The Design of the Century

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Abstract

We construct a 2-chromatic Steiner system $S(2,4,100)$ in which every block contains three points of one colour and one point of the other colour. The existence of such a design has been open for over 25 years.

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1 The background

A *Steiner system* $S(t,k,v)$ is an ordered pair $(V,\mathcal{B})$ where V is a set of cardinality v, the *base set*, and $\mathcal{B}$ is a collection of k-subsets of V, the *blocks*, which collectively have the property that every t-element subset of V is contained in precisely one block. Elements of V are called *points*. In this paper we are principally concerned with the case in which $t = 2$ and $k = 4$. Steiner systems $S(2,4,v)$ exist if and only if $v \equiv 1$ or $4 \pmod{12}$ [4]; such values of v are called *admissible*. Given a Steiner system $S(2,4,v)$, we may ask whether it is possible to colour each point of the base set V with one of two colours, say red or blue, so that no block is monochromatic. A Steiner system $S(2,4,v)$ having this property is said to be *2-chromatic* or to have a *blocking set*. It was shown in [5] that 2-chromatic $S(2,4,v)$s exist for all admissible v with the possible exception of three values, $v = 37$, 40 and 73. Existence for these three values was established in [3]. Perhaps we should also remark here that it is known that for all $v \geq 25$ there exists a Steiner system $S(2,4,v)$ which is not 2-chromatic, [8].

In a 2-chromatic $S(2,4,v)$ let c and $v-c$ be the cardinalities of the red and blue colour classes, respectively. If b_1, b_2 and b_3 are the numbers of blocks with colour patterns $RRRB$, $RRBB$ and $RBBB$, respectively, then by counting pairs we have:

$$\begin{aligned} 3b_1 + b_2 &= \frac{c\,(c-1)}{2}, \\ b_2 + 3b_3 &= \frac{(v-c)(v-c-1)}{2}, \\ 3b_1 + 4b_2 + 3b_3 &= c\,(v-c). \end{aligned}$$

Solving the equations for b_2 gives $b_2 = (4vc - 4c^2 + v - v^2)/4$, which is non-negative for

$$\frac{v-\sqrt{v}}{2} \leq c \leq \frac{v+\sqrt{v}}{2}.$$

Furthermore, in the extreme cases where $\{c, v-c\} = \{(v-\sqrt{v})/2, (v+\sqrt{v})/2\}$ it follows that $b_2 = 0$; i.e. every block contains three points of one colour and one of the other colour. Moreover, the monochromatic triples of each colour appearing in the blocks form Steiner systems $S(2,3,(v-\sqrt{v})/2)$ and $S(2,3,(v+\sqrt{v})/2)$. An $S(2,3,w)$ is usually called a *Steiner triple system* and denoted by STS(w); they exist if and only if $w \equiv 1$ or 3 (mod 6), [6]. A modern account of Kirkman's work is given in [1]. ¿From the preceding discussion, it is easy to deduce that a 2-chromatic $S(2,4,v)$ having all blocks containing three points of one colour and one of the other colour can exist only if v is of the form $(12s+2)^2$ or $(12s+10)^2$, $s \geq 0$.

The smallest non-trivial case is therefore $v = 100$, and has become known as "the Design of the Century". Its existence, and possible construction, has been a problem in Design Theory for over 25 years. An early reference is [7]. In this paper we construct the design. We make no claim for uniqueness and, indeed, we think it highly unlikely.

2 The method

The cardinalities of the two colour classes are 55, the red points, and 45, the blue points. Denote the former by $A_0, A_1, \ldots, A_{54}$ and the latter by $\infty, B_0, B_1, \ldots, B_{43}$. We will seek an $S(2,4,100)$ having an automorphism σ of order 11 defined by

$$\sigma : A_i \mapsto A_{i+5 \pmod{55}},\ B_j \mapsto B_{j+4 \pmod{44}},\ \infty \mapsto \infty.$$

Our method is based on a simple backtrack algorithm with four distinct stages.

Stage 1. Select systems STS(55) and STS(45), both having automorphism σ, on the red and blue points respectively. The latter is an example of a 4-rotational STS(v); such systems exist for $v \equiv 1, 9, 13$ or 21 (mod 24), [2].

Stage 2. The blue system has 30 orbits under the automorphism. We partition these into five classes of six orbits, and label each class with a different point from the set $\{A_0, A_1, A_2, A_3, A_4\}$. Within each class we then assign the label to one block of each of the six orbits in such a way that the blocks to which the label is assigned form a partial parallel class; i.e. the blocks are pairwise disjoint. The assignment of red points to the other blocks of blue points is completely determined by σ. It is clear that this assignment ensures that there are no repeated pairs of a blue point with a red point.

Stage 3. The red system has 45 orbits under the automorphism. We next deal with the blue point ∞. In the course of performing stage 2 of the algorithm the point ∞ will have been paired with two of the five subsets $\{A_{i+j} : j = 0, 5, 10, \ldots, 50\}$, $i = 0, 1, 2, 3, 4$. We assign ∞ to all blocks of a single orbit whose red points cover the remaining three subsets.

Stage 4. This leaves 44 orbits of the red system. As in stage 2 we partition these into four classes of 11 orbits and label each class with a different point from the set $\{B_0, B_1, B_2, B_3\}$. Within each class, we then assign the label, say X, to one block of each of the 11 orbits in such a way that the blocks to which X is assigned form a partial parallel class, say $\mathcal{P}$. We attempt to do this while satisfying the further constraint that none of the 22 red points with which X has already been paired in stage 2 occur in $\mathcal{P}$. This latter is, of course, a very severe constraint. Again, the assignment of the blue points to the other blocks of red points is completely determined by σ.

Finally, we make a brief remark about our implementation of the algorithm. Stages 3 and 4 execute very quickly on a modern computer system and we always ran the backtracking to completion. However, for each particular choice of systems STS(55) and STS(45), we did not run the backtracking of stage 2 to completion, preferring instead to return to stage 1 after a certain period of time and select new systems.

3 The design

Listed below are 75 blocks which, under the mapping σ, give "the Design of the Century". As described in the last section, the construction of the design involved significant computing. However, it is perfectly feasible, although perhaps a little tedious, to check the design by hand, and the dedicated reader is invited to do this.

B0	B1	B9	A0	B4	B6	B23	A0	B8	B11	B13	A0
B12	B20	B10	A0	B36	B5	B7	A0	B2	B3	B38	A0
B0	B4	B33	A1	B40	B3	B6	A1	B8	B24	B5	A1
B16	B7	B11	A1	B1	B2	B21	A1	B9	B13	B42	A1
B0	B6	B24	A2	B8	B19	B40	A2	B4	B31	B3	A2
B9	B14	B43	A2	B1	B7	B29	A2	B5	B26	B2	A2
B0	B14	B21	A3	B4	B35	B41	A3	B1	B10	B31	A3
B5	B23	∞	A3	B2	B7	B18	A3	B22	B34	B3	A3
B28	B1	B14	A4	B4	B22	B26	A4	B0	B38	∞	A4
B29	B39	B2	A4	B37	B5	B19	A4	B3	B11	B35	A4
A25	A29	A19	B0	A20	A32	A5	B0	A35	A48	A18	B0
A15	A39	A11	B0	A41	A43	A8	B0	A21	A42	A13	B0
A31	A14	A28	B0	A16	A9	A12	B0	A17	A23	A49	B0
A37	A44	A33	B0	A22	A34	A38	B0				
A35	A36	A13	B1	A10	A17	A18	B1	A25	A44	A11	B1
A20	A43	A19	B1	A5	A37	A39	B1	A15	A6	A12	B1
A30	A23	A28	B1	A26	A29	A34	B1	A21	A38	A48	B1
A27	A42	A9	B1	A32	A49	A7	B1				
A5	A7	A21	B2	A20	A25	A46	B2	A35	A41	A11	B2
A40	A54	A15	B2	A30	A47	A19	B2	A36	A37	A23	B2
A26	A39	A24	B2	A16	A32	A53	B2	A31	A12	A22	B2
A17	A8	A9	B2	A13	A28	A49	B2				
A40	A43	A32	B3	A35	A44	A15	B3	A10	A20	A38	B3
A5	A27	A47	B3	A25	A6	A13	B3	A21	A26	A36	B3
A31	A42	A11	B3	A16	A28	A34	B3	A41	A9	A29	B3
A12	A17	A48	B3	A8	A24	A54	B3	A0	A11	A37	∞

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