\mathcal{D}_2$		$\mathcal{D}_3$	
$\mathcal{P}_{1,1}$	$1_12_13_1 = B_{1,1,1}$	$\mathcal{P}_{1,2}$	$1_25_29_2 = B_{1,1,2}$	$\mathcal{P}_{1,3}$	$1_32_33_3 = B_{1,1,3}$
	$4_15_16_1 = B_{2,1,1}$		$2_26_27_2 = B_{2,1,2}$		$1_34_37_3 = B_{2,1,3}$
	$7_18_19_1 = B_{3,1,1}$		$3_24_28_2 = B_{3,1,2}$		$2_36_37_3 = B_{3,1,3}$
$\mathcal{P}_{2,1}$	$1_14_17_1 = B_{1,2,1}$	$\mathcal{P}_{2,2}$	$1_24_27_2 = B_{1,2,2}$	$\mathcal{P}_{2,3}$	$4_35_36_3 = B_{1,2,3}$
	$2_15_18_1 = B_{2,2,1}$		$2_25_28_2 = B_{2,2,2}$		$1_36_38_3 = B_{2,2,3}$
	$3_16_19_1 = B_{3,2,1}$		$3_26_29_2 = B_{3,2,2}$		$2_34_39_3 = B_{3,2,3}$
$\mathcal{P}_{3,1}$	$1_15_19_1 = B_{1,3,1}$	$\mathcal{P}_{3,2}$	$1_26_28_2 = B_{1,3,2}$	$\mathcal{P}_{3,3}$	$1_35_39_3 = B_{1,3,3}$
	$2_16_17_1 = B_{2,3,1}$		$2_24_29_2 = B_{2,3,2}$		$7_38_39_3 = B_{2,3,3}$
	$3_14_18_1 = B_{3,3,1}$		$3_25_27_2 = B_{3,3,2}$		$3_35_37_3 = B_{3,3,3}$
$\mathcal{P}_{4,1}$	$1_16_18_1 = B_{1,4,1}$	$\mathcal{P}_{4,2}$	$1_22_23_2 = B_{1,4,2}$	$\mathcal{P}_{4,3}$	$2_35_38_3 = B_{1,4,3}$
	$2_14_19_1 = B_{2,4,1}$		$4_25_26_2 = B_{2,4,2}$		$3_36_39_3 = B_{2,4,3}$
	$3_15_17_1 = B_{3,4,1}$		$7_28_29_2 = B_{3,4,2}$		$3_34_38_3 = B_{3,4,3}$

$\mathcal{E}_1$	$\mathcal{E}_2$	$\mathcal{E}_3$
$1_12_13_14_15_16_17_18_19_1$	$1_22_23_24_25_26_27_28_29_2$	$1_32_33_34_35_36_37_38_39_3$

For items (d), four suitable $TD(3,3)$ designs are listed below. In each case the three columns give the three groups, while the nine blocks are formed from the rows and the diagonals.

$\mathcal{T}_1$	$\mathcal{T}_2$	$\mathcal{T}_3$	$\mathcal{T}_4$
$B_{1,1,1}\ B_{1,1,2}\ B_{1,1,3}$	$B_{1,2,1}\ B_{1,2,2}\ B_{1,2,3}$	$B_{1,3,1}\ B_{1,3,2}\ B_{1,3,3}$	$B_{1,4,1}\ B_{1,4,2}\ B_{3,4,3}$
$B_{2,1,1}\ B_{2,1,2}\ B_{2,1,3}$	$B_{2,2,1}\ B_{2,2,2}\ B_{3,2,3}$	$B_{2,3,1}\ B_{3,3,2}\ B_{2,3,3}$	$B_{2,4,1}\ B_{3,4,2}\ B_{2,4,3}$
$B_{3,1,1}\ B_{3,1,2}\ B_{3,1,3}$	$B_{3,2,1}\ B_{3,2,2}\ B_{2,2,3}$	$B_{3,3,1}\ B_{2,3,2}\ B_{3,3,3}$	$B_{3,4,1}\ B_{2,4,2}\ B_{1,4,3}$

The 39 blocks of the resulting $A(2,3,3)$ design are listed below. The final three blocks are those from the designs $\mathcal{E}_i$ and the others are formed from the blocks of the $TD(3,3)$ designs, for example the block $2_15_18_13_26_29_24_35_36_3$ is given by the block $B_{2,2,1}B_{3,2,2}B_{1,2,3}$ of $\mathcal{T}_2$.

$1_12_13_11_25_29_21_32_33_3$	$1_12_13_12_26_27_22_36_37_3$	$1_12_13_13_24_28_21_34_37_3$
$4_15_16_12_26_27_21_34_37_3$	$4_15_16_13_24_28_21_32_33_3$	$4_15_16_11_25_29_22_36_37_3$
$7_18_19_13_24_28_22_36_37_3$	$7_18_19_11_25_29_21_34_37_3$	$7_18_19_12_26_27_21_32_33_3$
$1_14_17_11_24_27_24_35_36_3$	$1_14_17_12_25_28_21_36_38_3$	$1_14_17_13_26_29_22_34_39_3$
$2_15_18_12_25_28_22_34_39_3$	$2_15_18_13_26_29_24_35_36_3$	$2_15_18_11_24_27_21_36_38_3$
$3_16_19_13_26_29_21_36_38_3$	$3_16_19_11_24_27_22_34_39_3$	$3_16_19_12_25_28_24_35_36_3$
$1_15_19_11_26_28_21_35_39_3$	$1_15_19_13_25_27_23_35_37_3$	$1_15_19_12_24_29_27_38_39_3$
$2_16_17_13_25_27_27_38_39_3$	$2_16_17_12_24_29_21_35_39_3$	$2_16_17_11_26_28_23_35_37_3$
$3_14_18_12_24_29_23_35_37_3$	$3_14_18_11_26_28_27_38_39_3$	$3_14_18_13_25_27_21_35_39_3$
$1_16_18_11_22_23_23_34_38_3$	$1_16_18_17_28_29_22_35_38_3$	$1_16_18_14_25_26_23_36_39_3$
$2_14_19_17_28_29_23_36_39_3$	$2_14_19_14_25_26_23_34_38_3$	$2_14_19_11_22_23_22_35_38_3$
$3_15_17_14_25_26_22_35_38_3$	$3_15_17_11_22_23_23_36_39_3$	$3_15_17_17_28_29_23_34_38_3$
$1_12_13_14_15_16_17_18_19_1$	$1_22_23_24_25_26_27_28_29_2$	$1_32_33_34_35_36_37_38_39_3$

It is routine to verify that every pair of points appears precisely four times and that the design is therefore a $2-(27,9,4)$ design. $\square$

It was noted in the construction that for each i, $1 \le i \le q$, the ordering of the classes $\mathcal{P}_{j,i}$ for $1 \le j \le [{n-1 \atop d-1}]_q$ is arbitrary. If the ordering for $i = 1$ is fixed, then the number of ways of entering the classes of the remaining designs $\mathcal{D}_i$ $(i > 1)$ into the table shown in Figure 1(a) is $\left(([{n-1 \atop d-1}]_q)!\right)^{q-1}$. We will refer to these arrangements as *alignments* of the classes $\mathcal{P}_{j,i}$. Two alignments will be regarded as the same if one can be obtained from the other by re-ordering the rows of the table. Thus alignments are determined by the entries in each row, irrespective of the order of the rows. If the designs $\mathcal{D}_i$ are semi-simple for $1 \le i \le q-1$ and the partitioning of the design $\mathcal{D}_q$ has no repeated classes $\mathcal{P}_{j,q}$, then the $\left(([{n-1 \atop d-1}]_q)!\right)^{q-1}$ alignments of the classes are all distinct. If the design $\mathcal{D}_q$ is simple, then it will have no repeated classes however it is partitioned. Given a set of $s = tu$ distinct objects, the number of distinct ways of grouping these objects into u subsets each of size t is $s!/((t!)^u u!)$. Consequently if $\mathcal{D}_q$ is simple, then there are $\dfrac{(q^{n-d}[{n-1 \atop d-1}]_q)!}{((q^{n-d})!)^{[{n-1 \atop d-1}]_q}([{n-1 \atop d-1}]_q)!}$ distinct ways of partitioning the design into classes of size q^{n-d}.

We wish to use the construction recursively, so we make the following observations in the form of a lemma using the terminology employed in the construction.

Lemma 2.1

(*a*) *Suppose that* $\mathcal{D}_q$ *is simple and that each* $\mathcal{E}_i$ *is simple. Then the resulting* $A(d,n,q)$ *design is simple.*

(*b*) *Suppose that, in addition to the conditions of* (*a*), *each* $\mathcal{D}_i$ $(1 \le i \le q-1)$ *is semi-simple. Then varying any of the ingredients* $\mathcal{D}_i$, $\mathcal{E}_i$ *or* $\mathcal{T}_j$, *varying the partitioning of* $\mathcal{D}_q$, *or varying the alignments of the classes* $\mathcal{P}_{j,i}$ *results in distinct designs.*

(*c*) *Suppose that the conditions of both* (*a*) *and* (*b*) *are satisfied and also that*

(*i*) $\mathcal{D}_q$ *is a resolved design and that the classes* $\mathcal{P}_{j,q}$ *are its parallel classes, and*

(*ii*) *each of the designs* $\mathcal{E}_i$ *and* $\mathcal{T}_j$ *is resolvable.*

Then the resulting $A(d,n,q)$ *design is resolvable.*

Proof (a) If each $\mathcal{E}_i$ is simple, there is no repetition of blocks lying in $\mathcal{B}_b$. Furthermore, each of the blocks of $\mathcal{B}_b$ contains only points x_i with a common suffix i, whereas each of the blocks of $\mathcal{B}_a$ contains points with all suffices $i = 1, 2, \ldots, q$. Hence no block of $\mathcal{B}_b$ appears in $\mathcal{B}_a$. If $\mathcal{D}_q$ is simple then each block C_q of this design appears in one and only one of the classes $\mathcal{P}_{j,q}$ $(1 \le j \le [{n-1 \atop d-1}]_q)$. Consequently each such block C_q appears as a sub-block of q^{n-d} blocks of length q^d generated by one and only one of the designs $\mathcal{T}_j$. But the complementary sub-blocks of these q^{n-d} blocks are all distinct. Hence no block of $\mathcal{B}_a$ is repeated. It follows that $\mathcal{B}$ has no repeated blocks.

(b) Now consider two designs, $\mathcal{R}$ and $\mathcal{R}'$, resulting from the construction, the first formed from $\mathcal{D}_i$, $\mathcal{E}_i$ and $\mathcal{T}_j$, and the second from $\mathcal{D}'_i$, $\mathcal{E}'_i$ and $\mathcal{T}'_j$ where $1 \le i \le q$ and $1 \le j \le [{n-1 \atop d-1}]_q$. We will assume that $\mathcal{D}_q, \mathcal{D}'_q$ are simple, each $\mathcal{E}_i, \mathcal{E}'_i$ is simple and, for $i < q$, each $\mathcal{D}_i, \mathcal{D}'_i$ is semi-simple. If X is any structure in $\mathcal{R}$, then we will denote by X' the corresponding structure in $\mathcal{R}'$.

Choose any i such that $1 \le i \le q$. In each of $\mathcal{R}$ and $\mathcal{R}'$, delete all the blocks that contain points from $V \setminus V_i$. The remaining blocks are those of $\mathcal{E}_i$ (respectively, $\mathcal{E}'_i$). Hence if $\mathcal{E}_i$ and $\mathcal{E}'_i$ are different designs, then $\mathcal{R} \ne \mathcal{R}'$.

Subsequently we assume that $\mathcal{E}_i = \mathcal{E}'_i$ for $1 \le i \le q$. In each of $\mathcal{R}$ and $\mathcal{R}'$, delete all the blocks of $\mathcal{B}_b$ $(= \mathcal{B}'_b)$ so that only the blocks of $\mathcal{B}_a$ $(\mathcal{B}'_a)$ remain. Pick any block in $\mathcal{B}_a$ and let C_q be the sub-block containing all the points of V_q that appear in this block. Next list all the blocks of $\mathcal{B}_a$ that contain C_q as a sub-block. Since $\mathcal{D}_q$ is simple, this collection of blocks determines parallel classes $\mathcal{P}_{j,1}, \mathcal{P}_{j,2}, \ldots, \mathcal{P}_{j,q-1}$ in the designs $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_{q-1}$ (respectively). Furthermore, since the designs $\mathcal{D}_i$ $(1 \leq i \leq q-1)$ are semi-simple, there are precisely q^{n-d} different sub-blocks C_q which give rise to the same set of parallel classes. It is therefore possible to reconstruct the resolution of $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_{q-1}$ and the partition of $\mathcal{D}_q$ uniquely, along with the alignment of these classes. The same argument applies to $\mathcal{B}'_a$. Hence if $\mathcal{D}_i$ and $\mathcal{D}'_i$ are different resolved designs for any i $(1 \leq i \leq q-1)$, or if $\mathcal{D}_q$ and $\mathcal{D}'_q$ are differently partitioned designs, or if the classes are aligned differently, then $\mathcal{R} \neq \mathcal{R}'$.

So now assume that $\mathcal{D}_i = \mathcal{D}'_i$ for $1 \leq i \leq q-1$ as resolved designs, that $\mathcal{D}_q = \mathcal{D}'_q$ are partitioned identically, and that the alignments of the classes are identical. The argument given above determines the groups of each $\mathcal{T}_j$ in $\mathcal{R}$ and each $\mathcal{T}'_j$ in $\mathcal{R}'$, but also the blocks of $\mathcal{T}_j$ and $\mathcal{T}'_j$. If these blocks differ, then $\mathcal{R} \neq \mathcal{R}'$.

(c) If each of the designs $\mathcal{E}_i$ is resolvable then the blocks of $\mathcal{B}_b$ can be resolved into parallel classes. Consider the blocks of $\mathcal{B}_a$ formed from a single resolvable $\mathcal{T}_j$ with groups $\mathcal{P}_{j,i}$ $(1 \leq i \leq q)$, where each $\mathcal{P}_{j,i}$ is on the point set V_i and comprises blocks $B_{k,j,i}$ $(1 \leq k \leq q^{n-d})$ as described in the construction. The q^{n-d} blocks of $\mathcal{B}_a$ formed from a parallel class of $\mathcal{T}_j$ contain each point of each set V_i precisely once, and hence form a parallel class of blocks on the point set V. Thus $\mathcal{B}_a$ can be resolved into parallel classes.

This completes the proof of the lemma. □

We will make use of the following notation introduced in [3]. Denote by $NTD(k,n)$ the number of distinct $TD(k,n)$ designs and by $NSA(d,n,q)$ the number of distinct simple $A(d,n,q)$ designs. The number of semi-simple resolutions of resolvable $A(d,n,q)$ designs is denoted by $NS^*RA(d,n,q)$ and the number of simple resolved $A(d,n,q)$ designs by $NSRA(d,n,q)$. In the current paper we also want to consider the number of resolvable designs of a particular type. We will use $NSrA(d,n,q)$ to denote the number of simple resolvable $A(d,n,q)$ designs, and $NrTD(k,n)$ to denote the number of resolvable $TD(k,n)$ designs. There are some obvious inequalities between these various quantities, for example as indicated in the Introduction, $NSA(d,n,q) \geq NSrA(d,n,q)$ and $NSRA(d,n,q) \geq NSrA(d,n,q)$.

The lemma gives the following two inequalities.

$$\begin{aligned} NSA(d,n,q) \quad \geq \quad & (NSA(d,n-1,q))^{q} \times (NS^{*}RA(d-1,n-1,q))^{q-1} \\ & \times (NSA(d-1,n-1,q)) \times \left(NTD(q,q^{n-d})\right)^{\left[{n-1 \atop d-1}\right]_q} \\ & \times \left(\left(\begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_q\right)!\right)^{q-1} \\ & \times \frac{(q^{n-d}\left[{n-1 \atop d-1}\right]_q)!}{((q^{n-d})!)^{\left[{n-1 \atop d-1}\right]_q}(\left[{n-1 \atop d-1}\right]_q)!} \end{aligned} \tag{1}$$

$$\begin{aligned} NSrA(d,n,q) \quad \geq \quad & (NSrA(d,n-1,q))^{q} \times (NS^{*}RA(d-1,n-1,q))^{q-1} \\ & \times (NSRA(d-1,n-1,q)) \times \left(NrTD(q,q^{n-d})\right)^{\left[{n-1 \atop d-1}\right]_q} \\ & \times \left(\left(\begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_q\right)!\right)^{q-1} \end{aligned} \tag{2}$$

In connection with applying (1) and (2) recursively, note that an $A(n,n,q)$ design consists of a single block on q^n points, so the design is unique, resolvable, and has a unique resolution. Also, an $rA(1,n,q)$ design is a resolvable $2-(q^n,q,1)$ design, which is a resolvable Steiner system on q^n points with block length q, and $NS^{*}RA(1,n,q)$ is just the number of resolutions of such systems, all of which are simple. Use can also be made of the fact that for $1 \leq d \leq n-1$, $AG_d(n,q)$ has full automorphism group $A\Gamma L(n,q)$, and so $NSrA(d,n,q)$, $NSA(d,n,q)$, $NS^{*}RA(d,n,q)$ and $NSRA(d,n,q)$ are all at least $(q^n)!/|A\Gamma L(n,q)|$. Furthermore, if $q=p^s$, where p is prime, then $|A\Gamma L(n,q)| = sq^{\frac{n(n+1)}{2}}\prod_{i=1}^{n}(q^i-1)$. Note also that $s \leq p^s$ and so $s \leq q$, while $\prod_{i=1}^{n}(q^i-1) < \prod_{i=1}^{n} q^i = q^{\frac{n(n+1)}{2}}$, giving $|A\Gamma L(n,q)| < q^{n^2+n+1}$ and hence $(q^n)!/|A\Gamma L(n,q)| > (q^n)!/q^{n^2+n+1}$. We next derive some estimates for $NrTD(q,m)$ and consequently also for $NTD(q,m)$.

Lemma 2.2 *Suppose that q is a prime power and that $m=q^r$ where $r \geq 1$. If $q \geq 3$ then $NTD(q,m) \geq NrTD(q,m) \geq (m-1)!(m!)^{q-2} = (m!)^{q-1}/m$. If $q=2$ then $NTD(q,m) = NrTD(q,m) = 1$.*

Proof The conditions ensure the existence of an $rTD(q,m)$ design, $\mathcal{T}$. Denote the q groups of $\mathcal{T}$ by G_i, $1 \leq i \leq q$. If $q=2$ there are just two groups and every pair $(g_1,g_2) \in G_1 \times G_2$ forms a block. Consequently there is only one $TD(2,m)$ and this is resolvable. (Of course this design may have many distinct resolutions.)

Now suppose that $q \geq 3$. Let the points of G_i be $g_{i,1}, g_{i,2}, \ldots, g_{i,m}$. Suppose that π_2, π_2' are permutations of $\{g_{2,2}, g_{2,3}, \ldots, g_{2,m}\}$ and that for $i \geq 3$, π_i, π_i' are permutations of G_i. Put $\pi = \prod_{i=2}^{q} \pi_i$ and $\pi' = \prod_{i=2}^{q} \pi_i'$. For $1 \leq j \leq m$, let B_j be the block of $\mathcal{T}$ containing the points $g_{1,j}$ and $g_{2,1}$. Note that both $\pi(B_j)$ and $\pi'(B_j)$ contain the pair $\{g_{1,j}, g_{2,1}\}$. Now suppose that for some $i \geq 3$, $\pi_i \neq \pi_i'$. Then there exists $g_{i,k}$ such that $\pi_i(g_{i,k}) \neq \pi_i'(g_{i,k})$. The point $g_{i,k}$ must occur in one of the blocks B_j, say B_{j_k}. Then $\pi(B_{j_k}) \neq \pi'(B_{j_k})$, so that $\pi(\mathcal{T}) \neq \pi'(\mathcal{T})$.

Next suppose that $\pi_i = \pi_i'$ for $i \geq 3$, but $\pi_2 \neq \pi_2'$. Then there exists $g_{2,k}$ $(k \geq 2)$ such that $\pi_2(g_{2,k}) \neq \pi_2'(g_{2,k})$. But then if B is the block of $\mathcal{T}$ containing $g_{1,1}$ and $g_{2,k}$, this block must also contain some point $g_{3,j}$ (note $q \geq 3$). Hence $\pi(B)$ contains $g_{1,1}, \pi_3(g_{3,j})$ and $\pi_2(g_{2,k})$, while $\pi'(B)$ contains $g_{1,1}, \pi_3(g_{3,j})$ and $\pi_2'(g_{2,k})$. So again, $\pi(\mathcal{T}) \neq \pi'(\mathcal{T})$.

It follows that for $q \geq 3$, each of the $(m-1)!(m!)^{q-2}$ permutations π gives a distinct $rTD(q,m)$, each being an isomorphic copy of $\mathcal{T}$. Hence $NrTD(q,m) \geq (m-1)!(m!)^{q-2} = (m!)^{q-1}/m$. □

Note that if $q = 3$, then $NTD(3,m)$ is just the number of distinct Latin squares of side m; for large m this is $m^{m^2(1-o(1))}$. This is much larger than the estimate given by the previous lemma.

3 Applications

In this section we examine some consequences of applying inequalities (1) and (2). The bounds we obtain rely on some very crude estimates. We also note that if $j \geq i \geq 1$ then $q^{j-i} \leq (q^j - 1)/(q^i - 1)$, so that $\left[{n \atop d}\right]_q \geq q^{d(n-d)}$. The bounds that could be obtained by more exact use of (1) and (2) in particular cases would be much greater than those recorded here.

First we use (1) to obtain an explicit lower bound for the number of nonisomorphic simple $A(d,n,q)$ designs when $q \geq 3$ and $2 \leq d \leq n-2$.

Lemma 3.1 *Suppose that $q \geq 3$ and $2 \leq d \leq n-2$. Then the number of nonisomorphic simple $A(d,n,q)$ designs is at least*

$$\frac{((q^{n-1})!)^q \left(q^{n-d}\left[{n-1 \atop d-1}\right]_q\right)! \left(\left(\left[{n-1 \atop d-1}\right]_q\right)!\right)^{q-2} \left((q^{n-d})!\right)^{(q-3)\left[{n-1 \atop d-1}\right]_q}}{|A\Gamma L(n-1,q)|^{2q}}.$$

Proof Using (1), the previous lemma, and the fact that each of $NSA(d, n-1, q), NS^*RA(d-1, n-1, q)$ and $NSRA(d-1, n-1, q)$ is at least $(q^{n-1})!/|A\mathit{\Gamma} L(n-1, q)|$, we obtain

$$
\begin{aligned}
NSA(d,n,q) \quad \geq \quad & \frac{((q^{n-1})!)^{2q}}{|A\mathit{\Gamma} L(n-1,q)|^{2q}} \frac{\left(q^{n-d} \genfrac{[}{]}{0pt}{}{n-1}{d-1}_q\right)!}{((q^{n-d})!)^{\genfrac{[}{]}{0pt}{}{n-1}{d-1}_q}} \\
& \times \left(\left(\genfrac{[}{]}{0pt}{}{n-1}{d-1}_q\right)!\right)^{q-2} \left[\frac{\left((q^{n-d})!\right)^{q-1}}{q^{n-d}}\right]^{\genfrac{[}{]}{0pt}{}{n-1}{d-1}_q} .
\end{aligned}
$$

If $m \geq 8$ then $(m-1)! > m^{m/2}$ (this can be proved by induction using $7! = 5040, 8^4 = 4096$ and $(\frac{m+1}{m})^m < 3$ for $m \geq 1$). Since $q \geq 3$ and $n-d \geq 2$, $q^{n-d} \geq 9 > 8$, so that $(q^{n-d}-1)! > q^{(n-d)q^{n-d}/2}$. Hence

$$
\begin{aligned}
\left(\frac{(q^{n-d})!}{q^{n-d}}\right)^{\genfrac{[}{]}{0pt}{}{n-1}{d-1}_q} \quad & > \quad q^{(\frac{n-d}{2})q^{n-d}\genfrac{[}{]}{0pt}{}{n-1}{d-1}_q} \\
& \geq \quad q^{q^{n-d}\genfrac{[}{]}{0pt}{}{n-1}{d-1}_q} \\
& > \quad q^{q^{n-d}q^{(d-1)(n-d)}} \\
& = \quad q^{q^{d(n-d)}} .
\end{aligned}
$$

But $d(n-d)$ has minimum value $2(n-2)$ for $2 \leq d \leq n-2$, and $2(n-2) \geq n$ since $n \geq 4$. Thus

$$
\left(\frac{(q^{n-d})!}{q^{n-d}}\right)^{\genfrac{[}{]}{0pt}{}{n-1}{d-1}_q} \quad > \quad q^{q^n} .
$$

We also have $q^{q^n}((q^{n-1})!)^q > (q^n)!$. Hence

$$
\begin{aligned}
NSA(d,n,q) \quad \geq \quad & \frac{(q^n)!((q^{n-1})!)^q}{|A\mathit{\Gamma} L(n-1,q)|^{2q}} \frac{\left(q^{n-d} \genfrac{[}{]}{0pt}{}{n-1}{d-1}_q\right)!}{((q^{n-d})!)^{\genfrac{[}{]}{0pt}{}{n-1}{d-1}_q}} \\
& \times \left(\left(\genfrac{[}{]}{0pt}{}{n-1}{d-1}_q\right)!\right)^{q-2} \left[\left((q^{n-d})!\right)^{q-2}\right]^{\genfrac{[}{]}{0pt}{}{n-1}{d-1}_q} .
\end{aligned}
$$

Dividing by $(q^n)!$, the largest possible size of an isomorphism class, and cancelling $\left((q^{n-d})!\right)^{\genfrac{[}{]}{0pt}{}{n-1}{d-1}_q}$ gives the required result. $\square$

Some very crude estimates have been used in the previous lemma. More exact calculations in particular cases would greatly improve the lower bound.

Recursive use of (1) would also greatly improve the bound. Even so, noting that $(q^{n-1})!/|A\Gamma L(n-1,q)| > 1$, when $q \neq 2$ and $d \neq n-1$, it can be seen that the bound is better than that given by Clark, Jungnickel and Tonchev in [1], namely

$$\frac{\left(q^{n-d}\begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_q\right)!}{q!\,(q^{n-1}+q^{n-2}+\ldots+q+1)\,|A\Gamma L(n-1,q)|^q}.$$

Next we use (2) to obtain an explicit lower bound for the number of nonisomorphic simple resolvable $A(d,n,q)$ designs when $q \geq 3$ and $2 \leq d \leq n-2$.

Lemma 3.2 *Suppose that $q \geq 3$ and $2 \leq d \leq n-2$. Then the number of nonisomorphic simple resolvable $A(d,n,q)$ designs is at least*

$$\frac{((q^{n-1})!)^q\left(\left(\begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_q\right)!\right)^{q-1}\left((q^{n-d})!\right)^{(q-2)\begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_q}}{|A\Gamma L(n-1,q)|^{2q}}.$$

Proof Using (2) and arguing as in the previous lemma, we obtain first

$$NSrA(d,n,q) \;\geq\; \frac{\left((q^{n-1})!\right)^{2q}}{|A\Gamma L(n-1,q)|^{2q}}\left(\left(\begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_q\right)!\right)^{q-1}\left[\frac{\left((q^{n-d})!\right)^{q-1}}{q^{n-d}}\right]^{\begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_q},$$

and then

$$NSrA(d,n,q) \;\geq\; \frac{(q^n)!((q^{n-1})!)^q}{|A\Gamma L(n-1,q)|^{2q}}\left(\left(\begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_q\right)!\right)^{q-1}\left[\left((q^{n-d})!\right)^{q-2}\right]^{\begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_q}.$$

Dividing by $(q^n)!$ gives the required result. $\square$

Again, despite the crudeness of the estimates, when $q \neq 2$ and $d \neq n-1$ the bound is better than that given in [1], namely

$$\frac{\left((q^{n-d})!\right)^{\begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_q}\left(\begin{bmatrix} n-1 \\ d-1 \end{bmatrix}_q\right)!}{q!\,(q^{n-1}+q^{n-2}+\ldots+q+1)\,|A\Gamma L(n-1,q)|^q}.$$

Finally in this section we give two specific examples which illustrate the power of applying inequalities (1) and (2) recursively. Both examples relate to $A(3,5,2)$ designs and follow on from related examples in [3]. They provide direct comparisons with Examples 2.5 and 3.5 of [1].

Example 3.1 (The construction of $SA(3,5,2)$ designs)

By using inequality (1) and noting that $\left[{4 \atop 2}\right]_2 = 35$, we have

$$NSA(3,5,2) \geq (NSA(3,4,2))^2 \times NS^*RA(2,4,2) \times NSA(2,4,2) \times 140!/(4!)^{35}.$$

By reapplying inequality (1) and noting that $\left[{3 \atop 2}\right]_2 = 7$, we obtain

$$NSA(3,4,2) \geq NS^*RA(2,3,2) \times NSA(2,3,2) \times 14!/(2!)^7,$$

and noting that $\left[{3 \atop 1}\right]_2 = 7$, we also have

$$NSA(2,4,2) \geq (NSA(2,3,2))^2 \times NS^*RA(1,3,2) \times NSA(1,3,2) \times 28!/(4!)^7.$$

It was shown in [3, Example 2.1] that $NS^*RA(2,4,2) \geq \binom{N^*}{7}$, where $N^* \geq 16!/(5^2(4!)^5) = 105105$.

Up to isomorphism, there are four $2-(8,4,3)$ designs, all of which are simple, but only one of which is resolvable [2]. The resolvable design is $AG_2(3,2)$ with a unique resolution and full automorphism group $A\Gamma L(3,2)$ of order 1344. Hence there are $8!/1344 = 30$ distinct $S^*RA(2,3,2)$ designs. The other three nonisomorphic $2-(8,4,3)$ designs have automorphism group orders 48, 21 and 12. Consequently there are $8!(\frac{1}{1344} + \frac{1}{48} + \frac{1}{21} + \frac{1}{12}) = 6150$ distinct $SA(2,3,2)$ designs.

An $SA(1,3,2)$ design is a $2-(8,2,1)$ design which is necessarily simple and comprises all pairs from a set of eight points. Consequently $NSA(1,3,2) = 1$. A resolution of such a design is equivalent to a 1-factorization of the complete graph K_8. Since there are 6240 distinct 1-factorizations of K_8 [2], we have $NS^*RA(1,3,2) = 6240$.

By combining all the preceding inequalities and values we obtain

$$\begin{aligned} NSA(3,5,2) \quad \geq \quad & \left[30 \times 6150 \times 14!/2^7\right]^2 \times \binom{105105}{7} \\ & \times \left[6150^2 \times 6240 \times 28!/24^7\right] \times 140!/24^{35}. \end{aligned}$$

Dividing this by 32!, the largest possible size of an isomorphism class, gives the number of nonisomorphic simple $A(3,5,2)$ designs as at least 1.755405×10^{248}. This compares with the previous best estimate of 10^{180} given in [1]. □

Example 3.2 (The construction of $SrA(3,5,2)$ designs)

In this case we use inequality (2) which gives the following bounds:

$$NSrA(3,5,2) \geq (NSrA(3,4,2))^2 \times NS^*RA(2,4,2) \times NSRA(2,4,2) \times 35!,$$

and

$$NSrA(3,4,2) \geq NS^*RA(2,3,2) \times NSRA(2,3,2) \times 7!$$

As noted in the previous example, $NS^*RA(2,4,2) \geq \binom{105105}{7}$, while from [3] we have $NSRA(2,4,2) \geq (7!)^2 \times 6240^2 \times 30^2 \times 24^7$. And again as above $NS^*RA(2,3,2) = NSRA(2,3,2) = 30$.

By combining the preceding inequalities and values we obtain

$$\begin{aligned} NSrA(3,5,2) \quad &\geq \quad \left[30^2 \times 7!\right]^2 \times \binom{105105}{7} \\ &\qquad \times \left[(7!)^2 \times 6240^2 \times 30^2 \times 24^7\right] \times 35!. \end{aligned}$$

Dividing this by 32! gives the number of nonisomorphic simple resolvable $A(3,5,2)$ designs as at least 9.272669×10^{76}. This compares with the previous best estimate of 1.6×10^{27} given in [1]. □

Acknowledgement We thank the referees for some helpful suggestions that have improved the clarity of our exposition, particularly regarding the inclusion of Example 2.1 to illustrate the main construction.

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