All admissible 3-$(v, 4, \lambda)$ directed designs exist

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Abstract

In a t-(v, k, λ) *directed design* the blocks are ordered k-tuples and every ordered t-tuple of distinct points occurs in exactly λ blocks (as a subsequence). We show that a simple 3-$(v, 4, 2)$ directed design exists for all v. This completes the proof that the necessary condition $\lambda v \equiv 0 \pmod 2$ for the existence of a 3-$(v, 4, \lambda)$ directed design is sufficient.

1 Introduction

A t-(v, k, λ) *directed design* is a pair $(\mathcal{P}, \mathcal{B})$ where $\mathcal{P}$ is a set of v elements, called *points*, and $\mathcal{B}$ is a collection of ordered k-tuples of distinct elements of $\mathcal{P}$, called *blocks*, with the property that every ordered t-tuple of distinct elements of $\mathcal{P}$ occurs in exactly λ blocks (as a subsequence). A t-(v, k, λ) directed design with no repeated blocks is called *simple*. A t-$(v, k, 1)$ directed design is necessarily simple. Background information on directed designs is given in [2] and [3].

We usually specify a directed design by listing its blocks. For example, the following blocks form a 3-$(4, 4, 1)$ directed design:

$$(1, 2, 3, 4),\ (2, 1, 4, 3),\ (3, 1, 4, 2),\ (4, 2, 3, 1),\ (3, 2, 4, 1),\ (4, 1, 3, 2).$$

Here, for example, the block $(1, 2, 3, 4)$ contains the ordered triples $(1, 2, 3)$, $(1, 2, 4)$, $(1, 3, 4)$ and $(2, 3, 4)$.

A t-(v, k, λ) directed design is *cyclic* if it has an automorphism which permutes its points in a cycle of length v. The base blocks below, developed modulo 6, form a cyclic 3-$(6,4,1)$ directed design. This design is given by Soltankhah [13].

$$(0,1,3,5),\ (0,4,2,1),\ (0,3,1,2),\ (0,5,1,4),\ (0,5,2,3).$$

The following result (which is straightforward to prove) gives necessary conditions for the existence of a t-(v,k,λ) directed design.

Result 1.1 *Let $\mathcal{D}$ be a t-(v,k,λ) directed design. Then $\mathcal{D}$ is an s-(v,k,λ_s) directed design for $0 \le s < t$ where*

$$\lambda_s = \lambda \frac{\binom{v-s}{t-s} t!}{\binom{k-s}{t-s} s!}.$$

Hence λ_s must be an integer for $s = 0,1,2,\ldots,t-1$.

2-(v,k,λ) directed designs have been studied quite extensively. For such designs, the necessary conditions of Result 1.1 reduce to $2\lambda v(v-1) \equiv 0 \pmod{k(k-1)}$ and $2\lambda(v-1) \equiv 0 \pmod{k-1}$. It has been shown [1, 7, 12, 15, 16] that for $k \in \{3,4,5,6\}$ these necessary conditions are sufficient, with two exceptions, namely that no directed designs with parameters 2-$(15,5,1)$ or 2-$(21,6,1)$ exist.

In this paper, we are concerned with 3-$(v,4,\lambda)$ directed designs. For these, the necessary conditions of Result 1.1 reduce to the condition $\lambda v \equiv 0 \pmod 2$. It has been shown, by Soltankhah [13] building on work of Levenshtein [9], that this necessary condition is sufficient for all values of v, except possibly $v \equiv 3$ and $11 \pmod{12}$.

Both Levenshtein and Soltankhah make use of the following result involving t-(v,K,λ) designs. A t-(v,K,λ) *design* is a pair $(\mathcal{P},\mathcal{B})$ where $\mathcal{P}$ is a set of v elements, called *points*, and $\mathcal{B}$ is a collection of subsets of $\mathcal{P}$, called *blocks*, with the property that the size of every block is in the set K and every t-element subset of $\mathcal{P}$ is contained in exactly λ blocks. A t-(v,K,λ) *design* with no repeated blocks is called *simple*.

Result 1.2 **(Replacement Lemma)** *If there exist a t-(v,K,λ_1) design and a t-(k',k,λ_2) directed design for each $k' \in K$, then there exists a t-$(v,k,\lambda_1\lambda_2)$ directed design. A sufficient condition for the resulting directed design to be simple is that all original designs be simple and either $K = \{k\}$ or $\lambda_1 = 1$.*

Proof Replacing each block of the t-(v,K,λ_1) design with a copy of a directed t-(k',k,λ_2) design with point set the points of that block gives a t-$(v,k,\lambda_1\lambda_2)$ directed design. The claim about simplicity is clear. □

Levenshtein's contribution to the result we mentioned earlier was to prove, using the replacement lemma, that a 3-$(v, 4, 1)$ directed design exists for all even v. His proof is essentially as follows. Hanani [4, 5] has shown that there exists a 3-$(v, 4, 1)$ design for $v \equiv 2$ or 4 (mod 6), and a 3-$(v, \{4, 6\}, 1)$ design for $v \equiv 0$ (mod 6). Hence, provided that there exist a 3-$(4, 4, 1)$ directed design and a 3-$(6, 4, 1)$ directed design, it follows using the replacement lemma that a 3-$(v, 4, 1)$ directed design exists for all even v. These two small designs do indeed exist: we gave them as examples earlier.

In a similar way, Soltankhah [13] uses the replacement lemma to deduce the existence of simple 3-$(v, 4, 2)$ directed designs for $v \equiv 1$ or 5 (mod 12) from the existence of simple 3-$(v, 4, 2)$ designs for these values of v. Except for the case $v = 13$, the existence of these latter designs follows from Theorem 1 of Khosrovshahi and Ajoodani-Namini [8]. The argument relies on the existence of a *large set* of mutually disjoint 2-$(u, 3, 1)$ designs; these exist for $u \equiv 1$ or 3 (mod 6), $u \neq 7$ [10, 11, 17]. The missing simple 3-$(13, 4, 2)$ design, corresponding to $u = 7$, appears in Hanani [5].

Soltankhah [13] also uses the replacement lemma to show that there exists a simple 3-$(v, 4, 2)$ directed design for all even v. In addition, she proves, using more complicated methods, that a simple 3-$(v, 4, 2)$ directed design exists for $v \equiv 7$ or 9 (mod 12).

Since λ_1 copies of a 3-$(v, 4, \lambda)$ directed design form a 3-$(v, 4, \lambda_1\lambda)$ directed design, these results imply the result we mentioned earlier; that is, there exists a 3-$(v, 4, \lambda)$ directed design for all v and λ satisfying the necessary condition $\lambda v \equiv 0$ (mod 2), except possibly in the cases $v \equiv 3$ or 11 (mod 12).

In the next section we deal with the two remaining cases.

2 Main Theorem

In this section we complete the proof of the following theorem.

Theorem 2.1 *There exists a simple* 3-$(v, 4, 2)$ *directed design for all* v.

This theorem, together with Levenshtein's theorem stating that a 3-$(v, 4, 1)$ directed design exists for all even v, immediately gives the following result.

Theorem 2.2 *There exists a* 3-$(v, 4, \lambda)$ *directed design for all* v *and* λ *satisfying the necessary condition* $\lambda v \equiv 0$ (mod 2).

Our method, which was suggested by Soltankhah [14], is to use the replacement lemma to deduce Theorem 2.1 from the following theorem of Hanani [6].

Result 2.3 *There exists a* 3-$(v, \{4,5,6,7,9,11,13,15,19,23,27,29,31\}, 1)$ *design for all* v.

Thus we need to show that a simple 3-$(v, 4, 2)$ directed design exists for all values of v in the set $\{4,5,6,7,9,11,13,15,19,23,27,29,31\}$. All these values except $v = 11$, 15, 23 and 27 are covered by the results of Soltankhah [13] that we mentioned earlier. We now exhibit a simple 3-$(v, 4, 2)$ directed design for each of the four remaining values of v.

Developing the 9 base blocks below using the automorphism group $\{z \mapsto a^2z + b : a, b \in Z_{11}, a \neq 0\}$ yields a simple 3-(11, 4, 2) directed design.

$$(0,1,2,4),\quad (0,1,2,5),\quad (0,1,3,5),\quad (0,1,6,9),\quad (0,1,7,8),$$
$$(0,1,7,10),\quad (0,1,8,10),\quad (1,0,7,4),\quad (1,0,8,3).$$

Developing the 91 base blocks below modulo 15 yields a simple 3-(15, 4, 2) directed design.

$$(0,1,2,3),\quad (0,1,3,4),\quad (0,1,8,7),\quad (0,1,8,10),\quad (0,1,9,14),$$
$$(0,1,10,11),\quad (0,1,11,12),\quad (0,1,12,13),\quad (0,1,13,14),\quad (0,2,1,4),$$
$$(0,2,4,1),\quad (0,2,5,6),\quad (0,2,5,7),\quad (0,2,8,11),\quad (0,2,9,12),$$
$$(0,2,10,12),\quad (0,3,1,5),\quad (0,3,1,6),\quad (0,3,2,6),\quad (0,3,2,10),$$
$$(0,3,7,9),\quad (0,3,8,12),\quad (0,3,10,13),\quad (0,3,11,8),\quad (0,3,12,11),$$
$$(0,4,3,14),\quad (0,4,7,2),\quad (0,4,7,3),\quad (0,4,8,13),\quad (0,4,10,5),$$
$$(0,4,12,5),\quad (0,4,13,1),\quad (0,4,14,2),\quad (0,5,2,8),\quad (0,5,4,6),$$
$$(0,5,4,8),\quad (0,5,9,2),\quad (0,5,10,3),\quad (0,5,11,14),\quad (0,5,12,1),$$
$$(0,5,12,7),\quad (0,5,13,1),\quad (0,5,14,11),\quad (0,6,2,7),\quad (0,6,3,7),$$
$$(0,6,4,11),\quad (0,6,10,2),\quad (0,6,11,1),\quad (0,6,12,4),\quad (0,6,12,14),$$
$$(0,6,14,13),\quad (0,7,1,5),\quad (0,7,6,8),\quad (0,7,8,1),\quad (0,7,14,11),$$
$$(0,7,14,12),\quad (0,8,2,9),\quad (0,8,3,9),\quad (0,8,5,3),\quad (0,9,1,6),$$
$$(0,9,4,10),\quad (0,9,8,14),\quad (0,9,11,7),\quad (0,9,13,3),\quad (0,10,6,5),$$
$$(0,10,7,3),\quad (0,10,8,2),\quad (0,10,9,4),\quad (0,10,12,6),\quad (0,11,2,14),$$
$$(0,11,6,5),\quad (0,11,6,10),\quad (0,11,7,13),\quad (0,11,8,4),\quad (0,11,9,3),$$
$$(0,11,9,5),\quad (0,12,7,4),\quad (0,12,9,2),\quad (0,12,11,3),\quad (0,12,14,10),$$
$$(0,13,8,6),\quad (0,13,8,12),\quad (0,13,9,11),\quad (0,13,10,4),\quad (0,13,11,4),$$
$$(0,13,12,2),\quad (0,13,14,5),\quad (0,14,7,13),\quad (0,14,8,4),\quad (0,14,8,6),$$
$$(0,14,12,9).$$

Developing the 21 base blocks below using the automorphism group $\{z \mapsto a^2z + b : a, b \in Z_{23}, a \neq 0\}$ yields a simple 3-(23, 4, 2) directed design.

$$(0,1,2,3),\quad (0,1,3,4),\quad (0,1,4,5),\quad (0,1,5,6),\quad (0,1,6,10),$$
$$(0,1,8,11),\quad (0,1,8,14),\quad (0,1,11,19),\quad (0,1,12,22),\quad (0,1,14,12),$$
$$(0,1,17,15),\quad (0,1,20,18),\quad (1,0,3,14),\quad (1,0,6,4),\quad (1,0,11,3),$$
$$(1,0,15,8),\quad (1,0,17,4),\quad (1,0,19,6),\quad (1,0,20,5),\quad (1,0,21,22),$$
$$(1,0,22,18).$$

Developing the 25 base blocks below using the automorphism group $\{z \mapsto a^2z + b : a, b \in \mathrm{GF}(27), a \neq 0\}$ yields a simple 3-$(27, 4, 2)$ directed design. Here $p + qx + rx^2$ with $p, q, r \in \mathrm{GF}(3)$ is represented as $p + 3q + 9r$. The irreducible polynomial used is $x^3 - x - 1$.

$$
\begin{array}{ccccc}
(0,1,2,3), & (0,1,2,4), & (0,1,4,5), & (0,1,6,9), & (0,1,6,16), \\
(0,1,7,12), & (0,1,7,13), & (0,1,8,13), & (0,1,8,25), & (0,1,10,17), \\
(0,1,10,25), & (0,1,11,15), & (0,1,11,24), & (0,1,15,20), & (0,1,17,20), \\
(0,1,23,24), & (0,1,26,21), & (1,0,7,23), & (1,0,8,15), & (1,0,8,25), \\
(1,0,10,20), & (1,0,12,17), & (1,0,14,25), & (1,0,20,21), & (1,0,24,23).
\end{array}
$$

This completes the proof of Theorem 2.1.

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