Small surface trades in triangular embeddings

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Abstract

We enumerate all possible trades which involve up to six faces of the face set of a triangular embedding of a simple connected graph. These are classified by the underlying combinatorial trade on the associated block design, and by the geometrical arrangement of the faces necessary to avoid creation of a pseudosurface in the trading operation. The relationship of each of these trades to surface orientability is also established.

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1 Introduction

The concept of a trade is well established in combinatorial design theory and there are several published listings of small trades in various types of design. Below we give precise definitions sufficient for our purposes. A good overview is given in [2] and the listings we make use of appear in [6]. The purpose of this current paper is to investigate and to catalogue small surface trades in triangular embeddings. By applying such trades one may generally move between nonisomorphic embeddings of the same graph. Underlying any such surface trade there is a combinatorial trade on some (possibly partial) twofold triple system. However, the existence of a combinatorial trade amongst the triples formed by a set of triangular faces does not ensure the existence of a corresponding surface trade since applying the trade may transform the surface into a pseudosurface. The geometrical arrangement of the faces is important both for the feasibility of the trade and for questions of orientability.

In a recent paper, four of the present authors gave some results concerning small trades in triangular embeddings [4]. However, that paper focused on a different (although related) issue, namely the minimum non-zero number of faces in which two triangular embeddings of K_n, the complete graph on n vertices, can differ. In order to answer that question, a number of small surface trades containing four or six triangular faces were presented. In the current paper we give a definitive catalogue of such trades on up to six triangular faces and we identify those which potentially can form part of an orientable embedding and those which cannot.

For basic facts about graph embeddings, including their description by means of rotation schemes, we refer the reader to [5]. We assume throughout that G is a simple connected graph on n vertices, with vertex set V, embedded in a surface S. The surface may be orientable or non-orientable but we exclude from consideration pseudosurfaces (these result from a surface by making finitely many identifications of finite sets of points on that surface). We further assume that all the faces of the embedding are triangles. The embedding of G determines a *partial twofold triple system*, PTTS$(n) = (V, \mathcal{B})$, where $\mathcal{B}$ is the collection of triples of points of V formed by the vertices of the triangular faces; this has the property that every pair of points corresponding to an edge of G appears in precisely two triples (triangular faces of the embedding), but the edges of the complementary graph do not appear in any triple. When G is a complete graph K_n, the resulting PTTS(n) is known as a *twofold triple system*, TTS(n). To avoid needless repetition, it is convenient to regard a TTS(n) as a special case of a PTTS(n). A combinatorial trade on a PTTS(n) may be defined as follows.

Suppose that T_1 and T_2 are disjoint sets of triples taken from a finite base set U. If every pair of points of U occurs in the triples of T_1 with precisely the same multiplicity (0, 1 or 2) with which it appears in the triples of T_2, then the pair $\mathcal{T} = \{T_1, T_2\}$ is called a (combinatorial) *trade*. The *volume* of the trade $\mathcal{T}$, $vol(\mathcal{T})$, is the common cardinality of T_1 and T_2, and the *foundation* of the trade

$\mathcal{T}$, $found(\mathcal{T})$, is the set of points of U which appear amongst the triples of T_1 (or T_2).

The point of the foregoing definition is that if $P_1 = (V, \mathcal{B}_1)$ is a PTTS(n) whose triples include those of T_1, then by replacing these triples with those of T_2, we form another PTTS(n), $P_2 = (V, \mathcal{B}_2)$ say, and the triples of P_1 and P_2 cover exactly the same pairs of points from V with the same multiplicities.

Now consider the effect of making a trade on an embedding. Suppose that M_1 is a triangular embedding of the simple connected graph G in some surface S and that $P_1 = (V, \mathcal{B}_1)$ is the associated PTTS(n). Further suppose that $\mathcal{T} = \{T_1, T_2\}$ is a trade with $found(\mathcal{T}) \subseteq V$ and that $T_1 \subseteq \mathcal{B}_1$. Put $\mathcal{B}_2 = (\mathcal{B}_1 \setminus T_1) \cup T_2$, so that $P_2 = (V, \mathcal{B}_2)$ is a PTTS(n) covering all the edges of G precisely twice and no other pairs from V. If we now regard the triples from $\mathcal{B}_2$ as triangular faces and sew these faces together along the common edges, then this operation may or may not result in an embedding M_2 of G; the reason that the process may fail to yield an embedding is that the sewing operation may yield a pseudosurface. However, when the operation succeeds in producing a surface embedding, then we say that $\mathcal{T}$ forms a *surface trade* on the embedding M_1 of the graph G.

A variety of interesting questions may be posed concerning trades and embeddings. For example, does every combinatorial trade on a PTTS(n) yield at least one surface trade? Is it possible to characterize those combinatorial trades which, no matter how they lie on the surface, always transform a surface embedding into a surface embedding (rather than into a pseudosurface embedding)? Which surface trades are guaranteed to preserve orientability? How many different surface trades with foundation less than n must a triangular embedding of K_n possess? And if $b = b(n)$ denotes the minimum integer such that any two triangular embeddings of K_n may be transformed into one another by a trade of volume at most b, how does b vary with n? In order to make progress with such questions it is helpful to have a catalogue of small surface trades and to conduct some preliminary investigation of their properties. This is the purpose of the current paper.

Apart from the trivial case $G = K_3$, no triangular embedding of a simple connected graph G can give rise to a PTTS(n) with a repeated triple. Furthermore, in this trivial case, it is clear that no trade exists. We may therefore assume that $G \neq K_3$, and that the associated PTTS(n) does not contain any repeated triples. We consider here the case of trades $\mathcal{T}$ on PTTS(n)s with $vol(\mathcal{T}) \leq 6$. Up to isomorphism, there are precisely five such trades, one having $vol(\mathcal{T}) = 4$ and the other four having $vol(\mathcal{T}) = 6$. These five trades are all given in [6], where it is shown that there are no further trades $\mathcal{T} = \{T_1, T_2\}$ having $vol(\mathcal{T}) \leq 6$ except possibly for trades with foundation sizes 8 or 9 having at least one repeated pair of points occurring amongst the triples of T_1 (and T_2). However, these additional possibilities are easily excluded as follows.

Let n_i denote the number of points of T_1 having multiplicity $i > 0$. For a trade to exist, we must have $n_1 = 0$. If $|found(\mathcal{T})| \geq 8$ and $vol(\mathcal{T}) \leq 5$ then, by counting points and the occurrences of points in triples, we obtain $\sum_{i \geq 2} n_i \geq 8$ and $\sum_{i \geq 2} i n_i \leq 15$, which is clearly impossible. So we only have to consider the cases when $vol(\mathcal{T}) = 6$ and $|found(\mathcal{T})| = 8$ or 9.

If $|found(\mathcal{T})| = 9$ then $\sum_{i=2}^{6} n_i = 9$ and $\sum_{i=2}^{6} in_i = 18$, giving $n_i = 0$ for $i \geq 3$ and $n_2 = 9$. But if the repeated pair is $\{x, y\}$ then T_1 contains triples $\{x, y, a\}$ and $\{x, y, b\}$ and no further triples containing x or y. Consequently no disjoint set T_2 of triples covering the same pairs as T_1 can exist.

If $|found(\mathcal{T})| = 8$, then $\sum_{i=2}^{6} n_i = 8$ and $\sum_{i=2}^{6} in_i = 18$, giving $n_i = 0$ for $i \geq 5$ and $n_3 + 2n_4 = 2$. There are two numerical solutions to these equations given by $(n_2, n_3, n_4) = (7, 0, 1)$ or $(6, 2, 0)$. In the $(7, 0, 1)$ case, four triples of T_1 contain some point x and there must be another repeated point a amongst the eight occurrences of other points in these four triples. So T_1 contains triples $\{x, a, b\}$ and $\{x, a, c\}$ and no other triples containing a. As before, T_2 cannot exist. In the $(6, 2, 0)$ case, suppose that x and y are the two points which each occur in three triples of T_1 and that $S = \{a, b, c, d, e, f\}$ is the set of the remaining points. If a repeated pair contains at least one point of S, say a, then we have triples $\{a, \alpha, \beta\}$ and $\{a, \alpha, \gamma\}$ in T_1 and no other triples in T_1 containing a. So once again T_2 cannot exist. There remains only the possibility that T_1 has two triples $\{x, y, \alpha\}$ and $\{x, y, \beta\}$, and similarly that T_2 contains $\{x, y, \gamma\}$ and $\{x, y, \delta\}$, where $\alpha, \beta, \gamma, \delta \in S$. But then T_1 must also contain two further triples $\{x, \gamma, \delta\}$ and $\{y, \gamma, \delta\}$, and so the pair $\{\gamma, \delta\}$ is repeated and, by the earlier argument, this is not possible.

The five trades are listed below; for clarity and conciseness we omit commas and set brackets $\{,\}$ from triples, so that, for example, 624 is the triple $\{6, 2, 4\}$. The first three have common names as given. In each case T_1 is isomorphic with T_2.

1. (Pasch or quadrilateral trade) $T_1 = \{123, 145, 624, 635\}$,
   $T_2 = \{124, 135, 623, 645\}$.
2. (6-cycle trade) $T_1 = \{123, 145, 167, 834, 856, 872\}$,
   $T_2 = \{134, 156, 172, 823, 845, 867\}$.
3. (Semihead trade) $T_1 = \{127, 136, 145, 235, 246, 347\}$,
   $T_2 = \{126, 135, 147, 237, 245, 346\}$.
4. (Trade-X) $T_1 = \{123, 124, 156, 256, 345, 346\}$,
   $T_2 = \{125, 126, 134, 234, 356, 456\}$.
5. (Trade-Y) $T_1 = \{124, 125, 136, 137, 267, 345\}$,
   $T_2 = \{126, 127, 134, 135, 245, 367\}$.

Surface trades are not new. For example, in Figure 1 of [1] (which relates to triangulations of the projective plane), the pair {a, b} gives a geometrical realization of trade-X, the pair {c, d} a realization of a Pasch trade, and the pair {e, f} a realization of a semihead trade. However, in the current paper we examine each of the five combinatorial trades in turn and determine the precise geometrical circumstances in which a surface trade results.

2 Pasch trades

Consider the possibility of the triangular faces (defined by their vertex triples) 123, 145, 624, 635 of an embedding M being traded with the triangular faces 124, 135, 623, 645 to form an embedding M'. Initially we ignore the question of orientability. At the point 1, and up to reversal, there are two possibilities for the rotation in M, namely

(a) $1:\ 23\cdots45\cdots$ or

(b) $1:\ 23\cdots54\cdots$,

where $\cdots$ denotes undetermined sections of the rotation.

In M' there are faces 124 and 135, but in case (b) the partial rotations $4\cdots2$ and $3\cdots5$ preclude these unless these partial rotations are "empty", i.e. case (b) has the form $1:\ 2354$. In this case M also contains the faces 124 and 135, and so M' would have two copies of each of these faces. So we may exclude case (b). Applying similar reasoning at the other vertices shows that the (partial) rotations in M and in M' at the points $1, 2, \ldots, 6$ are, up to reversals, as follows:

M	M'
$1:\ 23\cdots45\cdots$	$1:\ 24\cdots35\cdots$
$2:\ 31\cdots64\cdots$	$2:\ 36\cdots14\cdots$
$3:\ 12\cdots56\cdots$	$3:\ 15\cdots26\cdots$
$4:\ 51\cdots62\cdots$	$4:\ 56\cdots12\cdots$
$5:\ 14\cdots36\cdots$	$5:\ 13\cdots46\cdots$
$6:\ 24\cdots35\cdots$	$6:\ 23\cdots45\cdots$

Table 2.1: (Partial) Pasch surface trade.

Next we consider the question of orientability. Assuming a consistent orientation of M and starting with $1:\ 23\cdots45\cdots$, we require $2:\ 31\cdots64\cdots$ and $4:\ 51\cdots62\cdots$. However, these give respectively $6:\ 42\cdots$ and $6:\ 24\cdots$, contradicting orientability. Therefore a consistent orientation of M (and similarly M') is not possible. Thus a surface trade based on the combinatorial Pasch trade is necessarily between nonorientable embeddings.

We have shown the necessity of Table 2.1 for the existence of a Pasch surface trade, but we have not demonstrated that such a trade exists. In order to do this, we make an observation which in fact applies to all the arrangements of facial triangles identified as potential surface trades in this paper; namely that there do indeed exist triangular embeddings containing these trades. That is to say, in this case, the partial rotation schemes M and M' shown in Table 2.1 may be completed to form a triangular embedding of some simple connected graph G, with similar completions in the other cases. To show this, take the rows of the partial rotation scheme for M with the undetermined sections eliminated and then determine any resulting non-triangular faces. From each such face, eliminate multiple vertices (if any) by the insertion of additional triangles involving new faces as illustrated below in Figure 2.1, where the twice repeated vertex x is eliminated from the face F by the insertion of new vertices x_1 and x_2.

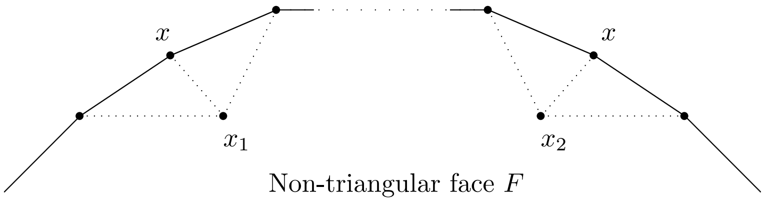

Figure 2.1: Eliminating multiple vertices from face F.

Having completed this elimination, for a non-triangular face without multiple vertices, insert a new vertex into the interior of that face and join it by non-intersecting edges to all the vertices on the boundary, thereby forming a triangular embedding of some simple connected graph.

Application of this algorithm to the case of the Pasch trade given in Table 2.1 give the rotations M and M' as shown below in Table 2.2

M		M'	
$1:$	$23x45y$	$1:$	$24x35y$
$2:$	$31y64z$	$2:$	$36y14z$
$3:$	$12z56x$	$3:$	$15z26x$
$4:$	$51x62z$	$4:$	$56x12z$
$5:$	$14z36y$	$5:$	$13z46y$
$6:$	$24x35y$	$6:$	$23x45y$
$x:$	1364	$x:$	1364
$y:$	1265	$y:$	1265
$z:$	2354	$z:$	2354

Table 2.2: Example of a Pasch surface trade.

In general, it is clear that this algorithm will preserve orientability in the sense that if a partial rotation scheme is potentially orientable, then the resulting triangular embedding M will be orientable. This does not however ensure that the traded embedding M' is orientable. We examine this aspect for potentially orientable partial rotation schemes as these arise. It is always possible to render both M and M' nonorientable by gluing on a nonorientable triangular embedding which shares a common face with M and M'.

3 6-cycle trades

Consider the possibility of the triangular faces 123, 145, 167, 834, 856, 872 of an embedding M being traded with the triangular faces 134, 156, 172, 823, 845, 867 to form an embedding M'. Initially we ignore the question of orientability. At the point 1, and up to reversal, there are eight possibilities for the rotation in M. These are all of the form $1:\ 23\cdots ab\cdots cd\cdots$, where $\{\{a,b\},\{c,d\}\} = \{\{4,5\},\{6,7\}\}$. Arguing as in the Pasch case, four of the eight possibilities may be excluded to leave the remaining four:

(1a) $1:\ 23\cdots 54\cdots 76\cdots$,

(1b) $1:\ 23\cdots 67\cdots 45\cdots$,

(1c) $1:\ 23\cdots 67\cdots 54\cdots$,

(1d) $1:\ 23\cdots 76\cdots 45\cdots$.

We then find that the permutations (2 4 6)(3 5 7) and (2 6 4)(3 7 5) preserve the six specified faces of M (and of M') and respectively map case (1a) to case (1c) and to case (1d). So, up to isomorphism, we may assume that the rotation at the point 1 in M has one of the forms (1a) or (1b).

Similarly, the possible rotations at the point 8 in M are:

(8a) $8:\ 34\cdots 65\cdots 27\cdots$,

(8b) $8:\ 34\cdots 72\cdots 56\cdots$,

(8c) $8:\ 34\cdots 72\cdots 65\cdots$,

(8d) $8:\ 34\cdots 27\cdots 56\cdots$.

So there are eight possible combinations of the rotation at 1 and the rotation at 8 in M. The permutation (2 3)(4 7)(5 6) applied to (1a, 8a) gives (1a, 8d), and applied to (1b, 8a) gives (1b, 8d). The permutation (1 8)(2 3 4 5 6 7) applied to (1a, 8b) gives (1b, 8a), and the permutation (1 8)(2 6)(3 5) applied to (1a, 8b) gives (1b, 8c). Therefore, up to isomorphism there are at most four combinations of rotations at the points 1 and 8 in M. These are (1a, 8a), (1a, 8b), (1a, 8c) and (1b, 8b). Observe that the patterns of partial rotation sections is different in each of these cases. In the (1a, 8a) case these sections are $3\cdots 5$, $4\cdots 7$, $6\cdots 2$, $4\cdots 6$, $5\cdots 2$ and $7\cdots 3$, so that no section $\alpha\cdots\beta$ is repeated. In the (1a, 8b) case there is exactly one repeated section (ignoring direction). In the (1a, 8c) case there are three repeated sections (ignoring direction) but it is not possible to obtain a consistent direction for all three. In the (1b, 8b) case there are again three repeated sections and a consistent direction can be obtained. It follows that the four cases are nonisomorphic.

Next consider the rotations at the points $2, 3, \dots, 7$. The possibilities at the point 2 in M are

$$2:\ 31\cdots 87\cdots \quad \text{or} \quad 2:\ 31\cdots 78\cdots,$$

and these must trade to the rotations in M'

$$2: \ 38\cdots17\cdots \quad \text{or} \quad 2:38\cdots71\cdots.$$

By the same argument given for Pasch trades, the only possibility is $2: \ 31\cdots87\cdots$ in M trading to $2: \ 38\cdots17\cdots$ in M'. Similar arguments apply at the points 3, 4, 5, 6 and 7. So we have the following table.

M	M'
$2: \ 31\cdots87\cdots$	$2: \ 38\cdots17\cdots$
$3: \ 48\cdots12\cdots$	$3: \ 41\cdots82\cdots$
$4: \ 51\cdots83\cdots$	$4: \ 58\cdots13\cdots$
$5: \ 68\cdots14\cdots$	$5: \ 61\cdots84\cdots$
$6: \ 71\cdots85\cdots$	$6: \ 78\cdots15\cdots$
$7: \ 28\cdots16\cdots$	$7: \ 21\cdots86\cdots$

Table 3.1: Rotations at points $2, 3, \ldots, 7$ for 6-cycle surface trades.

These rotations must be combined with each of the four possibilities for the points 1 and 8 to give four nonisomorphic forms for a 6-cycle surface trade.

It is easy to check that each of the isomorphism classes in M trades to the same isomorphism class in M'. Furthermore, in an orientable surface the partial directed rotation $1: \ 23\cdots54\cdots76\cdots$ implies $3: 12\cdots$ and $4: \ 15\cdots$. In cases (1a, 8x) with x = a, b or c, if M is orientable this gives $3: 12\cdots48\cdots$ and $4: \ 15\cdots38\cdots$, contradicting the orientability of the triangular face 348. So the only possibility for an orientable 6-cycle surface trade is (1b, 8b).

Examples of each of the four 6-cycle surface trades may be constructed using the algorithm described in Section 2. In the (1b, 8b) case an example with both M and M' orientable may be constructed by taking M to be the well-known triangular embedding of the complete tripartite graph $K_{3,3,3}$ in a torus. A rotation scheme for M with tripartition $\{\{1,8,9\},\{2,4,6\},\{3,5,7\}\}$ is:

$1: \ 236745$	$2: \ 315879$	$3: \ 486129$
$8: \ 347256$	$4: \ 517839$	$5: \ 682149$
$9: \ 276543$	$6: \ 713859$	$7: \ 284169$

It is easy to derive the embedding M' and to verify that both M and M' are orientable. An example of the (1b, 8b) case with M orientable and M' non-orientable is given in [4], pages 158-160.

4 Semihead trades

Note firstly that in addition to the trade $\mathcal{T} = \{T_1, T_2\}$ given in the Introduction, there exists a second trade involving T_1, namely $\mathcal{T}^* = \{T_1, T_3\}$ where $T_3 = \{125, 137, 146, 236, 247, 345\}$. However, the permutation (1 2)(5 6) provides an isomorphism between $\mathcal{T}$ and $\mathcal{T}^*$, and so it suffices to consider only the trade $\mathcal{T}$. Therefore consider the possibility of the triangular faces 127, 136, 145, 235, 246, 347 of an embedding M being traded with the triangular faces 126, 135, 147, 237, 245, 346 to form an embedding M'. Initially we ignore the question of orientability. As in the 6-cycle case and up to reversal, there appear at first to be eight possibilities for the rotation at the point 1 in M, but these are reduced to four by employing the usual argument. A similar situation occurs with the rotations at the points 2, 3 and 4. However, at each of the points 5, 6 and 7 we obtain a single possibility. Thus there are $4^4 = 256$ possibilities for the partial rotations at the points $1, 2, \ldots, 7$ in M. These may be represented as (1w, 2x, 3y, 4x) for w, x, y, z $\in$ {a, b, c, d} where the rotations at the points 1, 2, 3 and 4 are:

(1a) $1:\ 27\cdots36\cdots45\cdots,$	(2a) $2:\ 17\cdots46\cdots35\cdots,$
(1b) $1:\ 27\cdots36\cdots54\cdots,$	(2b) $2:\ 17\cdots46\cdots53\cdots,$
(1c) $1:\ 27\cdots63\cdots45\cdots,$	(2c) $2:\ 17\cdots64\cdots35\cdots,$
(1d) $1:\ 27\cdots54\cdots63\cdots,$	(2d) $2:\ 17\cdots53\cdots64\cdots,$
(3a) $3:\ 16\cdots25\cdots47\cdots,$	(4a) $4:\ 15\cdots37\cdots26\cdots,$
(3b) $3:\ 16\cdots25\cdots74\cdots,$	(4b) $4:\ 15\cdots37\cdots62\cdots,$
(3c) $3:\ 16\cdots52\cdots47\cdots,$	(4c) $4:\ 15\cdots73\cdots26\cdots,$
(3d) $3:\ 16\cdots74\cdots52\cdots,$	(4d) $4:\ 15\cdots62\cdots73\cdots,$

and the remaining ones are

$$5:\ 14\cdots32\cdots, \qquad 6:\ 13\cdots24\cdots, \qquad 7:\ 12\cdots43\cdots.$$

A computer analysis of the 256 possibilities shows that there are precisely 28 isomorphism classes. By saying this, we mean that there are 28 nonisomorphic geometrical arrangements of the six triangular faces on the surface M which permit surface trades. The list below gives a representative of each class in M and the number of the class to which it trades. One might legitimately regard a pair of classes such as (3, 6) as being isomorphic with the pair (6, 3), and if the reader takes this view then the number of isomorphism classes of trade pairs reduces from 28 to 19.

class #	representative	trades to	class #	representative	trades to
1	(1a, 2a, 3a, 4a)	1	15	(1a, 2b, 3d, 4d)	11
2	(1a, 2a, 3a, 4b)	2	16	(1a, 2c, 3c, 4c)	16
3	(1a, 2a, 3b, 4b)	6	17	(1a, 2c, 3c, 4d)	12
4	(1a, 2a, 3b, 4c)	7	18	(1a, 2c, 3d, 4d)	10
5	(1a, 2a, 3b, 4d)	5	19	(1a, 2d, 3d, 4d)	9
6	(1a, 2a, 3c, 4b)	3	20	(1b, 2b, 3b, 4b)	26
7	(1a, 2a, 3c, 4d)	4	21	(1b, 2b, 3b, 4c)	21
8	(1a, 2a, 3d, 4b)	8	22	(1b, 2b, 3b, 4d)	23
9	(1a, 2b, 3b, 4b)	19	23	(1b, 2b, 3c, 4b)	22
10	(1a, 2b, 3b, 4c)	18	24	(1b, 2b, 3c, 4d)	24
11	(1a, 2b, 3b, 4d)	15	25	(1b, 2b, 3d, 4b)	27
12	(1a, 2b, 3c, 4c)	17	26	(1b, 2c, 3b, 4b)	20
13	(1a, 2b, 3c, 4d)	13	27	(1b, 2c, 3c, 4b)	25
14	(1a, 2b, 3d, 4c)	14	28	(1b, 2c, 3c, 4c)	28

Examples of each of the 28 semihead surface trades may be constructed using the algorithm described in Section 2. Further analysis of these 28 classes using the same technique as in the earlier sections shows that the only one compatible with orientability is #16, namely (1a, 2c, 3c, 4c). An example of this trade with both M and M' orientable is given in Table 4.1 where both M and M' are realizations of the well-known toroidal triangular embedding of K_7.

M	M'
1 : 273645	1 : 264735
2 : 715346	2 : 615437
3 : 617425	3 : 517246
4 : 516237	4 : 716325
5 : 147632	5 : 136742
6 : 135724	6 : 127534
7 : 126543	7 : 145623

Table 4.1: Example of an orientable to orientable semihead surface trade.

A further example with M orientable and M' nonorientable is given by the following pair of rotation schemes.

M	M'
$1:\ 27wx3645$	$1:\ 2647wx35$
$2:\ 715346xwy$	$2:\ 615437ywx$
$3:\ 61x7425$	$3:\ 51x7246$
$4:\ 516237$	$4:\ 716325$
$5:\ 147632$	$5:\ 136742$
$6:\ 1357yx24$	$6:\ 12xy7534$
$7:\ 12y6543xzw$	$7:\ 1456y23xzw$
$w:\ 17zy2x$	$w:\ 17zy2x$
$x:\ 1w26yz73$	$x:\ 1w26yz73$
$y:\ 2wzx67$	$y:\ 276xzw$
$z:\ 7xyw$	$z:\ 7xyw$

Table 4.2: Example of an orientable to nonorientable semihead surface trade.

It is easy to check the orientability of M in Table 4.2. Orientability of M' with the rotation at the point 1 in the direction shown requires the rotation at the point 2 to also be in the direction shown; and we then have oriented triangles $1wx$ and $2wx$, contradicting orientability.

5 Trade-X

Consider the possibility of the triangular faces 123, 124, 156, 256, 345, 346 of an embedding M being traded with the triangular faces 125, 126, 134, 234, 356, 456 to form an embedding M'. Initially we ignore the question of orientability. At the point 1, and up to reversal, there are two possibilities for the rotation in M, namely (a) $1:\ 324\cdots56\cdots$, and (b) $1:\ 324\cdots65\cdots$. These trade respectively to rotations in M' given by (a) $1:\ 526\cdots34\cdots$, and (b) $1:\ 526\cdots43\cdots$. A similar situation occurs with the rotations at the remaining points 2, 3, 4, 5 and 6. Thus there are $2^6 = 64$ possibilities for the partial rotations at the points $1, 2 \ldots, 6$ in M. These may be represented as (1u, 2v, 3w, 4x, 5y, 6z) for u, v, w, x, y, z $\in$ {a, b}, where the rotations at the points $1, 2 \ldots, 6$ are:

(1a) $1:\ 324\cdots56\cdots$,
(1b) $1:\ 324\cdots65\cdots$,

(2a) $2:\ 314\cdots56\cdots$,
(2b) $2:\ 314\cdots65\cdots$,

(3a) $3:\ 546\cdots12\cdots$,
(3b) $3:\ 546\cdots21\cdots$,

(4a) $4:\ 536\cdots12\cdots$,
(4b) $4:\ 536\cdots21\cdots$,

(5a) $5:\ 162\cdots34\cdots$,
(5b) $5:\ 162\cdots43\cdots$,

(6a) $6:\ 152\cdots34\cdots$,
(6b) $6:\ 152\cdots43\cdots$.

A computer analysis of the 64 possibilities shows that there are precisely seven isomorphism classes, and the list below gives a representative of each class in M. Each of the isomorphism classes in M trades to the same class in M'.

class #	representative
1	(1a, 2a, 3a, 4a, 5a, 6a)
2	(1a, 2a, 3a, 4a, 5a, 6b)
3	(1a, 2a, 3a, 4a, 5b, 6b)
4	(1a, 2a, 3a, 4b, 5a, 6b)
5	(1a, 2a, 3a, 4b, 5b, 6a)
6	(1a, 2b, 3a, 4b, 5a, 6b)
7	(1a, 2b, 3a, 4b, 5b, 6a)

Examples of each of these seven surface trades may be constructed using the algorithm described in Section 2. It is easy to verify by hand that the only one of the seven compatible with orientability is #3, namely (1a, 2a, 3a, 4a, 5b, 6b). The effect of #3 on an orientable embedding M is to produce an *orientable* embedding M'. To see this consider the partial rotation schemes for M and M' which are shown in Table 5.1.

M	M'
$1:\ 324\cdots56\cdots$	$1:\ 526\cdots34\cdots$
$2:\ 413\cdots65\cdots$	$2:\ 615\cdots43\cdots$
$3:\ 645\cdots21\cdots$	$3:\ 241\cdots65\cdots$
$4:\ 536\cdots12\cdots$	$4:\ 132\cdots56\cdots$
$5:\ 261\cdots34\cdots$	$5:\ 364\cdots21\cdots$
$6:\ 152\cdots43\cdots$	$6:\ 453\cdots12\cdots$

Table 5.1: Potentially orientable surface trade-X.

Observe that each of the partial rotation sections $\alpha\cdots\beta$ at each point γ in M appears in the same direction at γ in M'. For example, $6\cdots3$ appears in the rotation at the point 1 in M and in M'. It then follows that if M is orientable, so is M'. An example of a complete rotation scheme M corresponding to case #3 of trade-X is the orientable embedding of K_{19} given in [4], pages 157-8. A smaller example is provided by K_7 in a torus, as shown in [3].

It is worth noting that trade-X has a particularly simple geometric form. The six triangular faces in M form three pairs, with the two triangles in each pair sharing a common edge. The trade is effected by performing three diagonal flips in which the common edges are firstly regarded as diagonals of quadrilaterals and are then replaced by the alternative diagonals. This geometric interpretation makes it clear that the trading operation can never result in a pseudosurface, and that it will preserve the orientability of an orientable embedding.

6 Trade-Y

Consider the possibility of the triangular faces 124, 125, 136, 137, 267, 345 of an embedding M being traded with the triangular faces 126, 127, 134, 135, 245, 367 to form an embedding M'. Initially we ignore the question of orientability. At the point 1, and up to reversal, there are two possibilities for the rotation in M, namely (a) $1: 425\cdots637\cdots$, and (b) $1: 425\cdots736\cdots$. However, the permutation (6 7) preserves the six specified faces of M (and of M') and maps case (a) to case (b). So, up to isomorphism, we may assume that the rotation at the point 1 in M has the form $1: 425\cdots637\cdots$. There are two alternative rotations at the points 2 and 3 in M, namely (2a) or (2b) , and (3a) or (3b) where

(2a) $2: 415\cdots67\cdots$,	(3a) $3: 617\cdots45\cdots$,
(2b) $2: 415\cdots76\cdots$,	(3b) $3: 617\cdots54\cdots$.

At each of the remaining points 4, 5, 6 and 7, the usual arguments give a single possibility as follows:

$4: 12\cdots35\cdots$,	$6: 13\cdots27\cdots$,
$5: 12\cdots34\cdots$,	$7: 13\cdots26\cdots$.

So there are four possible combinations of rotations at the points $1, 2\ldots, 7$ in M and these are defined by (2x, 3y) for x, y $\in$ {a, b}. The permutation (2 3)(4 6)(5 7) applied to (2a, 3b) gives (2b, 3a), and by checking the patterns of partial rotation schemes, as in earlier cases, it is easy to show that the three cases (2a, 3a), (2a, 3b) and (2b, 3b) are nonisomorphic. It is also easy to check, by the same method, that each of these three isomorphism classes in M trades to the same isomorphism class in M'.

The cases (2a, 3a) and (2a, 3b) cannot appear in an orientable embedding because the directed partial rotation $1: 425\cdots637\cdots$ then gives $2: 514\cdots76\cdots$ and $6: 31\cdots72\cdots$, contradicting the orientability of the triangle 276. However, the case (2b, 3b) is potentially orientable and an example of this trade with both M and M' orientable is given in Table 6.1 where both M and M' are again realizations of the toroidal triangular embedding of K_7.

M	M'
1 : 425637	1 : 627435
2 : 514673	2 : 716453
3 : 716452	3 : 514672
4 : 217536	4 : 317526
5 : 123476	5 : 132476
6 : 315724	6 : 215734
7 : 132654	7 : 123654

Table 6.1: Example of an orientable to orientable surface trade-Y.

A further example with M orientable and M' nonorientable is given by the following pair of rotation schemes.

M	M'
$1:\ 425w637x$	$1:\ 627x435w$
$2:\ 514y67$	$2:\ 716y45$
$3:\ 716zw45$	$3:\ 514wz67$
$4:\ 21x53wy$	$4:\ 31x52yw$
$5:\ 12734xw$	$5:\ 13724xw$
$6:\ 31w72yz$	$6:\ 21w73zy$
$7:\ 13526wzx$	$7:\ 12536wzx$
$w:\ 15xy43z76$	$w:\ 15xy43z76$
$x:\ 17zyw54$	$x:\ 17zyw54$
$y:\ 24wxz6$	$y:\ 24wxz6$
$z:\ 36yx7w$	$z:\ 36yx7w$

Table 6.2: Example of an orientable to nonorientable surface trade-Y.

It is easy to check the orientability of M in Table 6.2. Orientability of M' with the rotation at the point 1 in the direction shown requires the rotations at the points 3 and 7 to also be in the directions shown; and we then have directed rotations $w:\ z34\cdots$ and $w:\ z76\cdots$, contradicting orientability.

Remark
As noted in [3], a large family of trades may be formed from face 2-colourable triangulations. Take any such triangulation of a surface or a pseudosurface, and consider the two colour classes. Each of the two resulting sets of triples covers precisely the same pairs and these sets therefore form a combinatorial trade. For example, the Pasch trade corresponds to a face 2-colourable triangulation of an octahedron. The 6-cycle and semihead trades have similar representations. In fact, any trade $\mathcal{T} = \{T_1, T_2\}$ on a PTTS in which T_1 (and hence also T_2) has no repeated pairs may be represented in this way.

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