On colourings of Steiner triple systems

A.D. Forbes, M.J. Grannell and T.S. Griggs
Department of Pure Mathematics
The Open University
Walton Hall
Milton Keynes MK7 6AA
UNITED KINGDOM
tonyforbes@ltkz.demon.co.uk
m.j.grannell@open.ac.uk
t.s.griggs@open.ac.uk

This is a preprint of an article accepted for publication in Discrete Mathematics ©2002 (copyright owner as specified in the journal).

Dedicated to Professor Alex Rosa
on the occasion of his 65th birthday.

Abstract

A Steiner triple system, STS(v), is said to be χ-*chromatic* if the points can be coloured using χ colours, but no fewer, such that no block is monochromatic. All known 3-chromatic STS(v) are also *equitably colourable*, i.e. there exists a 3-colouring in which the cardinalities of the colour classes differ by at most one. We present examples of 3-chromatic STS(v) which do not admit equitable 3-colourings. We also present further examples of systems with unique and balanced colourings.

AMS classification: 05B07
Keywords: Steiner triple system; Chromatic number; Colouring.

1 Introduction

A particular interest of Alex Rosa is colourings of block designs. His work in this area is seminal, [14], [15] and his research on this topic is probably the best in the literature. A survey paper, co-authored with C.J. Colbourn [17], appeared in a collection of surveys in Design Theory edited by J.H. Dinitz and D.R. Stinson in 1992. The recent book, "Triple systems" [6], also devotes two chapters specifically to this topic. Therefore it is a pleasure and probably appropriate to be able to offer this small contribution relating to what is one of the most fundamental concepts in this area; the chromatic number of a Steiner triple system. First we recall some of the basic definitions and results.

A *Steiner triple system* of order v, briefly STS(v), is a pair (V, B) where V is a set of cardinality v whose elements are called *points*, and B is a collection of 3-element subsets of V, called *blocks* or *triples*, such that every 2-element subset of V appears in precisely one triple. Steiner triple systems exist if and only if $v \equiv 1$ or 3 (mod 6). Such values of v are called *admissible.* Let $S = (V, B)$ be an STS(v). A (weak) *colouring* of S is a mapping $\phi : V \to C$, where C is a set of cardinality m whose elements are called *colours*, such that $\mid \phi(T) \mid > 1$ for all $T \in B$, i.e. no triple is monochromatic. We say that S is *m-colourable.* The *chromatic number* of S, $\chi(S)$ is the smallest value of m for which S admits a colouring with m colours. We say that the Steiner triple system S is *χ-chromatic.*

An easy counting argument establishes that only the unique, trivial STS(3) is 2-chromatic. Steiner triple systems which are 3-chromatic exist for all $v \equiv$ 1 or 3 (mod 6), $v \geq 7$, (use the Bose and Skolem constructions; see also Theorem 3 on page 166 of [15] and the final section of this paper). The numbers of non-isomorphic Steiner triple systems of orders 7, 9, 13 and 15 are known; there are precisely 1, 1, 2 and 80 respectively and all are 3-chromatic, [12]. But more generally it is known that for all $m \geq 3$, there exists v_m such that for all admissible $v > v_m$, there exists an m-chromatic STS(v), see Theorem 18.6 on page 327 of [6]. Currently it is known that $v_4 = 15$ or 19, i.e. the only value of v for which the existence of a 4-chromatic STS(v) is in doubt is $v = 19$, [2], [10], and $v_5 \leq 127$, [8].

In this paper we will mainly, but not exclusively, be interested in 3-chromatic STS(v) and in particular the distribution of the colour classes. More generally we define an STS(v) to be *equitably m-chromatic* if it is m-chromatic and the cardinalities of the colour classes differ by at most one. This concept was introduced by A. Rosa in [16]. Although it is known that for every $m \geq 6$, there exists an m-chromatic STS(v) which does not admit an equitable m-colouring [11], the cases $m = 3$, 4 and 5 are in doubt. Indeed work point 18.9.4 on page 343 of [6] specifically asks whether every 3-chromatic STS(v) admits an equitable 3-colouring. An earlier suggestion on page 330 of [6] is that they do. However, we are able to answer this question in the negative. Below we present 3-chromatic STS(v) which do not admit equitable 3-colourings for $v = 25$, 27, 31, 33, 37 and 39. This is probably the major result of this paper but we also make contributions to knowledge about unique colourings and balanced colourings of Steiner triple systems (for precise definitions of these terms see later).

2 Counting lemmas and small cases

Let $S = (V, B)$ be a 3-chromatic STS(v). Both in this section and throughout the remainder of the paper we will denote the colour classes by $R(=$ red), $Y(=$ yellow) and $B(=$ blue) where $\mid R \mid = c_1 \geq \mid Y \mid = c_2 \geq \mid B \mid = c_3$. The following lemma is due to L. Haddad and V. Rődl [11].

Lemma 2.1 $v = c_1 + c_2 + c_3 \geq ((c_1 - c_2)^2 + (c_2 - c_3)^2 + (c_3 - c_1)^2)/2$.
Proof. See the original paper and also page 331 of [6]. □

Although the next two lemmas can be deduced from Lemma 2.1, it is easier to prove them directly. The parameter r is the usual *replication number* (= the number of blocks of a Steiner triple system in which each point appears). Trivially, $r = (v - 1)/2$.

Lemma 2.2 $c_1 \leq r + 1$.
Proof. Let $x \in R$. Then x occurs with the remaining $c_1 - 1$ points of the colour class R in different blocks. Hence $c_1 - 1 \leq r$. □

Lemma 2.3 *If* $v \neq 7$ *then* $c_1 \leq r$.
Proof. From Lemma 2.2, $c_1 \leq r+1$. Now suppose that $c_1 = r+1 = (v+1)/2$. Every pair of distinct points $x, y \in R$ appear in a different block and hence there are $(r + 1)r/2 = (v + 1)(v - 1)/8$ such blocks. There are $(v - 1)/2$ points which are not contained in the colour class R. Hence the number of

pairs $x, l, x \in R, l \notin R$ is $(v+1)(v-1)/4$ and these must all appear in the blocks described above. The remaining blocks of the STS(v) must therefore comprise a subsystem STS($(v-1)/2$) which moreover is 2-chromatic. Thus $(v-1)/2 = 3$ i.e. $v = 7$. □

We will use these lemmas extensively throughout this paper to determine possible colouring patterns for various values of v. In this section our interest will be mainly in the case $v = 15$, but for completeness we first give the results for $v = 7$, 9 and 13.

$v = 7$

Let the unique STS(7) be given by $V = \{0, 1, 2, 3, 4, 5, 6\}$ and
$B = \{\{0, 1, 3\}, \{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{4, 5, 0\}, \{5, 6, 1\}, \{6, 0, 2\}\}$.
The possible colouring patterns are $(c_1, c_2, c_3) = (4, 2, 1), (3, 3, 1)$ or $(3, 2, 2)$ and all are attainable as follows.

1. (c_1, c_2, c_3)=(4,2,1); $R = \{0, 1, 2, 5\}, Y = \{3, 4\}, B = \{6\}$.
2. (c_1, c_2, c_3)=(3,3,1); $R = \{0, 1, 2\}, Y = \{3, 4, 5\}, B = \{6\}$.
3. (c_1, c_2, c_3)=(3,2,2); $R = \{0, 1, 2\}, Y = \{3, 4\}, B = \{5, 6\}$.

$v = 9$

Let the unique STS(9) be given by $V = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ and
$B = \{\{0, 1, 2\}, \{3, 4, 5\}, \{6, 7, 8\}, \{0, 3, 6\}, \{1, 4, 7\}, \{2, 5, 8\},$
$\{0, 4, 8\}, \{1, 5, 6\}, \{2, 3, 7\}, \{0, 5, 7\}, \{1, 3, 8\}, \{2, 4, 6\}\}$.
The possible colouring patterns are $(c_1, c_2, c_3) = (4, 4, 1), (4, 3, 2)$ or $(3, 3, 3)$ and all are attainable as follows.

1. (c_1, c_2, c_3)=(4,4,1); $R = \{0, 1, 3, 4\}, Y = \{2, 5, 6, 7\}, B = \{8\}$.
2. (c_1, c_2, c_3)=(4,3,2); $R = \{0, 1, 3, 4\}, Y = \{2, 5, 6\}, B = \{7, 8\}$.
3. (c_1, c_2, c_3)=(3,3,3); $R = \{0, 1, 3\}, Y = \{2, 4, 7\}, B = \{5, 6, 8\}$.

$v = 13$

One of the STS(13)s is given by $V = \{0,1,2,3,4,5,6,7,8,9,10,11,12\}$ and B is the collection of 26 blocks obtained by the action of the cyclic group generated by the mapping $i \mapsto i+1 \pmod{13}$ on the two base blocks $\{0,1,4\}$ and $\{0,2,7\}$. The second STS(13) is formed by replacing the blocks $\{0,1,4\},\{0,2,7\},\{2,4,9\},\{7,9,1\}$ in the above system by the blocks $\{2,7,9\},\{1,4,9\},\{0,1,7\},\{0,2,4\}$.
The possible colouring patterns are $(c_1,c_2,c_3) = (6,5,2),(6,4,3),(5,5,3)$ or $(5,4,4)$ and all are attainable for both systems as follows.

1. (c_1,c_2,c_3)=(6,5,2); $R = \{1,2,4,6,7,8\}, Y = \{5,9,10,11,12\}, B = \{0,3\}$.
2. (c_1,c_2,c_3)=(6,4,3); $R = \{1,2,4,6,7,8\}, Y = \{9,10,11,12\}, B = \{0,3,5\}$.
3. (c_1,c_2,c_3)=(5,5,3); $R = \{0,1,2,3,9\}, Y = \{4,5,7,10,11\}, B = \{6,8,12\}$.
4. (c_1,c_2,c_3)=(5,4,4); $R = \{0,1,2,3,9\}, Y = \{4,5,7,10\}, B = \{6,8,11,12\}$.

$v = 15$

The possible colouring patterns are $(c_1,c_2,c_3) = (7,5,3),(7,4,4),(6,6,3),$ $(6,5,4)$ or $(5,5,5)$. The case of the STS(15) formed by the point-line design of the projective geometry PG(3,2), (♯1 in the listing of [12]), was considered by J. Pelikán [13]. Every 3-colouring of this system is equitable, i.e. it is 3-chromatic but only with colour classes of cardinality $(c_1,c_2,c_3) = (5,5,5)$. Moreover the 3-colouring is unique up to isomorphism, [8]. Beyond this there seems to have been no systematic investigation of the colouring patterns attainable in 3-colourings of the other STS(15)s. We have determined that 76 of the 80 STS(15)s can be 3-coloured with any of the 5 colouring patterns listed above. But system ♯7 is 3-chromatic only with colour classes $(c_1,c_2,c_3) = (6,5,4)$ or $(5,5,5)$ and systems ♯79 and ♯80 are 3-chromatic only with colour classes $(c_1,c_2,c_3) = (6,6,3),(6,5,4)$ or $(5,5,5)$.

3 Non-equitable 3-chromatic STS(v)

Below we present in a compact notation details of 3-chromatic STS(v) which do not admit equitable 3-colourings for v = 25, 27, 31, 33, 37 and 39. The same notation is used in [5] and [9]. The base set V = {`0,1,2,...,v-1`} and the blocks are represented by a string of symbols $s_1, s_2, \ldots, s_b$ where $b = v(v-1)/6 = \mid B \mid$. Using the usual lexicographical order, the symbol s_i is the largest element z_i in the i^{th} triple $\{x_i, y_i, z_i\}$ where $x_i < y_i < z_i$. The remaining two elements implicitly have the property that there is no pair $x'_i < y'_i$ such that $\{x'_i, y'_i\}$ does not appear in an earlier triple and either (i) $x'_i < x_i$, or (ii) $x'_i = x_i$ and $y'_i < y_i$. The integers `10, 11, 12,..., 35` will be represented by the lower case letters `a, b, c,..., z` respectively and the integers `36, 37, 38` by the upper case letters `A, B, C` respectively. All systems were constructed by the well-known technique of hill-climbing [19].

STS(25)

```
l49mijkbfo hn9abgjhin kfobakilhn gomnocdeij kmldecomjk
iecdbkjnnm fglooglfmh fgnnkjmljh iooijmnelm lmlohnonko
```

The 3-colouring is unique, modulo swapping colour classes R and Y, with pattern $(c_1, c_2, c_3) = (9, 9, 7)$ and is given by $R = \{\texttt{0,1,2,a,h,i,j,n,o}\}$,
$Y = \{\texttt{3,4,5,b,c,e,f,g,m}\}$, $B = \{\texttt{6,7,8,9,d,k,l}\}$.
The number of blocks of each colour type is $RRY = 18$, $RRB = 18$, $RYY = 19$, $RBB = 10$, $YYB = 17$, $YBB = 11$, $RYB = 7$.

STS(27) ♯1

```
n9pmd8blej oiqco6ga9f qjplm6ko8d ignjpqfbkd ighqnmpngh
ieqjoh9iqj mlnobcnlmp qkfgilqpjm ofqpnekpon mqkpnonohl
qlmpkimopq kjlonpq
```

The 3-colouring is unique with pattern $(c_1, c_2, c_3) = (10, 9, 8)$ and is given by $R = \{\texttt{4,8,9,b,f,i,j,k,n,q}\}$, $Y = \{\texttt{3,5,7,a,c,d,e,o,p}\}$,
$B = \{\texttt{0,1,2,6,g,h,l,m}\}$.
The number of blocks of each colour type is $RRY = 22$, $RRB = 23$, $RYY = 19$, $RBB = 13$, $YYB = 17$, $YBB = 15$, $RYB = 8$.

STS(27) ♯2

```
k56bedjclo mpq4mal8ij ogqnpiefpm nkhljqdhoc pfenql7neo
qkglmpqlbg kopjnbmajp okikjdiqnp qbihonnfhl hglpoiohmq
ppqmnqmmkl kqnooqp
```

The 3-colouring is unique with pattern $(c_1, c_2, c_3) = (10, 9, 8)$ and is given by $R = \{\texttt{0,1,2,8,b,e,g,i,l,n}\}$, $Y = \{\texttt{5,9,a,c,d,j,o,p,q}\}$, $B = \{\texttt{3,4,6,7,f,h,k,m}\}$.
The number of blocks of each colour type is $RRY = 24$, $RRB = 21$, $RYY = 17$, $RBB = 15$, $YYB = 19$, $YBB = 13$, $RYB = 8$.
The fact that both systems $\sharp 1$ and $\sharp 2$ are uniquely 3-colourable but have different statistics of block colour types shows that they are non-isomorphic.

STS(31)

```
tlkcmrbniu sojpqoahfd mpljgksurj 7kfmcqeuts prqtprdsmu
gniobeiamp ntlsucsled iunqrijuls kmotgohjtu kqpqkfhtus
drmjntcrgj stphlrqtur pqktosjlon qphnsqoutq ruoptrusup rtuns
```

The 3-colouring is unique, modulo swapping colour classes R and Y, with pattern $(c_1, c_2, c_3) = (11, 11, 9)$ and is given by $R = \{\texttt{0,3,7,8,9,a,e,f,l,m,p}\}$, $Y = \{\texttt{2,4,6,b,c,h,i,j,o,q,u}\}$, $B = \{\texttt{1,5,d,g,k,n,r,s,t}\}$.
The number of blocks of each colour type is $RRY = 26$, $RRB = 29$, $RYY = 30$, $RBB = 16$, $YYB = 25$, $YBB = 20$, $RYB = 9$.

STS(33) $\sharp$1

```
epd8rniqvk lsjowuogva sckltmuwqr phqlawemsu ijtvrk7ole
pjwrntsvue ohmfplnswt pmwnvkjtqs ovtjmqrwui lgksdhlprt
uudqfpjwsv rpdnotvcth wluoqvlruw gtmsropgnv qtwokquswp
mvvrspkuqw notutwsrvw vuvrwu
```

The 3-colouring is unique with pattern $(c_1, c_2, c_3) = (12, 11, 10)$ and is given by $R = \{\texttt{4,6,a,b,d,h,k,p,q,s,u,v}\}$, $Y = \{\texttt{0,2,3,7,8,9,c,j,o,r,t}\}$, $B = \{\texttt{1,5,e,f,g,i,l,m,n,w}\}$.
The number of blocks of each colour type is $RRY = 35$, $RRB = 31$, $RYY = 26$, $RBB = 24$, $YYB = 29$, $YBB = 21$, $RYB = 10$.

STS(33) $\sharp$2

```
8qhaoclifj srntuwvj7u tcbkwrpims qms6rjgdln kwuotgfnts
pciqlvuwkh wjetumpnrv wghtdvnmls r9uwjmqios pvdosjfmnv
```

```
pubvlihrnt qrwtnovqup lsqkmuqtko wrvroukswp vpiltshruj
qvtqvworwv utrstqvwuw wsvpwu
```

The 3-colouring is unique with pattern $(c_1, c_2, c_3) = (12, 11, 10)$ and is given by $R = \{\texttt{4,7,a,b,d,g,j,k,l,r,v,w}\}$, $Y = \{\texttt{1,2,6,9,e,f,h,m,o,p,s}\}$, $B = \{\texttt{0,3,5,8,c,i,n,q,t,u}\}$.
The number of blocks of each colour type is $RRY = 33$, $RRB = 33$, $RYY = 28$, $RBB = 22$, $YYB = 27$, $YBB = 23$, $RYB = 10$.
The fact that both systems $\sharp 1$ and $\sharp 2$ are uniquely 3-colourable but have different statistics of block colour types shows that they are non-isomorphic.

STS(37)

```
isjekfcgmh nAozyxvwje wflnmhqcpk uzyxAkfcgm onbhqlrzyx
Algmhncudr otszyAmhdc injsovutzA nlidjeryxw vuAidjekAp
zyxwvjAkfg rqzyxwkflA tsrzyxlgvo utszymvrAx wtzAvtzyxw
uupvqwrxsy ztrqwxsytz owxsytzoux sytzAupytz upvAzoupvq
upqwvqwrAw rxAxsvyAAv wA
```

The 3-colouring is unique, modulo swapping colour classes R and Y, with pattern $(c_1, c_2, c_3) = (13, 13, 11)$ and is given by
$R = \{\texttt{0,2,c,e,f,g,h,i,j,k,m,n,A}\}$, $Y = \{\texttt{1,3,4,5,6,7,8,9,a,b,o,t,v}\}$, $B = \{\texttt{d,l,p,q,r,s,u,w,x,y,z}\}$.
The number of blocks of each colour type is $RRY = 35$, $RRB = 43$, $RYY = 44$, $RBB = 23$, $YYB = 34$, $YBB = 32$, $RYB = 11$.

STS(39)

```
kelfmgnhoi pjxwCBAzyl fmgnhoipjd rsCBAzymgn hoipjdksut
CBAznhqipj wkoetvuCBA oipjdkeluy xwvCBpjdke lfvAzyxwCd
kelfmwCBAz yxelfwgxyr CBAzfcnmyt sCBAzrhnuz vtCBAgomAx
vuCBitBzyx vCvACBwzyx wysztAuBvr CysztAuBxC wqztAuBvCr
qxqnBvCxwA BvCwxryCmq ysuxqxrysz rsztysztAq AuwuBwv
```

This system has two 3-colourings, both with pattern $(c_1, c_2, c_3) = (14, 13, 12)$.
The first of these is given by $R = \{\texttt{0,1,2,3,4,5,6,7,8,9,a,b,u,x}\}$, $Y = \{\texttt{c,d,e,f,g,h,j,k,l,m,n,o,p}\}$, $B = \{\texttt{i,q,r,s,t,v,w,y,z,A,B,C}\}$.
The number of blocks of each colour type is $RRY = 62$, $RRB = 29$,

$RYY = 23$, $RBB = 49$, $YBB = 55$, $YBB = 17$, $RYB = 12$.
The second colouring is obtained by moving the element `b` from the colour class R to the colour class Y (and re-naming R as Y and vice-versa). It is given by $R = \{$`b,c,d,e,f,g,h,j,k,l,m,n,o,p`$\}$,
$Y = \{$`0,1,2,3,4,5,6,7,8,9,a,u,x`$\}$, $B = \{$`i,q,r,s,t,v,w,y,z,A,B,C`$\}$.
The number of blocks of each colour type is $RRY = 33$, $RRB = 58$, $RYY = 52$, $RBB = 20$, $YYB = 26$, $YBB = 46$, $RYB = 12$.

4 The cases $v = 19$ and 21

The results of the previous two sections naturally raise the question of whether there exist 3-chromatic STS(v) which do not admit equitable 3-colourings for $v = 19$ and 21? In the former case we are able to answer the question in the negative, but cannot resolve completely the situation for the latter. We present our results next, beginning with three simple lemmas.

Lemma 4.1 *Suppose there exists a 3-chromatic STS(v) with colour classes R,Y,B as above of cardinalities* c_1, c_2, c_3 *respectively with* $c_1 \geq c_2 \geq c_3$*. If* $c_1 > \binom{c_3}{2}$ *then the STS(v) can be recoloured with colour classes of cardinalities* $c_1 - 1, c_2, c_3 + 1$*.*
Proof. There are exactly $\binom{c_3}{2}$ triples of the form $\{x, l, m\}, l \in B, m \in B$. Thus if $c_1 > \binom{c_3}{2}$ then for some $z \in R$, there is no triple of the form $\{z, l, m, \}, l \in B, m \in B$. The point z may then be recoloured blue. □

Lemma 4.2 *Suppose there exists a 3-chromatic STS(v) with colour classes R,Y,B as above of cardinalities* c_1, c_2, c_3 *respectively with* $c_1 \geq c_2 \geq c_3$*. If* $c_2 > \binom{c_3}{2}$ *then the STS(v) can be recoloured with colour classes of cardinalities* $c_1, c_2 - 1, c_3 + 1$*.*
Proof. As Lemma 4.1. □

Lemma 4.3 *Suppose there exists a 3-chromatic STS(v) with colour classes R,Y,B as above of cardinalities* c_1, c_2, c_3 *respectively with* $c_1 \geq c_2 \geq c_3$*. If* $c_1 + c_2 > \binom{c_3}{2}$ *then the STS(v) can be recoloured with colour classes of cardinalities* $c_1 - 1, c_2, c_3 + 1$ *or* $c_1, c_2 - 1, c_3 + 1$*.*

Proof. By the same argument as used in Lemma 4.1, there are exactly $\binom{c_3}{2}$ triples of the form $\{x, l, m\}, l \in B, m \in B$. Thus if $c_1 + c_2 > \binom{c_3}{2}$ then for some $z \in R \cup Y$, there is no triple of the form $\{z, l, m\}, l \in B, m \in B$. The point z may then be recoloured blue. $\square$

In the next two theorems we will use the notation $(c_1, c_2, c_3) \to (c_1', c_2', c_3')$ to denote that a 3-colouring of an STS(v) with colour classes of cardinalities c_1, c_2, c_3 can be transformed into a 3-colouring with colour classes of cardinalities c_1', c_2', c_3'.

Theorem 4.1 *Every 3-chromatic STS(19) is equitably 3-colourable.*
Proof. The possible colouring patterns, as determined by Lemma 2.1, are $(c_1, c_2, c_3) = (9, 6, 4)$ or $(9, 5, 5)$ or $(8, 7, 4)$ or $(8, 6, 5)$ or $(7, 7, 5)$ or $(7, 6, 6)$. We prove the theorem by showing that a 3-colouring with any of these colouring patterns can be transformed into a 3-colouring with $(c_1, c_2, c_3) = (7, 6, 6)$.

(i) Either $(9, 5, 5) \to (9, 6, 4)$ or $(9, 5, 5) \to (8, 6, 5)$ by Lemma 4.3.
(ii) $(9, 6, 4) \to (8, 6, 5)$ by Lemma 4.1.
(iii) $(8, 7, 4) \to (7, 7, 5)$ by Lemma 4.1.
(iv) $(7, 7, 5) \to (7, 6, 6)$ by Lemma 4.3.

(v) To complete the proof we show that either $(8, 6, 5) \to (7, 7, 5)$ or $(8, 6, 5) \to (7, 6, 6)$. Consider a 3-chromatic STS(19) with colour classes R(= red), Y(= yellow) and B(= blue) of cardinalities $c_1 = 8$, $c_2 = 6$, $c_3 = 5$ respectively. Denote the red points by $r_i, 1 \leq i \leq 8$, the yellow points by $y_i, 1 \leq i \leq 6$ and the blue points by $b_i, 1 \leq i \leq 5$.
There are $\binom{8}{2} = 28$ triples of the form $\{r_i, r_j, l\}, 1 \leq i < j \leq 8, l \notin R$, and hence $(8 \times 9) - (28 \times 2) = 16$ triples of the form $\{r_i, m, n\}, 1 \leq i \leq 8, m \notin R, n \notin R$. Moreover each point $r_i, 1 \leq i \leq 8$, appears in precisely two of these triples.
It will now be possible to recolour a red point as either yellow, giving $(c_1, c_2, c_3) = (7, 7, 5)$ or blue, giving $(c_1, c_2, c_3) = (7, 6, 6)$ unless for each $i : 1 \leq i \leq 8$ the two triples are of the form $\{r_i, y_j, y_{j'}\}, 1 \leq j < j' \leq 6$ and $\{r_i, b_k, b_{k'}\}, 1 \leq k < k' \leq 5$. But this would account for 8 pairs of yellow points and 8 pairs of blue points leaving $\binom{6}{2} - 8 = 7$ pairs of yellow points and $\binom{5}{2} - 8 = 2$ pairs of blue points, a total of 9 pairs in all, to appear in the remaining blocks.

But there are $57-(28+16)=13$ remaining blocks, none of which contains a red point, and these collectively must contain 13 pairs of either yellow points or blue points. So the situation described cannot arise and the theorem is proved. $\square$

Theorem 4.2 *Every 3-chromatic STS(21) has a 3-colouring in which the cardinalities of the colour classes are* $(c_1, c_2, c_3) = (8, 7, 6)$ *or* $(7, 7, 7)$.
Proof. The possible colouring patterns, as determined by Lemma 2.1, are $(c_1, c_2, c_3) = (10, 6, 5)$ or $(9, 8, 4)$ or $(9, 7, 5)$ or $(9, 6, 6)$ or $(8, 8, 5)$ or $(8, 7, 6)$ or $(7, 7, 7)$.

(i) We first show that either $(10, 6, 5) \rightarrow (9, 7, 5)$ or $(10, 6, 5) \rightarrow (9, 6, 6)$.
Consider a 3-chromatic STS(21) with colour classes $R(=$ red$)$, $Y(=$ yellow$)$ and $B(=$ blue$)$ of cardinalities $c_1 = 10, c_2 = 6, c_3 = 5$ respectively. Denote the red points by $r_i, 1 \leq i \leq 10$, the yellow points by $y_i, 1 \leq i \leq 6$ and the blue points by $b_i, 1 \leq i \leq 5$.
There are $\binom{10}{2} = 45$ triples of the form $\{r_i, r_j, l\}, 1 \leq i < j \leq 10, l \notin R$, and hence $(10 \times 10) - (45 \times 2) = 10$ triples of the form $\{r_i, m, n\}, 1 \leq i \leq 10$, $m \notin R, n \notin R$. Moreover each point $r_i, 1 \leq i \leq 10$, appears in precisely one of these triples.
There are $\binom{5}{2} = 10$ pairs of the form $\{b_j, b_k\}, 1 \leq j < k \leq 5$. We can recolour a red point as blue, giving $(c_1, c_2, c_3) = (9, 6, 6)$, unless all 10 triples which contain one red point are of the form $\{r_i, b_j, b_k\}, 1 \leq i \leq 10, 1 \leq j < k \leq 5$. But in this latter case all $\binom{6}{2} = 15$ triples of the form $\{y_i, y_j, l\}, 1 \leq i < j \leq 6, l \notin Y$ must be of the form $\{y_i, y_j, b_k\}, 1 \leq k \leq 5$, and hence we can recolour a red point as yellow giving $(c_1, c_2, c_3) = (9, 7, 5)$.

(ii) Either $(9, 7, 5) \rightarrow (9, 6, 6)$ or $(9, 7, 5) \rightarrow (8, 7, 6)$ by Lemma 4.3.
(iii) $(9, 8, 4) \rightarrow (8, 8, 5)$ by Lemma 4.1.
(iv) $(8, 8, 5) \rightarrow (8, 7, 6)$ by Lemma 4.3.

(v) To complete the proof we show that $(9, 6, 6) \rightarrow (8, 7, 6)$.
Consider a 3-chromatic STS(21) with colour classes $R(=$ red$)$, $Y(=$ yellow$)$ and $B(=$ blue$)$ of cardinalities $c_1 = 9, c_2 = 6, c_3 = 6$ respectively. Denote the red points by $r_i, 1 \leq i \leq 9$, the yellow points by $y_i, 1 \leq i \leq 6$ and the blue points by $b_i, 1 \leq i \leq 6$.
There are $\binom{9}{2} = 36$ triples of the form $\{r_i, r_j, l\}, 1 \leq i < j \leq 9, l \notin R$ and

hence $(9 \times 10) - (36 \times 2) = 18$ triples of the form $\{r_i, m, n\}, 1 \leq i \leq 9$, $m \notin R, n \notin R$. Moreover each point $r_i, 1 \leq i \leq 9$, appears in precisely two of these triples.
It will now be possible to recolour a red point as either yellow, giving $(c_1, c_2, c_3) = (8, 7, 6)$, or blue, and then recolouring all yellow points as blue and vice-versa, again giving $(c_1, c_2, c_3) = (8, 7, 6)$ unless for <u>each</u> $i : 1 \leq i \leq 9$, the two triples are of the form $\{r_i, y_j, y_{j'}\}, 1 \leq j < j' \leq 6$ and $\{r_i, b_k, b_{k'}\}, 1 \leq k < k' \leq 6$. But this would account for 9 pairs of yellow points and 9 pairs of blue points leaving $\binom{6}{2} - 9 = 6$ pairs of both yellow points and blue points, a total of 12 pairs in all, to appear in the remaining blocks.
But there are $70 - (36 + 18) = 16$ remaining blocks, none of which contains a red point and these collectively must contain 16 pairs of either yellow points or blue points. So the situation described above cannot arise and the theorem is proved. □

5 Unique and balanced colourings

In [17], A. Rosa and C.J. Colbourn introduce the concept of a *uniquely colourable* Steiner triple system. This is defined to be an m-chromatic STS(v), (V, B), in which any m-colouring of the system produces the same partition of the set V into colour classes. The authors of [2] state that in the course of searching for 4-chromatic Steiner triple systems of small orders, they “discovered a uniquely colourable 3-chromatic STS of order 33”. However they do not list the system although presumably it is the one given in Table IV on page 301 of [4]. It appears to be the only known example of a uniquely colourable STS(v). The examples given in Section 3 for $v = 25$, 27, 31, 33 and 37 are also uniquely colourable, as well as being non-equitably 3-chromatic. However we are able also to add to the spectrum of known uniquely colourable and equitably 3-chromatic STS(v), the values $v = 25$, 27, 31, 37, 49, 55 and 57 as well as a new system for the value $v = 33$. We list these systems next in the same compact notation as before with the extension that the integers `39, 40, 41,..., 56` are represented by the upper case letters `D, E, F,..., U` respectively.

STS(25) ♯1

```
96hfcgeolj mn38genioj kml5kaigfm lno7kfgmbe nolbohkenj
m89mifojnc hjolnmmfhi jodnohlkkf lgnglndkmi nhimjlokoo
```

The unique 3-colouring, modulo swapping colour classes Y and B, is given by $R = \{\texttt{0,3,4,6,e,g,j,k,o}\}$, $Y = \{\texttt{1,2,5,7,8,b,c,l}\}$,
$B = \{\texttt{9,a,d,f,h,i,m,n}\}$.
The number of blocks of each colour type is $RRY = 17$, $RRB = 19$, $RYY = 15$, $RBB = 13$, $YYB = 13$, $YBB = 15$, $RYB = 8$.

STS(25) ♯2

```
29omijklnf ghabljhico fmnblakijn mhgomnldeo ijkdhecajk
necobkijom fgdnngkmlh fgljkmoojn imkjhlelim kmononolno
```

The unique 3-colouring, modulo swapping colour classes Y and B, is given by $R = \{\texttt{1,5,a,b,d,e,g,h,k}\}$, $Y = \{\texttt{0,8,f,i,l,m,n,o}\}$,
$B = \{\texttt{2,3,4,6,7,9,c,j}\}$.
The number of blocks of each colour type is $RRY = 17$, $RRB = 19$, $RYY = 15$, $RBB = 13$, $YYB = 13$, $YBB = 15$, $RYB = 8$.
Although both systems ♯1 and ♯2 have the same statistics of block colour types, calculation of other invariants shows that they are non-isomorphic.

STS(27) ♯1

```
qpdnjgible mkoikhom8c dljpnjo7hc amqnlkah9g onpflqedpq
filmkimgqf nplqjpkfon mnjkeolkfg lipdqnjphi jomqpohgoi
nmokmhmqpq qpnqpqo
```

The unique 3-colouring, modulo swapping the three colour classes, is given by $R = \{\texttt{0,1,c,d,e,k,l,n,p}\}$, $Y = \{\texttt{2,3,8,9,b,f,g,i,q}\}$,
$B = \{\texttt{4,5,6,7,a,h,j,m,o}\}$.
The number of blocks of each colour type is $RRY = 18$, $RRB = 18$, $RYY = 18$, $RBB = 18$, $YYB = 18$, $YBB = 18$, $RYB = 9$.

STS(27) ♯2

```
fb89ijqelg nmpdeh7ocj kimnqkfcnm ioqgjpp9lc fohjnqmcdj
ilkoqagfoh pnqb9eihqp hnpjolklmn qpohegpqml gdmokqpnfi
qopmlkponn klqpmqo
```

The unique 3-colouring, modulo swapping the three colour classes, is given by $R = \{\mathtt{0,8,a,c,f,g,h,j,o}\}$, $Y = \{\mathtt{1,2,4,5,6,9,e,i,q}\}$, $B = \{\mathtt{3,7,b,d,k,l,m,n,p}\}$.
The number of blocks of each colour type is $RRY = 16$, $RRB = 20$, $RYY = 20$, $RBB = 16$, $YYB = 16$, $YBB = 20$, $RYB = 9$.
The fact that both systems ♯1 and ♯2 are uniquely 3-colourable but have different statistics of block colour types shows that they are non-isomorphic.

STS(31) ♯1

```
c48bsnlktq gurmp5lbkh oitpjmqsus hqnpetgork lupfdjochr
unmtglrmts doqinumesa lnpquotcqn frijtubumg kpotgpiuhj
trsfjkrqlq hurokfpion uuotsrqsml rtnmlouslo tsqststpqp surru
```

The unique 3-colouring, modulo swapping colour classes Y and B, is given by $R = \{\mathtt{1,6,7,d,j,l,m,n,q,r,t}\}$, $Y = \{\mathtt{0,3,5,c,e,g,i,k,p,s}\}$, $B = \{\mathtt{2,4,8,9,a,b,f,h,o,u}\}$.
The number of blocks of each colour type is $RRY = 29$, $RRB = 26$, $RYY = 21$, $RBB = 24$, $YYB = 24$, $YBB = 21$, $RYB = 10$.

STS(31) ♯2

```
36rkhejtpd lisuqn6cpq kihjgrout7 spcgumiqhr otamdpfkns
tluqrojfpq gemnlurthd jnuilsomik gptjusqdqm cinrsuoluk
jstnslmtue qogsrrtolr sjktupmntq ukonpouqso uprmtsuptn qrrts
```

The unique 3-colouring, modulo swapping colour classes Y and B, is given by $R = \{\mathtt{2,7,8,9,e,f,j,p,s,t,u}\}$, $Y = \{\mathtt{0,6,a,b,c,h,l,m,n,o}\}$, $B = \{\mathtt{1,3,4,5,d,g,i,k,q,r}\}$.
The number of blocks of each colour type is $RRY = 24$, $RRB = 31$, $RYY = 26$, $RBB = 19$, $YYB = 19$, $YBB = 26$, $RYB = 10$.

STS(31) ♯3

```
74cb8aopqr nlktu589cb hurinqmts6 9acsjmlipo ru7abjqlhk
rupt8bcmog ljutqs9cfs qportunaej otpmnsubqk ojrutpscpl
nsqtoruirt kmupqsgtkj mspruuhsmq tnornmhusp ortltinuts urqsq
```

The unique 3-colouring, modulo swapping colour classes Y and B, is given by $R = \{\texttt{2,4,6,9,a,e,g,h,p,s,u}\}$, $Y = \{\texttt{0,3,5,8,l,n,o,q,r,t}\}$, $B = \{\texttt{1,7,b,c,d,f,i,j,k,m}\}$.
The number of blocks of each colour type is $RRY = 31$, $RRB = 24$, $RYY = 19$, $RBB = 26$, $YYB = 26$, $YBB = 19$, $RYB = 10$.
This system contains an STS(13) on the set $\{\texttt{0,1,2,3,4,5,6,7,8,9,a,b,c}\}$ upon which this 3-colouring induces an equitable 3-colouring.
The fact that systems ♯1, ♯2 and ♯3 are uniquely 3-colourable but have different statistics of block colour types shows that they are non-isomorphic.

STS(33)

```
ed8bc9amwi lvqrtu3579 bdgtjrnsuq w689cekusq mopvwa7bde
rimtuwsqvc desqvkumwt readhvowts prucdlmnto wurvbethwr
mquvseulrn vstpwcipoq lnuwvjnkpv oswuorlsnp wtvptjlqrv
swwouvkpqt snkqrpuvwt vswuot
```

The unique 3-colouring, modulo swapping the three colour classes, is given by $R = \{\texttt{0,2,4,9,c,i,j,n,r,s,t}\}$, $Y = \{\texttt{1,5,6,8,d,h,k,l,m,o,v}\}$,
$B = \{\texttt{3,7,a,b,e,f,g,p,q,u,w}\}$.
The number of blocks of each colour type is $RRY = 26$, $RRB = 29$, $RYY = 29$, $RBB = 26$, $YYB = 26$, $YBB = 29$, $RYB = 11$.
This system contains an STS(15) on the set
$\{\texttt{0,1,2,3,4,5,6,7,8,9,a,b,c,d,e}\}$.

STS(37) ♯1

```
idjekflgmh nozyxwvAje kflgmhncpq zyxwAkflqm hnrAwpiszy
xlgmhncidr utszyAmhnc wdtxrszvuA ncidjetyxw vuAidjekAs
pywvzjekfA rqzyxwkflA tsrzyxlgAz vutsymuAxw vtzxouyzwv
jhvqxsytzr vqwruytzox wrsytzouxt syxzoupyzo upvAoupvqz
tpvqwvqwrA wrxAxsAyAA Au
```

The unique 3-colouring, modulo swapping colour classes Y and B, is given by $R = \{\mathtt{0,1,2,3,4,5,6,7,8,9,r,t,A}\}$, $Y = \{\mathtt{a,b,d,e,f,g,h,i,j,k,l,n}\}$, $B = \{\mathtt{c,m,o,p,q,s,u,v,w,x,y,z}\}$.
The number of blocks of each colour type is $RRY = 46$, $RRB = 32$, $RYY = 26$, $RBB = 40$, $YYB = 40$, $YBB = 26$, $RYB = 12$.

STS(37) ♯2

```
378Aabtrsw uvzxyqop4b A8arstuowx yzvpq96aAs trvwuyzxpq
o68bAqopzx ywuvtrs9Ab opqxyzuvwr stapqoyzxv wustr9bzxy
trsqopwuvu yxrstopqzv wAyzvstrpq oxwuwuzqvp trsoxyxvwo
pqrstuyzvw upqostryzx fjkAmngnAk mlimAiknAl AnmlnArvwA
yzszAwyxuy AuwzAxAzyx zA
```

The unique 3-colouring, modulo swapping colour classes Y and B is given by $R = \{\mathtt{0,3,6,9,c,e,f,i,o,r,u,v,A}\}$, $Y = \{\mathtt{1,4,a,d,g,j,l,q,t,w,x,z}\}$, $B = \{\mathtt{2,5,7,8,b,h,k,m,n,p,s,y}\}$.
The number of blocks of each colour type is $RRY = 43$, $RRB = 35$, $RYY = 29$, $RBB = 37$, $YYB = 37$, $YBB = 29$, $RYB = 12$.
This system contains three STS(13)s on the sets $\{\mathtt{0,1,2,3,4,5,6,7,8,9,a,b,A}\}$, $\{\mathtt{c,d,e,f,g,h,i,j,k,l,m,n,A}\}$ and $\{\mathtt{o,p,q,r,s,t,u,v,w,x,y,z,A}\}$ respectively.
The fact that systems ♯1 and ♯2 are uniquely 3-colourable but have different statistics of block colour types shows that they are non-isomorphic.

STS(49)

```
Ehpiqjrksl tmunvwMLKJ IHGFpiqjrk sltmunvgxG yLKJIHMqjr
kslAmunvgo yGMzLKJIHr ksltmunvgo hzCBALKJIM sltmunvgoh
pAEDCBLKJM tmunvgohpi BGFEDCLKMu nJgxhpiqCL HIGFEDMvgo
hpiqjDKJIH GFEMChpiqj rMHxLKJIGF piqjrkMtLz yKIHGqjrks
MBAzLKJIHr kslMCDBALK JIsltMFEDC BLKJtmMHGF EDCLKuMJIH
GFEDLMLKJI HGFEExFyGz HvIBJCKDLF yGzHAIBJCK DLwGzHAIBJ
CKDLwEHAIB JCKDLwExIB oKDLwExFJJ CKDLwExFyK DLwExFyGwE
xFyGzLExFy zAFyGzHAMG zHAIMHAIBM IBJMJCMKMM HM
```

The unique 3-colouring, modulo swapping colour classes Y and B, is given by $R = \{\texttt{0,1,2,3,4,6,7,8,9,a,b,c,d,e,f,H,L}\}$,
$Y = \{\texttt{5,h,i,j,k,l,m,o,p,q,r,s,t,u,v,M}\}$,
$B = \{\texttt{g,n,w,x,y,z,A,B,C,D,E,F,G,I,J,K}\}$.
The number of blocks of each colour type is $RRY = 95$, $RRB = 41$, $RYY = 33$, $RBB = 87$, $YYB = 87$, $YBB = 33$, $RYB = 16$.

STS(55)

```
rjsktlumvn eoxyqOAzER QPNMLKSskt luzvnwoxpy qmiBCRQPON
MLStlumvnw oxpyqzirCE DRQPONMSum vnwoxpyqzi rjDGFERQPO
NSvnwoxpyq zirjsEIHGF RQPOSwoxpy qzirjskFKJ IHGRQPSxpy
qzirjKktGC MLJIHRQSyq HirjsktlpO NMLKJIRSzi rjsktluIQP
ONMLKJSrjs ktlumSBRQP ONMLKsktlu zvSODCRQPN MLtlmvnSCD
FERQPONMum vnwSHGFERQ PONvnwoSJI HGFRQPOwox SHELKJIGRQ
PxpSDNMLJI HRQySPONML KJIRSDRQPN MLKJJBKCLD MENFOGPHQI
RKCLDMENFO GPHQIRALDM ENFOGPHQIR AJMENFOGPH QIRAJBNFGP
HQIRAJBKOG PHQIRAJBKC PHQIRAJBKC LQIRAJBKCL ORAJBKCLDM
JBKCLDMESK wLMENSLDME NFSMNFOSNF OGSOGPSPHS QSSKO
```

The unique 3-colouring, modulo swapping colour classes Y and B, is given by $R = \{\texttt{1,2,3,4,5,6,7,8,9,a,b,c,d,e,f,g,h,D,O}\}$,
$Y = \{\texttt{0,i,j,k,l,m,n,o,p,q,r,s,t,v,w,x,y,S}\}$,
$B = \{\texttt{u,z,A,B,C,E,F,G,H,I,J,K,L,M,N,P,Q,R}\}$.
The number of blocks of each colour type is $RRY = 126$, $RRB = 45$, $RYY = 36$, $RBB = 117$, $YYB = 117$, $YBB = 36$, $RYB = 18$.

STS(57)

A cyclic Steiner triple system whose blocks are the set of triples obtained by action of the group generated by the mapping $i \mapsto i + 1 \pmod{57}$ on the set of starter blocks $\{0, 1, 3\}$, $\{0, 4, 9\}$, $\{0, 6, 13\}$, $\{0, 8, 26\}$, $\{0, 10, 33\}$, $\{0, 11, 32\}$, $\{0, 12, 40\}$, $\{0, 14, 41\}$, $\{0, 15, 35\}$, $\{0, 19, 38\}$.
The unique 3-colouring is given by the three residue classes mod 3.
The number of blocks of each colour type is $RRY = 152$, $RRB = 19$, $RYY = 19$, $RBB = 152$, $YYB = 152$, $YBB = 19$, $RYB = 19$.

Another concept is that of a *balanced colouring*, considered by C.J. Colbourn, L. Haddad and V. Linek, [4]. An m-chromatic STS(v) is said to be *m-balanced* if every m-colouring of the system is equitable. In [4], the following result is proved.

Theorem 5.1 *With the possible exceptions of* $v \in \{19, 21, 37, 49, 55, 57, 67, 69, 85, 109, 139\}$*, for all admissible* $v \geq 15$*, there exists a 3-balanced STS(v).*
Proof. See the original paper. □

The systems listed immediately above, which are both uniquely colourable and equitably 3-chromatic, are 3-balanced. The first STS(27) is perhaps of particular interest in that the block colourings are also balanced. In any 3-colouring of a Steiner triple system the number of blocks which contain a point from each of the three colour classes is determined by the cardinalities of the colour classes. However, the number of bichromatic blocks of the six types is not determined and they are not necessarily all equal. But in this particular system the number of bichromatic blocks of each type is 18. The STS(37), STS(49), STS(55) and STS(57) remove these values of v from the list of possible exceptions in Theorem 5.1. Indeed we are able to remove all values $v \geq 21$ from the list. Below are given two examples of a 3-balanced but not uniquely colourable STS(21). The systems STS(31) $\sharp$3, STS(33), STS(37) $\sharp$2 and STS(49) given in this section enable us to construct 3-balanced STS(v) for the remaining values of v, by a straightforward application of Lemma 3.1 of [4]. For $v = 67, 85$ and 109 we put $\epsilon = 1, u = 4$ and use 3-GDDs of type $6^3, 8^3$ and 8^4 respectively. For $v = 69$ we put $\epsilon = 0, u = 5$ (for which parameters the lemma also holds) and use a 3-GDD of type 6^3. For $v = 139$ we put $\epsilon = 1, u = 0$ and use a 3-GDD of type $10^3 16^1$.

STS(21) $\sharp$1

```
5ackfejidh 6bajfkegi7 cbkgfhj8di fhgk9edihj
afejikbgck jchgkdihej fkjgkhiijk
```

A 3-colouring is given by $R = \{$`0,1,2,3,8,e,i`$\}$, $Y = \{$`4,6,7,c,f,j,k`$\}$, $B = \{$`5,9,a,b,d,g,h`$\}$. The system is a Kirkman triple system.
The number of blocks of each colour type is $RRY = 11$, $RRB = 10$, $RYY = 10$, $RBB = 11$, $YYB = 11$, $YBB = 10$, $RYB = 7$.

STS(21) ♯2

```
4a9bijkfgh 5ajkibghfb 9kighfj5cd eijkdecbjk
iecdakij8f ghghfhfjij kjkikigehk
```

A 3-colouring is given by $R = \{\mathtt{0,1,2,6,9,e,g}\}$, $Y = \{\mathtt{3,b,c,d,f,h,j}\}$, $B = \{\mathtt{4,5,7,8,a,i,k}\}$.
The number of blocks of each colour type is $RRY = 10$, $RRB = 11$, $RYY = 11$, $RBB = 10$, $YYB = 10$, $YBB = 11$, $RYB = 7$.
Systems ♯1 and ♯2 are non-isomorphic.

Finally in this section we should record that both the perfect STS(25) given on page 207 of [6] and the affine STS(27) are 3-balanced, [4], but do not have unique colourings. An example of a 3-balanced but not uniquely colourable STS(37) is given below.

STS(37) ♯3

```
idjekflgmh nozyxwvAje kfl8hnmgpq zyxwAkflgm hnciqsrzyx
Algmhncidr utszyAmhnc ibjswvutzA ncidjetyxw vuAidqekAz
pyxwvekfAj rqzuwykmlA jtsrzyxlgA xvusyzstAx wvuzjAzytw
vxupvwrsyt zxvqwrxytz owrxsytzou xsyzoupyto upvzzoupvq
upvqwAvqwr AtwrxAxsAy AA
```

A 3-colouring is given by $R = \{\mathtt{0,1,2,3,4,5,6,8,9,a,s,t,A}\}$,
$Y = \{\mathtt{7,b,c,d,e,g,h,i,k,l,m,n}\}$, $B = \{\mathtt{f,j,o,p,q,r,u,v,w,x,y,z}\}$.
The number of blocks of each colour type is $RRY = 46$, $RRB = 32$, $RYY = 26$, $RBB = 40$, $YYB = 40$, $YBB = 26$, $RYB = 12$.
Another colouring is obtained by moving the element t from the colour class R to the colour class B (and re-naming R as B and vice-versa).

6 The cases $v = 19$ and 21 revisited

In the Introduction, we remarked that the only value of v for which the existence of a 4-chromatic STS(v) is in doubt is $v = 19$. During the preparation of this paper we searched extensively for a 4-chromatic STS(19) but were unsuccessful. However the situation for $v = 21$ is different. The first 4-chromatic STS(21) was constructed by L. Haddad, [10], and plays a significant role in the determination of the spectrum of 4-chromatic STS(v). In the course of our investigations we discovered five further 4-chromatic STS(21)s, all obtained by making trades in Haddad's system. In the interests of completeness, we list this system below and the trades which produce the new 4-chromatic STS(21)s. Calculation of invariants of the systems shows that they are non-isomorphic.

STS(21) ♯1 (Haddad, [10])
$\{0,1,2\}$, $\{0,3,9\}$, $\{0,4,10\}$, $\{0,5,11\}$, $\{0,6,18\}$, $\{0,7,19\}$, $\{0,8,20\}$, $\{0,12,15\}$, $\{0,13,16\}$, $\{0,14,17\}$, $\{1,3,10\}$, $\{1,4,11\}$, $\{1,5,9\}$, $\{1,6,19\}$, $\{1,7,20\}$, $\{1,8,18\}$, $\{1,12,16\}$, $\{1,13,17\}$, $\{1,14,15\}$, $\{2,3,11\}$, $\{2,4,9\}$, $\{2,5,10\}$, $\{2,6,20\}$, $\{2,7,18\}$, $\{2,8,19\}$, $\{2,12,17\}$, $\{2,13,15\}$, $\{2,14,16\}$, $\{3,4,5\}$, $\{3,6,12\}$, $\{3,7,13\}$, $\{3,8,14\}$, $\{3,15,18\}$, $\{3,16,19\}$, $\{3,17,20\}$, $\{4,6,13\}$, $\{4,7,14\}$, $\{4,8,12\}$, $\{4,15,19\}$, $\{4,16,20\}$, $\{4,17,18\}$, $\{5,6,14\}$, $\{5,7,12\}$, $\{5,8,13\}$, $\{5,15,20\}$, $\{5,16,18\}$. $\{5,17,19\}$, $\{6,7,8\}$, $\{6,9,15\}$, $\{6,10,16\}$, $\{6,11,17\}$, $\{7,9,16\}$, $\{7,10,17\}$, $\{7,11,15\}$, $\{8,9,17\}$, $\{8,10,15\}$, $\{8,11,16\}$, $\{9,10,11\}$, $\{9,12,18\}$, $\{9,13,19\}$, $\{9,14,20\}$, $\{10,12,19\}$, $\{10,13,20\}$, $\{10,14,18\}$, $\{11,12,20\}$, $\{11,13,18\}$, $\{11,14,19\}$, $\{12,13,14\}$, $\{15,16,17\}$, $\{18,19,20\}$.

STS(21) ♯2
Replace the triples $\{1,6,19\}$, $\{1,12,16\}$, $\{6,10,16\}$, $\{10,12,19\}$ in system ♯1 by the triples $\{1,6,16\}$, $\{1,12,19\}$, $\{6,10,19\}$, $\{10,12,16\}$; a Pasch switch.

STS(21) ♯3
Replace the triples $\{0,1,2\}$, $\{0,5,11\}$, $\{1,4,11\}$, $\{2,4,9\}$, $\{2,5,10\}$, $\{9,10,11\}$ in system ♯1 by the triples $\{0,1,11\}$, $\{0,2,5\}$, $\{1,2,4\}$, $\{2,9,10\}$, $\{4,9,11\}$, $\{5,10,11\}$; a 6 block trade from an STS(9).

STS(21) ♯4
Replace the triples $\{0,7,19\}$, $\{0,8,20\}$, $\{1,6,19\}$, $\{1,7,20\}$, $\{1,8,18\}$, $\{2,6,20\}$, $\{2,8,19\}$, $\{6,7,8\}$, $\{18,19,20\}$ in system ♯1 by the triples $\{0,7,8\}$, $\{0,19,20\}$, $\{1,6,8\}$, $\{1,7,19\}$, $\{1,18,20\}$, $\{2,6,19\}$, $\{2,8,20\}$, $\{6,7,20\}$, $\{8,18,19\}$; a 9 block trade from an STS(9).

STS(21) ♯5
Replace the triples $\{0,1,2\}$, $\{0,6,18\}$, $\{0,7,19\}$, $\{1,6,19\}$, $\{1,7,20\}$, $\{2,6,20\}$, $\{2,7,18\}$, $\{2,8,20\}$, $\{6,7,8\}$, $\{18,19,20\}$ in system ♯1 by the triples $\{0,1,19\}$, $\{0,2,6\}$, $\{0,7,18\}$, $\{1,2,20\}$, $\{1,6,7\}$, $\{2,7,8\}$, $\{2,18,19\}$, $\{6,8,19\}$, $\{6,18,20\}$, $\{7,19,20\}$; a 10 block trade from an STS(9).

STS(21) ♯6
Replace the triples $\{0,1,2\}$, $\{0,7,19\}$, $\{0,8,20\}$, $\{1,6,19\}$, $\{1,7,20\}$, $\{1,8,18\}$, $\{2,6,20\}$, $\{2,7,18\}$, $\{2,8,19\}$, $\{6,7,8\}$, $\{18,19,20\}$ in system ♯1 by the triples $\{0,1,8\}$, $\{0,2,7\}$, $\{0,19,20\}$, $\{1,2,6\}$, $\{1,7,19\}$, $\{1,18,20\}$, $\{2,8,20\}$, $\{2,18,19\}$, $\{6,7,20\}$, $\{6,8,19\}$, $\{7,8,18\}$; an 11 block trade from an STS(9).

These results prompted the question of determining, particularly for the cases $v = 19$ and 21, the *range* of values which are taken by the chromatic number. More formally, for each admissible v define the set $K(v) = \{\chi$: there exists a χ-chromatic STS$(v)\}$. We know that $K(3) = \{2\}$ and that $K(7) = K(9) = K(13) = K(15) = \{3\}$. Further $\{3\} \subset K(19)$ and $\{3,4\} \subset K(21)$. Although we were unable to determine $K(19)$ and $K(21)$ precisely we were able to prove a set inclusion result about them.
The proofs depend on the concept of an *arc* or *independent set*. This is defined to be a subset $S \subset V$, of an STS(v), (V,B), having the property that $T \not\subset S$ for all $T \in B$, i.e. no triple is contained in the independent set. An independent set is *maximum* if it has the largest cardinality of any independent set in the design. Define $I(v) = \{\beta$: there exists an STS(v) with maximum independent set of cardinality $\beta\}$. Then it is known that $I(19) = \{7,8,9,10\}$ and $\{8,9,10\} \subset I(21)$, [3].

Theorem 6.1 $\{3\} \subset K(19) \subset \{3,4\}$.
Proof. It suffices to prove that every STS(19) is 4-colourable.
From the result about $I(19)$ quoted above, every STS(19) has an independent set of cardinality 7. Choose this set to be one of the colour classes, say $R(=$ red). This gives $\binom{7}{2} = 21$ triples with two red points and one non-red point and $(7 \times 9) - (21 \times 2) = 21$ triples with one red point and two non-red points. This leaves 12 non-red points and 15 triples of non-red points, (red-free blocks).
Now suppose that there exists a point x which is contained in 5 red-free blocks, say $\{x,a,b\}$, $\{x,c,d\}$, $\{x,e,f\}$, $\{x,g,h\}$, $\{x,i,j\}$ where in addition

we can assume that the triples of the STS(19) which contain the pairs $\{a,c\}$ or $\{a,d\}$ do not also contain the points e or f. We now assign the points a,c,d,e,f to a colour class $Y(=$ yellow), the points g,h,i,j to a colour class $B(=$ blue) and the points x,b,y to a colour class $G(=$ green), where y is the remaining non-red point. It is readily verified that there is no monochromatic triple in the colour classes Y, B or G.

If no such point x exists then since $(15 \times 3)/12 > 3$, there exists a point z which is contained in precisely 4 red-free blocks, say $\{z,a,b\}$, $\{z,c,d\}$, $\{z,e,f\}$, $\{z,g,h\}$. Let i,j,k be the other 3 non-red points. Further, we can assume that the triples of the STS(19) which contain the pairs $\{a,i\}$ or $\{b,i\}$ do not contain the points c or d. Assign the points a,b,c,d,i to a colour class $Y(=$ yellow), the points e,f,g,h to a colour class $B(=$ blue) and the points z,j,k to a colour class $G(=$ green). Again it is readily verified that there is no monochromatic triple in the colour classes Y, B or G. □

Theorem 6.2 $\{3,4\} \subset K(21) \subset \{3,4,5\}$.

Proof. It suffices to prove that every STS(21) is 5-colourable.

Unlike in the case of the previous theorem, it is not known what is the minimum cardinality of a maximum independent set in an STS(21). However it is easy to verify that every STS(21) has an independent set of cardinality 6 and this is sufficient to prove the theorem. Choose the independent set to be one of the colour classes, say $R(=$ red). This gives $\binom{6}{2} = 15$ triples with two red points and one non-red point and $(6 \times 10) - (15 \times 2) = 30$ triples with one red point and two non-red points. Thus leaves 15 non-red points and 25 triples of non-red points, (red-free blocks).

Now suppose that there exists a point x which is contained in 7 red-free blocks, say $\{x,a,b\}$, $\{x,c,d\}$, $\{x,e,f\}$, $\{x,g,h\}$, $\{x,i,j\}$, $\{x,k,l\}$, $\{x,m,n\}$ where in addition we can assume that the triples of the STS(21) which contain the pairs $\{a,c\}$ or $\{a,d\}$ do not contain the points e or f. We now assign the points a,c,d,e,f to a colour class $Y(=$ yellow), the points g,h,i,j to a colour class $B(=$ blue), the points k,l,m,n to a colour $G(=$ green) and the points x,b to a colour class $P(=$ pink).

If no such point x exists then suppose that there exists a point y which is contained in 6 red-free blocks, say $\{y,a,b\}$, $\{y,c,d\}$, $\{y,e,f\}$, $\{y,g,h\}$, $\{y,i,j\}$, $\{y,k,l\}$. Let m,n be the other 2 non-red points. We now assign the points a,b,c,d to a colour class $Y(=$ yellow), the points e,f,g,h to a colour class

$B(=$ blue), the points i, j, k, l to a colour class $G(=$ green) and the points y, m, n to a colour $P(=$ pink).
If no such points x or y exist then since $(25 \times 3)/15 = 5$, every point is contained in precisely 5 red-free blocks. Choose one of these, say z, and let the red-free blocks in which it occurs be $\{z, a, b\}$, $\{z, c, d\}$, $\{z, e, f\}$, $\{z, g, h\}$, $\{z, i, j, \}$. Let k, l, m, n be the other 4 non-red points. Further we can assume that the triples of the STS(21) which contain the pairs $\{a, c\}$ or $\{a, d\}$ do not contain the points e or f, and that the triple which contains the pair$\{b, k\}$ does not contain the point l. We now assign the points a, c, d, e, f to a colour class $Y(=$ yellow), the points g, h, i, j to a colour class $B(=$ blue), the points b, k, l to a colour class $G(=$ green) and the points z, m, n to a colour class $P(=$ pink). □

7 Netto systems

The so-called *Netto systems* are a class of Steiner triple systems which admit a group of automorphisms acting doubly homogeneously but not doubly transitively on the points. They exist for orders $v = q = p^n$ where $p \equiv 7 \pmod{12}$ is prime and n is odd. Probably the most elegant construction of them is described in [7]. Let r_1 and r_2 be the two primitive sixth roots of unity in the field GF(q). Then r_1 and r_2 are the roots of the equation $x^2 - x + 1 = 0$ and hence $r_1 + r_2 = 1$ and $r_1 r_2 = 1$. Define a relation $<$ in GF(q) by $u < v$ if $v - u$ is a non-zero square. Then, since $q \equiv -1 \pmod 4$, for distinct elements $u, v \in \mathrm{GF}(q)$ either $u < v$ or $v < u$ but not both. The points of a Netto system N(q) are the elements of GF(q). For a pair of distinct elements u, v with $u < v$ let $w = f(u, v) = ur_1 + vr_2$. Then it is easily verified that $v < w$ and $u = f(v, w)$ and $w < u$ and $v = f(w, u)$. The blocks of N(q) are the collection of all such triples $\{u, v, w\}$.
We have investigated the colourability of the Netto systems N(19), N(31) and N(43). The system N(19) is 3-chromatic but only with colouring patterns $(c_1, c_2, c_3) = (7, 6, 6)$ or $(7, 7, 5)$; in fact it is one of only two STS(19)s we have discovered which is 3-chromatic with just these two patterns. If we represent N(19) on the set $V = \{0, 1, 2, \ldots, 18\}$ with blocks B obtained by the action of the cyclic group generated by the mapping $i \mapsto i + 1 \pmod{19}$ on the three base blocks $\{0, 1, 8\}$, $\{0, 2, 5\}$ and $\{0, 4, 13\}$, then a 3-colouring with pattern $(c_1, c_2, c_3) = (7, 6, 6)$ is given by $R = \{0, 1, 2, 3, 4, 12, 14\}$, $Y =$

$\{5, 7, 8, 11, 17, 18\}$, $B = \{6, 9, 10, 13, 15, 16\}$. A 3-colouring with pattern $(c_1, c_2, c_3) = (7, 7, 5)$ is given by $R = \{0, 1, 2, 3, 4, 12, 14\}$, $Y = \{6, 7, 9, 10, 11, 15, 16\}$, $B = \{5, 8, 13, 17, 18\}$.

The other STS(19) which is 3-chromatic only with colouring patterns $(c_1, c_2, c_3) = (7, 6, 6)$ or $(7, 7, 5)$ is also cyclic. It is system A2 in the listing given in [12]. Using the representation given there with the three base blocks $\{0, 1, 4\}$, $\{0, 2, 12\}$ and $\{0, 5, 13\}$, then a 3-colouring with pattern $(c_1, c_2, c_3) = (7, 6, 6)$ is given by $R = \{0, 1, 2, 3, 7, 8, 17\}$, $Y = \{4, 5, 9, 13, 14, 15\}$, $B = \{6, 10, 11, 12, 16, 18\}$. A 3-colouring with pattern $(c_1, c_2, c_3) = (7, 7, 5)$ is given by $R = \{0, 1, 2, 3, 7, 8, 17\}$, $Y = \{4, 5, 9, 13, 14, 15, 16\}$, $B = \{6, 10, 11, 12, 18\}$.

There are four non-isomorphic, cyclic Steiner triple systems of order 19 and for completeness we give the 3-colouring patterns which are achievable for the remaining two systems. For system A1 with the three base blocks $\{0, 1, 4\}$, $\{0, 2, 9\}$ and $\{0, 5, 11\}$, only 3-colourings with patterns $(c_1, c_2, c_3) = (7, 6, 6), (7, 7, 5)$ or $(8, 6, 5)$ can be achieved. Examples of these are $R = \{0, 1, 2, 3, 7, 8, 12\}$, $Y = \{4, 5, 9, 14, 17, 18\}$, $B = \{6, 10, 11, 13, 15, 16\}$; $R = \{0, 1, 2, 3, 7, 11, 12\}$, $Y = \{4, 8, 9, 13, 16, 17, 18\}$, $B = \{5, 6, 10, 14, 15\}$; $R = \{0, 1, 2, 3, 8, 11, 12, 17\}$, $Y = \{4, 5, 7, 13, 15, 16\}$, $B = \{6, 9, 10, 14, 18\}$ respectively. For system A3 with the three base blocks $\{0, 1, 8\}$, $\{0, 2, 5\}$ and $\{0, 4, 10\}$, again only 3-colourings with patterns $(c_1, c_2, c_3) = (7, 6, 6), (7, 7, 5)$ or $(8, 6, 5)$ can be achieved. Examples of these are $R = \{0, 1, 2, 3, 4, 12, 14\}$, $Y = \{5, 8, 10, 11, 15, 17\}$, $B = \{6, 7, 9, 13, 16, 18\}$; $R = \{0, 1, 2, 3, 4, 12, 15\}$, $Y = \{7, 8, 13, 14, 16, 17, 18\}$, $B = \{5, 6, 9, 10, 11\}$; $R = \{0, 1, 2, 4, 6, 11, 14, 17\}$, $Y = \{3, 7, 10, 15, 16, 18\}$, $B = \{5, 8, 9, 12, 13\}$ respectively.

The system N(31) is 3-balanced. If we represent N(31) on the set $V = \{0, 1, 2, \dots, 30\}$ with blocks B obtained by the action of the group generated by the mapping $i \mapsto i + 1 \pmod{31}$ on the five base blocks $\{0, 1, 6\}$, $\{0, 2, 12\}$, $\{0, 3, 16\}$, $\{0, 4, 24\}$ and $\{0, 8, 17\}$, then a 3-colouring with pattern $(c_1, c_2, c_3) = (11, 10, 10)$ is given by $R = \{0, 1, 2, 3, 4, 11, 18, 19, 23, 27, 29\}$, $Y = \{5, 6, 8, 12, 15, 17, 20, 25, 26, 30\}$, $B = \{7, 9, 10, 13, 14, 16, 21, 22, 24, 28\}$.

The system N(43) is 4-chromatic. If we represent N(43) on the set $V = \{0, 1, 2, \dots, 42\}$ with blocks B obtained by the action of the cyclic group generated by the mapping $i \mapsto i + 1 \pmod{43}$ on the seven base blocks $\{0, 1, 37\}$, $\{0, 2, 14\}$, $\{0, 3, 21\}$, $\{0, 4, 19\}$, $\{0, 5, 35\}$, $\{0, 9, 32\}$ and $\{0, 10, 26\}$, then a 4-

colouring is given by $R = \{16, 17, 18, 19, 23, 25, 28, 29, 32, 33, 34, 36, 40, 42\}$, $Y = \{3, 9, 14, 15, 21, 24, 27, 30, 35, 37, 38, 39, 41\}$,
$B = \{4, 5, 6, 7, 13, 20, 22, 26, 31\}$, $G = \{0, 1, 2, 8, 10, 11, 12\}$.

8 Concluding remarks

This paper has primarily been concerned with 3-chromatic Steiner triple systems and more particularly with the cardinalities of the colour classes. A 3-colouring may be either equitable or non-equitable. Hence 3-chromatic STS(v) can be partitioned into three types; (i) those which have both equitable and non-equitable 3-colourings, (ii) those which have only equitable 3-colourings, which are called 3-balanced, and (iii) those which have only non-equitable 3-colourings, which by analogy we will call *3-unbalanced*.

The spectrum of the first type is easy to determine; it is all admissible $v \geq 7$. For $v \equiv 3 \pmod 6$ use the Bose construction, [1], also given on page 25 of [6]. Let $Q = (S, \bigotimes)$ be a commutative idempotent quasigroup (Latin square) of odd order. Let the points of the Steiner triple system $V = S \times \{0, 1, 2\}$. The blocks B are the union of the two sets $\{\{x_0, x_1, x_2\} : x \in S\}$ and $\{\{x_i, y_i, (x \bigotimes y)_{i+1}\} : x \in S, y \in S, x \neq y, i \in Z_3\}$, where here, and in the next construction, we write x_i to denote the ordered pair (x, i). Clearly if we use the subscripts to determine the colour classes then an equitable 3-colouring of the system is obtained. A non-equitable colouring may be obtained by recolouring any one of the points with subscript 0 as the colour of the class with subscript 1.

For $v \equiv 1 \pmod 6$ use the Skolem construction, [18], also given on page 26 of [6]. Let $Q = (S, \bigotimes)$ be a commutative half-idempotent quasigroup (Latin square) of even order. Let the points of the Steiner triple system be $V = (S \times \{0, 1, 2\}) \cup \{\infty\}$. The blocks B are the union of the three sets $\{\{x_0, x_1, x_2\} : x \in S, x \bigotimes x = x\}$, $\{\{\infty, x_i, (x \bigotimes x)_{i+1}\} : x \in S, x \bigotimes x \neq x, i \in Z_3\}$ and $\{\{x_i, y_i, (x \bigotimes y)_{i+1}\} : x \in S, y \in S, x \neq y, i \in Z_3\}$. Again, an equitable 3-colouring is determined by the subscripts with the point ∞ assigned to any of the colour classes, say the colour class of the points with subscript 0. A non-equitable colouring may then be obtained by recolouring any one of the points with subscript 1 as the colour with subscript 2.

As stated in Section 5, the question of the existence of 3-balanced STS(v) was considered by C.J. Colbourn, L. Haddad and V. Linek, [4]. We have removed all values apart from $v = 19$ from their list of possible exceptions in Theorem 5.1 and so the current state of knowledge is that the spectrum is all admissible $v \geq 15$ with the possible exception of $v = 19$. However we were unable to find a 3-balanced STS(19) and indeed conjecture that none exist. To prove this, one would need an argument, similar to that given in the proof of Theorem 4.1, that any 3-colouring with pattern $(c_1, c_2, c_3) = (7, 6, 6)$ can be transformed into a 3-colouring with a different pattern. We give this as our first problem.

Problem 1 *Does there exist a 3-balanced STS(19)?*

In Section 3, we exhibited the first known examples of 3-unbalanced STS(v); for $v = 25$, 27, 31, 33, 37 and 39. It was previously known that 3-unbalanced STS(v) do not exist for $v = 7$, 9, 13 and 15 and we succeeded in proving that none exist for $v = 19$. We conjecture that 3-unbalanced STS(v) exist for all admissible $v \geq 25$ and this is now an interesting unresolved problem. However, we were unable to find a 3-unbalanced STS(21) nor to prove a supplement to Theorem 4.2, that any 3-colouring with pattern $(c_1, c_2, c_3) = (8, 7, 6)$ can be transformed into a 3-colouring with pattern $(c_1, c_2, c_3) = (7, 7, 7)$. We give this as our second problem.

Problem 2 *Does there exist a 3-unbalanced STS(21)?*

A more difficult problem would appear to be the construction of STS(v) which are uniquely 3-colourable. The only known examples would appear to be for the values $v = 25$, 27, 31, 33 and 37 for both equitable and non-equitable colourings and $v = 49$, 55 and 57 for equitable colourings, all of which are given above. But it would not be unreasonable to conjecture that both types exist for all admissible $v \geq 25$.

Finally, it would be of interest to determine precisely the sets $K(19)$ and $K(21)$ defined in Section 6, and we give these as our final two problems. The first of these was also posed in [10].

Problem 3 *Does there exist a 4-chromatic STS(19)?*

Problem 4 *Does there exist a 5-chromatic STS(21)?*

Just for the record we conjecture that the answers to all four of our problems are in the negative.

References

[1] R.C. Bose, On the construction of balanced incomplete block designs, *Ann. Eugenics* **9** (1939), 353-399.

[2] M. de Brandes, K.T. Phelps and V. Rődl, Colouring Steiner triple systems, *SIAM J. Alg. Disc. Math.* **3** (1982), 241-249.

[3] M. de Brandes and V. Rődl, Steiner triple systems with small maximal independent sets, *Ars Combin.* **17** (1984), 15-19.

[4] C.J. Colbourn, L. Haddad and V. Linek, Balanced Steiner triple systems, *J. Combin. Theory Ser. A* **78** (1997), 292-302.

[5] C.J. Colbourn, S.S. Magliveras and D.R. Stinson, Steiner triple systems of order 19 with nontrivial automorphism group, *Math. Comput.* **59** (1992), 283-295 & S25-S27.

[6] C.J. Colbourn and A. Rosa, Triple systems, Clarendon, Oxford (1999).

[7] A. Delandtsheer, J. Doyen, J. Siemons and C. Tamburini, Doubly homogeneous $2-(v,k,1)$ designs, *J. Combin. Theory Ser. A* **43** (1986), 140-145.

[8] J. Fugère, L. Haddad and D. Wehlau, 5-chromatic Steiner triple systems, *J. Combin. Des.* **2** (1994), 287-299.

[9] T.S. Griggs and J.P. Murphy, 101 anti-Pasch Steiner triple systems of order 19, *J. Combin. Math. Combin. Comput.* **13** (1993), 129-141.

[10] L. Haddad, On the chromatic numbers of Steiner triple systems, *J. Combin. Des.* **7** (1999), 1-10.

[11] L. Haddad and V. Rődl, Unbalanced Steiner triple systems, *J. Combin. Theory Ser. A* **66** (1994), 1-16.

[12] R. Mathon, K.T. Phelps and A. Rosa, Small Steiner triple systems and their properties, *Ars Combin.* **15** (1983), 3-110.

[13] J. Pelikán, Properties of balanced incomplete block designs, in: *Combinatorial Theory and its applications*, Balatonfured, *Colloq. Math. Soc. J. Bolyai* **4** (1969), 869-889.

[14] A. Rosa, On the chromatic number of Steiner triple systems, in: *Combinatorial structures and their applications*, Gordon and Breach, New York (1970), 369-371.

[15] A. Rosa, Steiner triple systems and their chromatic number, *Acta Fac. Rerum Natur. Univ. Comen. Math.* **24** (1970), 159-174.

[16] A. Rosa, Colouring problems in combinatorial designs, *Congr. Num.* **56** (1987), 45-52.

[17] A. Rosa and C.J. Colbourn, Colorings of blocks designs, in: *Contemporary Design Theory: a collection of surveys* (ed. J.H. Dinitz and D.R. Stinson), John Wiley, New York (1992), 402-430.

[18] Th. Skolem, Some remarks on the triple systems of Steiner, *Math. Scand.* **6** (1958), 273-280.

[19] D.R. Stinson, Hill-climbing algorithms for the construction of combinatorial designs, *Ann. Discrete Math.* **26** (1995), 321-334.