Orientable biembeddings of cyclic Steiner triple systems from current assignments on Möbius ladder graphs

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Abstract

We give a characterization of a current assignment on the bipartite Möbius ladder graph with $2n+1$ rungs. Such an assignment yields an index one current graph with current group $\mathbb{Z}_{12n+7}$ that generates an orientable face 2-colorable triangular embedding of the complete graph K_{12n+7} or, equivalently, an orientable biembedding of two cyclic Steiner triple systems of order $12n+7$. We use our characterization to construct Skolem sequences that give rise to such current assignments. These produce many nonisomorphic orientable biembeddings of cyclic Steiner triple systems of order $12n+7$.

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1 Introduction

A Steiner triple system of order m, briefly STS(m), is an ordered pair $(V, \mathcal{B})$, where V is an m-element set (the *points*) and $\mathcal{B}$ is a set of 3-element subsets of V (the *blocks*) such that each 2-element subset of V appears in precisely one block. A necessary and sufficient condition for the existence of an STS(m) is that $m \equiv 1$ or $3 \pmod 6$. In this paper, without loss of generality, by a *cyclic* STS(m) we mean a system whose point set is $\mathbb{Z}_m$, and which has the property that if $\{x, y, z\}$ is a block then, for each $\tau \in \mathbb{Z}_m$, $\{x+\tau, y+\tau, z+\tau\}$ is also a block. A cyclic STS(m) exists for $m \equiv 1$ or $3 \pmod 6$ except for $m = 9$.

In a triangular embedding of a graph, a face may be described by specifying the triple (x, y, z) of vertices incident with that face. Given a face 2-colorable triangular embedding of the complete graph K_m in an orientable or nonorientable surface, the triangular faces in each of the two color classes form the blocks of two STS(m)s. Conversely, given two STS(m)s, say S_1 and S_2, one may ask whether there is a face 2-colorable triangular embedding of K_m in which the color classes correspond to isomorphic copies of S_1 and S_2. Such an embedding is called a *biembedding* of S_1 and S_2 and is described as orientable or nonorientable according as the nature of the surface. Euler's formula shows that an orientable biembedding of STS(m)s is only possible if $m \equiv 3$ or $7 \pmod{12}$.

Recent computational results [3, 4, 7] show that there is a reasonable evidence to support the following two deep conjectures:

(a) Every pair of STS(m)s, $m \equiv 1$ or $3 \pmod 6$ and $m \geq 9$, can be biembedded in a nonorientable surface.

(b) Each STS(m), $m \equiv 3$ or $7 \pmod{12}$ and $m \geq 3$, has a biembedding in an orientable surface.

In the present paper, bearing in mind conjecture (b), we study how to generate orientable biembeddings of cyclic STS($12n+7$)s.

Given two triangular embeddings f and f' of K_n, if ψ is a bijection between the vertices of K_n such that (x, y, z) is a face of f if and only if $(\psi(x), \psi(y), \psi(z))$ is a face of f', then ψ is said to be an *isomorphism* from f to f', and the two embeddings are said to be isomorphic. If $f = f'$ then such a mapping ψ is called an *automorphism* of the embedding.

Many of the results on orientable biembeddings of STS($12n+7$)s are results for small values of n, and were obtained by computer searches. The only known ways to obtain orientable face 2-colorable triangular embeddings of K_{12n+7} for unboundedly large n are to use recursive constructions [8, 9, 10], or to construct bipartite index one current graphs with the current group $\mathbb{Z}_{12n+7}$. This latter method, as described by Youngs in [14], gives orientable biembeddings of cyclic STS($12n+7$)s, and the biembeddings themselves have a cyclic automorphism of order $12n+7$. However, the construction of such current graphs is difficult and there has been no real progress since Youngs' paper. We also remark that any cyclic STS($12n+3$) has a short orbit generated by the triple $\{0, 4n+1, 8n+2\}$, and this precludes a similar construction using a bipartite index one current graph. This is because the biembedded systems would then have common blocks and so could not form an embedding in a surface. For this reason we concentrate here on biembeddings of STS($12n+7$)s.

By a ($12n+7$)-*current assignment* we mean a current assignment with current group $\mathbb{Z}_{12n+7}$ on the bipartite Möbius ladder graph with $2n+1$ rungs such that by choosing a suitable rotation of the graph we can obtain an index one current graph with current group $\mathbb{Z}_{12n+7}$ generating an orientable face 2-colorable triangular embedding of K_{12n+7}. Youngs constructed a ($12n+7$)-current assignment for every $n \geq 1$, thereby establishing the existence of an orientable biembedding of two cyclic STS($12n+7$)s for every $n \geq 1$.

The aim of the present paper is to characterize ($12n+7$)-current assignments and to use this characterization to construct many different ($12n+7$)-current assignments. It is then possible to construct many nonisomorphic orientable biembeddings of cyclic STS($12n+7$)s. Each of the biembeddings produced by this method has itself a cyclic automorphism of order $12n+7$. The characterization gives some insight into why it has been so difficult to construct ($12n+7$)-current assignments: each ($12n+7$)-current assignment is associated with a partition of $\mathbb{Z}_{12n+7}$ that has a property, which we call skewness, that seems to be hard to satisfy.

In Section 2 we describe how a bipartite index one current graph with current group $\mathbb{Z}_{12n+7}$ determines an orientable biembedding of two cyclic STS($12n+7$)s. A characterization of ($12n+7$)-current assignments is given in Section 3. The characterization has the following form. We define a class of so-called skew centered systems of order $12n+7$, each of which is a collection of ordered triples of elements of $\mathbb{Z}_{12n+7}$. We show in Theorem 2 that every skew centered system determines a ($12n+7$)-current assignment and that every $(12n+7)$-current assignment is determined by a skew centered system. This enables us in Theorem 3 to describe all orientable biembeddings of cyclic STS$(12n+7)$s which can be obtained from $(12n+7)$-current assignments.

In Section 4 we show how some $(12n+7)$-current assignments are generated by a class of Skolem sequences that we call skew. A Skolem sequence of order n, $n \equiv 0$ or $1 \pmod 4$, is a sequence $\{s_1, s_2, \ldots, s_{2n}\}$ of $2n$ integers satisfying the following two conditions.

(i) For each $k \in \{1, 2, \ldots, n\}$ there are precisely two elements of the sequence, say s_i and s_j, such that $s_i = s_j = k$.

(ii) If $s_i = s_j = k$ and $i < j$ then $j - i = k$.

There are several equivalent definitions of Skolem sequences and we use a pictorial representation that is appropriate for our purposes in Section 4. It was shown by Abrham [1] that the number of Skolem sequences of order n is at least $2^{\lfloor n/3 \rfloor}$, and computational evidence [2] suggests that for sufficiently large n there are at least Ab^n Skolem sequences of order n, where $A > 0$ and $b > 6$. We prove in Theorem 4 that, for a certain infinite set of values of n, there is a positive constant B such that the number of skew Skolem sequences of order n is at least $Bn^{\frac{1}{\log_2 9}}$, although we suspect that, for large values of $n \equiv 1 \pmod 4$, a substantial proportion of Skolem sequences of order n are in fact skew. Finally, in Theorem 5 these skew Skolem sequences are used to show that for certain values of n there are at least $C2^{4n}n^{\frac{1}{log_2 9}-1}$ (where $C > 0$ is a constant) nonisomorphic orientable biembeddings of cyclic STS$(24n+7)$s. We also observe that, by using Youngs' construction and employing results from [12], it can be shown that for every $n \geq 1$, there are at least $\frac{1}{4n+1}2^{4n-2}$ nonisomorphic orientable biembeddings of cyclic STS$(24n+7)$s.

2 Current graphs and orientable biembeddings of cyclic STS$(12n+7)$s

In this section we describe how a bipartite index one current graph with current group $\mathbb{Z}_{12n+7}$ determines an orientable biembedding of two cyclic STS$(12n+7)$s.

By a *generic triple* on $\mathbb{Z}_m$ we mean a cyclically ordered triple (β, γ, δ) of distinct nonzero elements of $\mathbb{Z}_m$ such that $\beta + \gamma + \delta = 0$. Generic triples are closely related to Heffter difference triples; see for example [6]. A generic triple (β, γ, δ) on $\mathbb{Z}_m$ *induces* an orbit of m blocks $\{\{x, x+\beta, x+\beta+\gamma\}$: $x = 0, 1, \ldots, m-1\}$. It is easy to see that the orbit is well-defined in that it does not depend on the choice of two consecutive elements β, γ in the cyclic triple (β, γ, δ), that the 2-element subsets contained in the m blocks are exactly all the 2-element subsets $\{y, z\}$ such that $y - z \in \{\pm\beta, \pm\gamma, \pm\delta\}$, and that each of these subsets appears in the blocks exactly once. Generic triples (β, γ, δ) and $(-\delta, -\gamma, -\beta)$ induce the same orbit.

By a *generic triple system* of order $12n + 7$, GTS$(12n + 7)$ for short, we mean a collection $\{(\beta_i, \gamma_i, \delta_i) : i = 1, 2, \ldots, 2n+1\}$ of generic triples on $\mathbb{Z}_{12n+7}$ such that the sets $\{\beta_i, \gamma_i, \delta_i\}$ are pairwise disjoint and their union contains exactly one element from every pair of inverse nonzero elements of $\mathbb{Z}_{12n+7}$. It is easy to see that the blocks induced by all generic triples of a GTS$(12n+7)$ form a cyclic STS$(12n + 7)$. This cyclic STS is said to be induced by the GTS. Indeed, for every two elements x and y of $\mathbb{Z}_{12n+7}$, exactly one of the two elements $x - y$ and $y - x$ appears in some generic triple of the GTS$(12n + 7)$ and this generic triple induces exactly one block containing the two elements.

Two GTS$(12n + 7)$s, F_1 and F_2, are *switch equivalent* if for every generic triple (β, γ, δ) of F_1, F_2 contains either (β, γ, δ) or $(-\delta, -\gamma, -\beta)$. To find a generic triple (β, γ, δ) inducing blocks $\{x + \tau, y + \tau, z + \tau\}$, $\tau = 0, 1, \ldots,$ $12n + 6$, of a cyclic STS we must represent the block $\{x, y, z\}$ as $\{c, c + \beta,$ $c + \beta + \gamma\}$. There are exactly two different cyclic orderings of the elements of $\{x, y, z\}$, namely (x, y, z) and (x, z, y). Hence there are exactly two different generic triples inducing the blocks, namely $(\beta', \gamma', \delta')$ and $(-\delta', -\gamma', -\beta')$, where $\beta' = y - x$, $\gamma' = z - y$ and $\delta' = x - z$. Consequently, every cyclic STS$(12n+7)$ is induced by exactly one GTS$(12n+7)$ up to switch equivalence.

Now we briefly review some material about index one current graphs in the form used in the paper. The reader is referred to [11, 13] for a more detailed development of the material sketched herein. We assume that the reader is familiar with current graphs, derived graphs and the derived embeddings generated by current graphs.

Let G be a connected trivalent graph with vertex set $V(G)$ and edge set $E(G)$. Each edge $e \in E(G)$ gives rise to two reverse arcs e^+ and e^-, and the set of all arcs, $A(G)$, is called the *arc set* of the graph G. By a *current assignment* λ on G with current group $\mathbb{Z}_{12s+7}$, we mean a function $\lambda : A(G) \to \mathbb{Z}_{12n+7} \setminus \{0\}$ such that $\lambda(e^-) = -\lambda(e^+)$ for every edge e. A *vertex rotation* D_v , $v \in V(G)$, is a cyclic permutation of the three arcs directed from the vertex v. A set of such rotations $D = \{D_v : v \in V(G)\}$ is called a *rotation of* G. A triple $\langle G, \lambda, D\rangle$ is called a *current graph*. It may be represented in a diagram of G with vertex rotations indicated by coloring the vertices so that black vertices denote a clockwise rotation, and white vertices an anticlockwise rotation. Each pair of reverse arcs is represented by one of the arcs with the current indicated.

Alternately applying D and the arc-reversing involution, we obtain a cycle (consisting of arcs of $A(G)$) called a *circuit* induced by the rotation of D. A current graph $\langle G, \lambda, D\rangle$ is said to be *index one* if D induces a single circuit containing all the arcs in $A(G)$. If (a_1, a_2, a_3) is the rotation of some vertex $v \in V(G)$ and $\lambda(a_i) = \varepsilon_i$ for $i = 1, 2, 3$, then the cyclically ordered triple $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ is called the *current rotation* at the vertex v. If $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$, then we say that Kirchhoff's Current Law (KCL) is satisfied at v.

Consider an index one current graph $\langle G, \lambda, D\rangle$ with current group $\mathbb{Z}_{12n+7}$ that satisfies the following conditions (A1 and A2).

(A1) The currents on the arcs of the current graph are pairwise distinct and form the set of all nonzero elements of the current group $\mathbb{Z}_{12n+7}$.

(A2) KCL is satisfied at each vertex.

It is shown in [11, 13] that such a current graph generates an orientable triangular embedding of K_{12n+7}. The elements of $\mathbb{Z}_{12n+7}$ give the vertices of the embedding. The rotation at 0 in the embedding is obtained from the single circuit induced by D by listing the currents assigned to the arcs in the order in which they appear in the circuit. The rotation at $x \in \mathbb{Z}_{12n+7}$ is then obtained by adding x (modulo $12n+7$) to all the entries in the rotation at 0. A vertex in the current graph with current rotation $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ induces $12n+7$ triangular faces of the embedding with vertex sets $\{x, x+\varepsilon_1, x+\varepsilon_1+\varepsilon_2\}$ for $x = 0, 1, \ldots, 12n+6$.

For every two vertices v and w of the current graph, some face induced by v shares a common edge with some face induced by w if and only if v and w are adjacent vertices. Hence, if the current graph is bipartite, then the derived embedding is face 2-colorable and the faces induced by the vertices of one vertex partition class all lie in the same color class.

If we consider the current rotation of a vertex as a generic triple, then the vertex sets of the faces induced by the vertex are exactly the blocks generated by the generic triple. Hence we have the following result.

(B) Let V and V' be the two vertex partition classes of a bipartite index one current graph with current group $\mathbb{Z}_{12n+7}$ satisfying (A1) and (A2). Then the set of current rotations of the vertices of V is a GTS$(12n+7)$ inducing a cyclic STS$(12n+7)$ that is orientably biembeddable with a cyclic STS$(12n+7)$ induced by the current rotations of the vertices from V'.

For a current rotation $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, the *reverse* current rotation is $(\varepsilon_1, \varepsilon_3, \varepsilon_2)$; a current rotation and the reverse current rotation considered as generic triples induce different blocks. Given a vertex with a current rotation, reversing the rotation of the vertex yields the reverse current rotation. In the proof of Theorem 3, given a bipartite index one current graph with two vertex partition classes V and V', the set of current rotations of the vertices of each vertex partition class is a GTS$(12n+7)$ inducing an orientably embedded cyclic STS$(12n+7)$. We will show that if we reverse the rotations of arbitrarily chosen vertices of V, then the rotations of the vertices of V' can be chosen in such a way that the resulting rotation of the graph induces exactly one circuit. As a consequence, if we reverse arbitrarily chosen generic triples of the GTS$(12n+7)$ corresponding to the current rotations of the vertices of V, we obtain a new GTS$(12n+7)$ also inducing an orientably embedded cyclic STS$(12n+7)$. Here we will need Lemma 1.

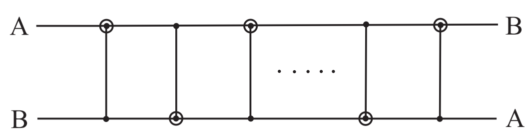

Figure 1: The graph ML$(2n+1)$ with $2n+1$ vertical edges.

Consider the bipartite graph with $2n+1$ vertical edges shown in Fig. 1 with the ends labelled by the same letter, A or B, identified. This graph is called the Möbius ladder graph with $2n+1$ rungs and is denoted by ML$(2n+1)$. The circled vertices form one of the two vertex partition classes of this graph. For a circled vertex, the *opposite vertex* is an uncircled vertex such that the both vertices are the end vertices of the same vertical edge.

Lemma 1 *For every choice of rotations of the circled vertices in Fig. 1, there are rotations of the uncircled vertices such that the resulting rotation of the graph induces exactly one circuit.*

Proof. Suppose rotations of the circled vertices are given. Then for each circled vertex, if the rotation of that vertex is clockwise (respectively, anticlockwise), choose the rotation of the opposite vertex to be anticlockwise (respectively, clockwise). It is easy to check the following: the resulting rotation of the graph induces exactly three circuits; there is a vertical edge e traversed by two different circuits; each of the end vertices of e lies in all three circuits. Now reverse the rotation of the uncircled vertex incident with e. As a result, the three circuits are "sewed" into one circuit. ■

3 Skew centered systems and orientable biembeddings of cyclic STS$(12n+7)$s

In this section we define a class of skew centered systems of order $12n+7$ each of which is a collection of ordered triples of elements of $\mathbb{Z}_{12n+7}$. We show in Theorem 2 that every skew centered system determines a current assignment on the graph ML$(2n+1)$ satisfying (A1) and (A2), and that every such current assignment is determined by a skew centered system. The elements of the ordered triples of a skew centered system A of order $12n+7$ form 3-element subsets $T_1, T_2, \dots, T_{2n+1}$ of $\mathbb{Z}_{12n+7}$. In the current assignment determined by A there is a vertex partition class of the graph ML$(2n+1)$ with the property that for every $T_i = \{\beta, \gamma, \delta\}$, there is a vertex of the partition class such that the set of currents on the arcs directed from the vertex is either $\{\beta, \gamma, \delta\}$ or $\{-\beta, -\gamma, -\delta\}$. Taking into account Lemma 1, we establish Theorem 3 that describes all orientable biembeddings of cyclic STS$(12n+7)$s that can be obtained from current assignments on the graph ML$(2n+1)$ satisfying (A1) and (A2).

By a *centered system* of order $12n+7$, denoted by CS$(12n+7)$, we mean a collection $\{\langle\beta_i, \gamma_i, \delta_i\rangle : i = 1, 2, \dots, 2n+1\}$ of ordered triples of elements of $\mathbb{Z}_{12n+7}$ such that:

(1) The sets $\{\beta_i, \gamma_i, \delta_i\}$ are pairwise disjoint and their union contains exactly one element from each pair of inverse nonzero elements of $\mathbb{Z}_{12n+7}$.

(2) $\beta_i + \gamma_i + \delta_i = 0$ for every i.

(3) There is an element $\Omega \in \mathbb{Z}_{12n+7}$ with the property that the set $\{\gamma_1, -\delta_1, \gamma_2, -\delta_2, \dots, \gamma_{2n+1}, -\delta_{2n+1}\}$ can be partitioned into pairs $\{x, y\}$ such that $x + y = \Omega$.

The sets $\{\beta_1, \beta_2, \ldots, \beta_{2n+1}\}$ and $\{\gamma_1, -\delta_1, \gamma_2, -\delta_2, \ldots, \gamma_{2n+1}, -\delta_{2n+1}\}$ are called the *label set* and the *base set*, respectively, of the CS$(12n+7)$. The element Ω is called the *pair sum* of the CS$(12n+7)$.

A CS$(12n+7)$ can be represented as a diagram in the following way. Let $c_1, c_2, \ldots, c_{4n+2}$ be the elements of the base set arranged in ascending order (in accordance with the linear order $0 < 1 < 2 < \cdots < 12n+6$). Place $4n+2$ vertices on a vertical line and label these vertices from top to bottom by the elements $c_1, c_2, \ldots, c_{4n+2}$, in that order. For every triple $\langle \beta, \gamma, \delta \rangle$ of the CS, draw a *diagram curve* with label β joining the vertices γ and $-\delta$ (note that $\beta + \gamma = -\delta$); the diagram curve represents the triple. When we say that a subset of diagram curves *covers* a subset $\mathcal{A}$ of the base set, we mean that $\mathcal{A}$ consists of all end vertices of the curves; such a nonempty subset $\mathcal{A}$ is called a *covered* subset. Figs. 2(a), (b), and (c) show three different CS(55)s; in each case the pair sum is 37 and the base set is $\{10, 11, \ldots, 27\}$.

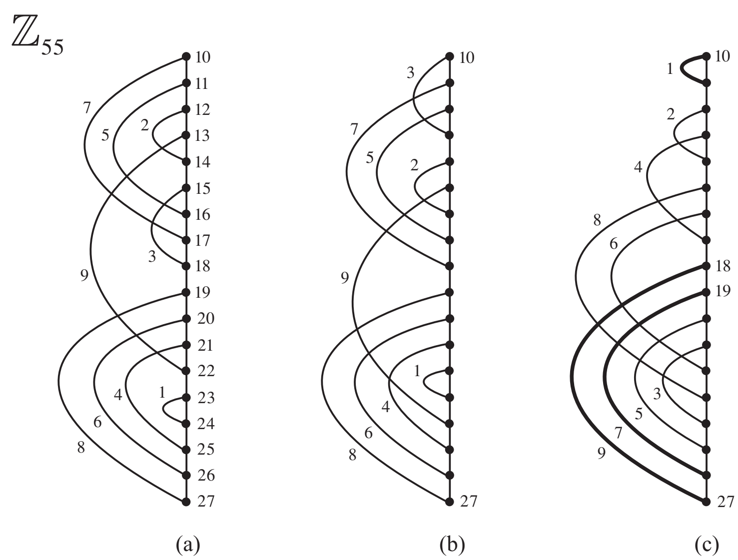

Figure 2: The diagrams of three different CS(55)s.

For each element x of the base set $\mathcal{H}$ denote by x^* the element from $\mathcal{H}$ such that $x+x^*$ equals the pair sum. For convenience, we will write x_i^* instead of $(x_i)^*$. By the definition of a CS$(12n+7)$, the base set does not contain two inverse elements of $\mathbb{Z}_{12n+7}$, hence $x \neq x^*$. A proper subset $\mathcal{M} \subset \mathcal{H}$ is *symmetrical* if $x \in \mathcal{M}$ implies $x^* \in \mathcal{M}$.

A CS$(12n+7)$ is *skew* if the base set does not contain a symmetrical covered proper subset; in other words, no proper subset of the set of the diagram curves covers a symmetrical subset. The CS(55)s in Figs. 2(a) and (b) are skew. The CS(55) in Fig. 2(c) is not skew; the end vertices of the thick curves form a symmetrical subset.

Starting with the diagram of a CS($12n+7$), we construct two new pictorial representations that will enable us easily to recognize whether or not the system is skew. Firstly, the *double diagram* of the CS($12n+7$) is formed by taking each diagram curve with end vertices, say v and w, and adding a new curve joining the vertices v^* and w^*. Note that if $\langle \beta, \gamma, \delta \rangle$ is a triple of the CS($12s+7$) then the corresponding new curve joins the vertices γ^* and $(-\delta)^*$. Secondly, the *auxiliary diagram* of the CS($12n+7$) is formed by taking each pair of vertices x and x^*, and adding a new curve joining these two vertices. In each of these diagrams (double or auxiliary) there are exactly $4n+2$ curves joining vertices of the base set, and every vertex of the base set is incident with exactly two curves. Hence, the curves form *diagram cycles* of even length in the corresponding diagram. In what follows, when we speak about a diagram cycle, it will be clear from the context which diagram, double or auxiliary, is meant. Fig. 3(a) (respectively, (b)) shows the cycles of the double (respectively, auxiliary) diagram constructed for the CS(55) shown at the left of Fig. 2. A diagram cycle may be described by the cyclic sequence $(x_1, x_2, \ldots, x_{2t})$ of vertices obtained by listing the incident vertices when traversing the cycle in some chosen direction. The sequences $(x_1, x_2, \ldots, x_{2t})$ and $(x_{2t}, \ldots, x_2, x_1)$ designate the same diagram cycle.

Subsequently, when speaking about curves of the double or auxiliary diagram Γ of a CS($12n+7$), only the diagram curves of the CS($12n+7$) entering into Γ will be referred to as diagram curves.

Theorem 1 (a) *A CS*($12n+7$) *is skew if and only if the double diagram of the CS*($12n+7$) *has exactly one cycle.*

(b) *A CS*($12n+7$) *is skew if and only if the auxiliary diagram of the CS*($12n+7$) *has exactly one cycle.*

Proof. Taking into account the definition of a skew CS($12n+7$), it suffices to show that given a diagram of a CS($12n+7$), the base set $\mathcal{H}$ contains a symmetrical covered proper subset if and only if the double (respectively, auxiliary) diagram has at least two cycles.

Suppose that $\mathcal{A}$ is a symmetrical covered proper subset of $\mathcal{H}$. Then, in the double (respectively, auxiliary) diagram, for every curve incident with a vertex of $\mathcal{A}$, both end vertices of the curve are in $\mathcal{A}$. Hence the curves incident with vertices of $\mathcal{A}$ form cycles. Since $\mathcal{A}$ and $\mathcal{H} \setminus \mathcal{A}$ are symmetrical covered proper subsets of $\mathcal{H}$, the diagram has at least two cycles.

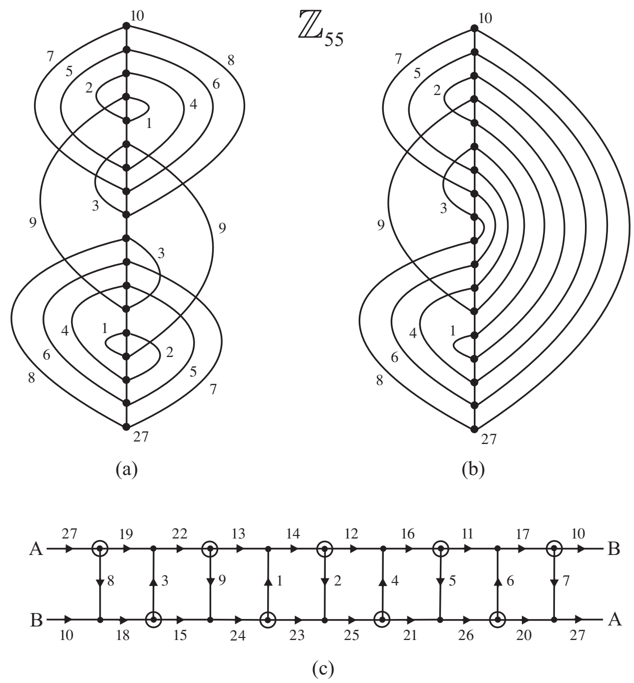

Figure 3: The cycles of the double and auxiliary diagrams (and the corresponding current assignment on the graph ML(9)) for the skew CS(55) in Fig. 2(a).

Now we show that if the double (respectively, auxiliary) diagram has at least two cycles, then $\mathcal{H}$ contains a symmetrical covered proper subset. It is easy to see that if $(x_1, x_2, \ldots, x_{2t})$ is a cycle of the double (respectively, auxiliary) diagram, then $\{x_1, x_2, \ldots, x_{2t}\}$ is a covered subset.

Suppose that the double diagram has at least two cycles. By the construction of the double diagram, if vertices y and z are joined by a curve, then the vertices y^* and z^* are joined by a curve as well. Hence, if $(x_1, x_2, \ldots, x_{2t})$ is a diagram cycle, then $(x_1^*, x_2^*, \ldots, x_{2t}^*)$ is also a diagram cycle. The sets $\mathcal{K} = \{x_1, x_2, \ldots, x_{2t}\}$ and $\mathcal{K}^* = \{x_1^*, x_2^*, \ldots, x_{2t}^*\}$ either are disjoint or are the same set. The symmetrical set $\mathcal{F} = \mathcal{K} \bigcup \mathcal{K}^*$ is covered. If $\mathcal{K}$ and $\mathcal{K}^*$ are disjoint, then $|\mathcal{F}| = |\mathcal{K}| + |\mathcal{K}^*| = 4t < 4n + 2 = |\mathcal{H}|$, hence $\mathcal{F}$ is a proper subset of $\mathcal{H}$. If $\mathcal{K}$ and $\mathcal{K}^*$ are the same set, then, since there are at least two cycles, we have $|\mathcal{F}| = |\mathcal{K}| < |\mathcal{H}|$, hence $\mathcal{F}$ is a proper subset of $\mathcal{H}$.

Suppose that the auxiliary diagram has at least two cycles. Every cycle of the diagram is of the form $(y_1, y_1^*, y_2, y_2^*, \ldots, y_t, y_t^*)$, hence the vertices of every cycle form a symmetrical covered proper subset of $\mathcal{H}$. ■

The double diagram of a skew CS($12n+7$) will be used to obtain a current assignment on the graph ML($2n+1$) based on the CS($12n+7$). The auxiliary diagrams will be used to construct many skew CS($12n+7$)s.

Lemma 2 *The single cycle*

$$(x_1, x_2, \ldots, x_{4n+2}) \tag{1}$$

of the double diagram of a skew CS$(12n+7)$ *is necessarily of the form*

$$(x_1, x_2, \ldots, x_{2n+1}, x_1^*, x_2^*, \ldots, x_{2n+1}^*).$$

Proof. Each two vertices of the diagram cycle are connected by two different subpaths of this cycle. Considering (1), there must be $x_{t+1} = x_1^*$ for some $t \in \{0, 1, \ldots, 4n+1\}$. Then $x_1, x_2, \ldots, x_t, x_{t+1}$ $(x_{t+1} = x_1^*)$ and $x_1^*, x_2^*, \ldots, x_t^*, x_{t+1}^*$ $(x_{t+1}^* = x_1)$ are two subpaths of the diagram cycle joining the vertices x_1 and $x_1^* = x_{t+1}$. If the two subpaths are different, then $(x_1, x_2, \ldots, x_t, x_1^*, x_2^*, \ldots, x_t^*)$ is the diagram cycle and $t = 2n+1$. If the two subpaths are the same subpath (traversed in opposite directions), we have $x_i^* = x_{t+2-i}$ for $i = 1, 2 \ldots, t+1$. Then either $x_j = x_j^*$ for some j (a contradiction) or $x_{j+1} = x_j^*$ (a contradiction, since the CS($12n+7$) is skew and, thus, there is no curve joining vertices x_j and $x_{j+1} = x_j^*$). ■

Given a current assignment on the graph ML($2n+1$) and a vertex partition class, we say that the currents on the arcs directed from the vertices of the class are *based* on a CS($12n+7$) if for every triple $\langle \beta, \gamma, \delta \rangle$ of the CS($12n+7$), there is a vertex of the class such that the currents on the arcs directed from the vertex are either β, γ, δ or $-\beta, -\gamma, -\delta$.

A skew CS($12n+7$) *determines* a current assignment on the graph ML($2n+1$). Suppose that

$$(x_1, x_2, \ldots, x_{2n+1}, x_1^*, x_2^*, \ldots, x_{2n+1}^*). \tag{2}$$

is the cycle of the double diagram of the CS($12n+7$), then the current assignment is given in Fig. 4. For example, Fig. 3(c) shows such a current assignment constructed from the diagram cycle in Fig. 3(a); ignore for now the circles on some of the vertices.

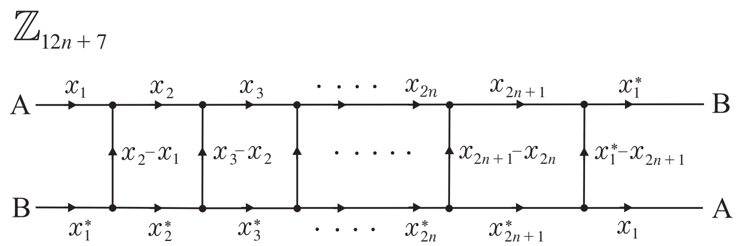

Figure 4: A current assignment on the graph ML$(2n+1)$ obtained from the double diagram cycle (2).

The cycle given by (2) is the same as that given by $(y_1, y_2, \ldots, y_{2n+1}, y_1^*, y_2^*, \ldots, y_{2n+1}^*) = (x_{2n+1}^*, \ldots, x_2^*, x_1^*, x_{2n+1}, \ldots, x_2, x_1)$. However, the current assignment on ML$(2n+1)$ that results from writing the cycle in this form looks slightly different. It can be obtained from that in Fig. 4 by reversing all currents. So, when we say that a skew CS$(12n+7)$ determines a current assignment on the graph ML$(2n+1)$, we mean that it does so up to such a reversal of the currents.

Since $x_{j+1} - x_j = -(x_{j+1}^* - x_j^*)$ for every $j = 1, 2, \ldots, 2n+1$ (with $x_{2n+2} = x_1$), KCL holds at every vertex. The set of currents on the $2n+1$ vertical arcs of Fig. 4 contains exactly one element from every pair $\{\beta, -\beta\}$, where β belongs to the label set of the CS$(12n+7)$. The set of currents on the horizontal arcs is the base set of the CS$(12n+7)$. Hence, the current assignment in Fig. 4 satisfies (A1) and (A2).

The cycle (2) can be written as $(y_1, z_1, y_2, z_2, \ldots, y_{2n+1}, z_{2n+1})$, where for $i = 1, 2, \ldots, 2n+1$, the vertices y_i and z_i are connected by a diagram curve of the CS$(12n+7)$, that is, either $\langle z_i - y_i, y_i, -z_i \rangle$ or $\langle y_i - z_i, z_i, -y_i \rangle$ is a triple of the CS$(12n+7)$. By "a vertex of the graph in Fig. 4 corresponding to the pair $\{y_i, z_i\}$" we mean a vertex incident with horizontal arcs carrying currents y_i and z_i; the currents on the arcs directed from the vertex are z_i, $-y_i$, and $y_i - z_i$. The vertices corresponding to all the pairs $\{y_j, z_j\}$, $j = 1, 2, \ldots, 2n+1$, form a vertex partition class of the bipartite graph. Hence, we obtain the following result.

(C) Given a current assignment on the graph ML$(2n+1)$ determined by a skew CS$(12n+7)$, there is a vertex partition class such that the currents on the arcs directed from the vertices of the class are based on the CS$(12n+7)$. The partition class is said to be *associated* with the CS$(12n+7)$.

Theorem 2 *A current assignment on the graph ML$(2n+1)$ satisfies (A1) and (A2) if and only if the current assignment is determined by a skew CS$(12n+7)$.*

Proof. It was shown above that every skew CS$(12n+7)$ determines a current assignment on the graph ML$(2n+1)$ satisfying (A1) and (A2). Now, to prove the theorem, it suffices to prove the following statement.

(D) Given a current assignment on the graph ML$(2n+1)$ satisfying (A1) and (A2), for each vertex partition class, there is a skew CS$(12n+7)$ determining the current assignment such that the currents on the arcs directed from the vertices of the class are based on the CS$(12n+7)$.

The statement asserts slightly more than we really need to prove the theorem, but we give the statement in such a form for later use.

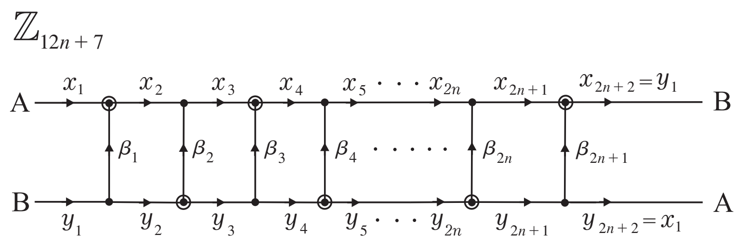

Figure 5: A current assignment satisfying (A1) and (A2).

Consider the current assignment given in Fig. 5 and satisfying (A1) and (A2). Since KCL holds at every vertex, $x_i + y_i = x_j + y_j$ for every i and j. Choose a vertex partition class (the vertices of the class are circled in the figure). Consider any vertex v of this partition class. If in Fig. 5 the arcs incident with v carry currents β, δ, δ', where β is the current on the vertical arc and is such that $\beta + \delta = \delta'$, then the triple $\langle \beta, \delta, -\delta' \rangle$ is *associated* with v. Consider the triples $\langle \beta_1, x_1, -x_2 \rangle$, $\langle \beta_2, y_3, -y_2 \rangle$, $\langle \beta_3, x_3, -x_4 \rangle$, $\langle \beta_4, y_5, -y_4 \rangle$, $\langle \beta_5, x_5, -x_6 \rangle$, $\dots$, $\langle \beta_{2n}, y_{2n+1}, -y_{2n} \rangle$, $\langle \beta_{2n+1}, x_{2n+1}, -y_1 \rangle$ associated with the vertices of the partition class. It is easy to see that the triples form a CS$(12n+7)$ with label set $\{\beta_1, \beta_2, \dots, \beta_{2n+1}\}$, the base set $\{x_1, x_2, \dots, x_{2n+1}, y_1, y_2, \dots, y_{2n+1}\}$, and the pair sum $x_1 + y_1$ (hence, $x_i = y_i^*$ for $i = 1, 2, \dots, 2n+1$). The currents on the arcs directed from the vertices of the partition class are based on the CS$(12n+7)$. Now we want to show that the cyclic sequence

$$(x_1, x_2, \dots, x_{2n+1}, y_1, y_2, \dots, y_{2n+1}) \tag{3}$$

of the currents on the horizontal arcs in Fig. 5 is the cycle of the double diagram of the CS($12n+7$). Let every two neighboring vertices of the cycle (3) be joined by a curve. The diagram of the CS($12n+7$) consists of curves joining the following pairs of vertices: $\{x_1, x_2\}$, $\{y_2, y_3\}$, $\{x_3, x_4\}$, $\{y_4, y_5\}$, $\{x_5, x_6\}$, ... , $\{y_{2n}, y_{2n+1}\}$, $\{x_{2n+1}, y_1\}$. So we see that the cycle (3) contains all curves of the diagram of the CS($12n+7$), and for every curve of the cycle, if the curve joins vertices v and w, then the cycle contains a curve joining the vertices v^* and w^*. Hence, the cycle (3) is the only cycle of the double diagram of the CS($12n+7$). By Theorem 1, the CS($12n+7$) is skew. ■

The two CS($12n+7$)s given by $A = \{\langle\beta_i, \gamma_i, \delta_i\rangle : i = 1, 2, \ldots, 2n+1\}$ and $A' = \{\langle\beta_i, (-\delta_i)^*, -\gamma_i^*, \rangle : i = 1, 2, \ldots, 2n+1\}$ will be said to be *equivalent*. Note that the curve with label β_i joins the vertices γ_i and $-\delta_i$ in the diagram of A and joins the vertices $(-\delta_i)^* = \Omega + \delta_i$ and $\gamma_i^* = -(-\gamma_i^*)$ in the diagram of A', where Ω denotes the pair sum. It follows that $\beta_i + (-\delta_i)^* + (-\gamma_i^*) = \beta_i + (\Omega + \delta_i) - (\Omega - \gamma_i) = 0$, and the double diagram of A consists of all diagram curves of the diagrams of A and A'. Considering how a current assignment on ML($2n+1$) is determined by a CS($12n+7$), and taking into account the proof of Theorem 2, we obtain the following statement.

(E) Two different CS($12n+7$)s determine the same current assignment on ML($2n+1$) if and only if the CS($12n+7$)s are equivalent.

By a *skew* GTS($12n+7$) we mean a GTS($12n+7$) with an associated skew CS($12n+7$), for every triple $\langle\beta, \gamma, \delta\rangle$ of which, the GTS has exactly one of the following generic triples: (β, γ, δ), $(-\delta, -\gamma, -\beta)$, (β, δ, γ), $(-\gamma, -\delta, -\beta)$. We say that the skew GTS($12n+7$) is *based* on the skew CS($12n+7$).

Theorem 3 *For every nonnegative integer n, every cyclic STS($12n+7$) induced by every skew GTS($12n+7$) is orientably biembeddable with a cyclic STS($12n+7$) induced by a skew GTS($12n+7$).*

Proof. Take X to be an arbitrary skew CS($12n+7$). Consider the current assignment on the graph ML($2n+1$) determined by X. By (C), there is a vertex partition $\{V, V'\}$ of the bipartite graph, where V is associated with X. Now, given an arbitrary skew GTS($12n+7$) based on X, taking into account (C), we can choose the rotations of the vertices of V and then, by Lemma 1, the rotations of the vertices of V' such that the resulting rotation of the graph ML($2n+1$) induces exactly one circuit and for each generic triple

(β, γ, δ) of the GTS, there is a vertex of V with current rotation (β, γ, δ) or $(-\delta, -\gamma, -\beta)$. Since generic triples (β, γ, δ) and $(-\delta, -\gamma, -\beta)$ induce the same blocks, then by (B), the resulting current graph shows that the cyclic STS$(12n + 7)$ induced by the skew GTS$(12n + 7)$ is orientably biembedded with a cyclic STS$(12n+7)$ X' induced by the current rotations of the vertices of V'. Taking (D) into account, we see that X' is itself induced by a skew GTS$(12n + 7)$. ■

4 Constructing skew centered systems

In this section we show how to use Theorem 1 (b) to construct many skew CS$(12n + 7)$s for some values of n. The inequivalent skew CS$(12n + 7)$s obtained give different index one current graphs which yield mutually nonisomorphic orientable biembeddings of cyclic STS$(12n + 7)$s (Theorem 5).

We will consider CS$(12n + 7)$s with the label set $\{1, 2, \ldots, 2n + 1\}$ and the base set $\{2n+2, 2n+3, \ldots, 6n+3\}$. Such CS$(12n+7)$s are called Skolem sequences of order $2n + 1$ (SK$(2n + 1)$, for short) in the literature and exist only for even n, see [5]. The CS(55)s in Fig. 2 are examples of SK(9)s.

The only known example of a current assignment on the graph ML$(4n+1)$ satisfying (A1) and (A2) is given in [14]. The current assignment is constructed for all $n \geq 1$ and is given in terms of zigzags. One can check that the current assignment is determined by the skew SK$(4n+1)$, $n \geq 1$, given in Fig. 6. This figure demonstrates the existence of a skew SK$(4n+1)$ for every $n \geq 1$; it is easy enough to check that the double diagram of the SK$(4n + 1)$ has exactly one cycle. In Fig. 6, and subsequently, we represent the fragment with $k + 1$ curves shown in Fig. 7(a) by the picture given in Fig. 7(b).

In what follows, for positive integers $a \leq b$, by the *interval* $[a, b]$ of order $b - a + 1$ we mean a set of $b - a + 1$ vertices on a vertical line labelled by integers $a, a + 1, a + 2 \ldots, b - 1, b$ in this order. For $a \leq a' \leq b' \leq b$, the interval $[a', b']$ of vertices is a *subinterval* of the interval $[a, b]$.

We will construct a skew SK$(4n + 1)$ in the form of the diagram of the SK$(4n + 1)$. To prove skewness we will use Theorem 1 (b) by showing that the auxiliary diagram has exactly one cycle. The diagram is composed of some fragments: the fragments $P(s + 1, 3s + 1)$ and $Q(s + 1, 3s + 1)$, and fragments representing skew SK(s)s. Each of the fragments is a family of curves covering some interval of vertices on a vertical line. The curves are labeled by positive integers; if a curve has label δ, then the curve joins vertices x and $x + \delta$. The labeled curves are taken to be the diagram curves of the

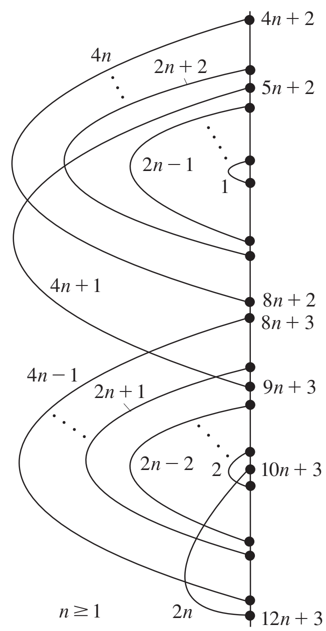

Figure 6: A skew SK($4n + 1$) for every $n \geq 1$.

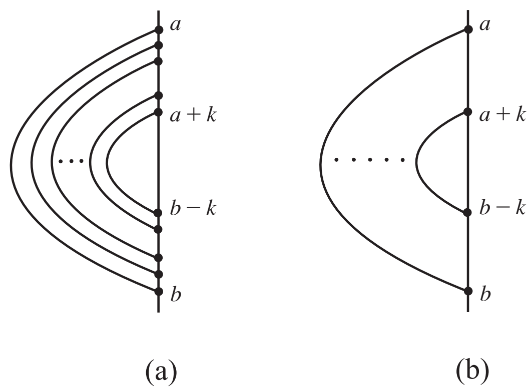

Figure 7: Representation of a fragment.

resulting skew SK($4n + 1$). The fragments are combined in such a way that the auxiliary diagram has exactly one cycle.

Given an interval of vertices on a vertical line, a set of curves joining pairs of the vertices is called a *normal* set of curves if the following conditions hold:

(1) for every vertex, the number of curves incident with the vertex is either

one or two;

(2) the curves do not form cycles.

The curves of a normal set of curves form *paths of the normal set*; the end vertices of these paths are vertices incident with exactly one curve. In our pictorial representations of a normal set of curves, we indicate the two end vertices of each path by connecting them with a dashed curve

We define $P(s, 3s-2)$ to be the family of curves shown in Fig. 8(a) that covers the interval $[a, a+4s-3]$. This family has exactly $2s-1$ curves which have labels $s, s+1, \ldots, 3s-2$, respectively. Fig. 8(b) is a schematic representation of the family $P(s, 3s-2)$, and Fig. 8(c) shows $P(5, 13)$ for the case $a = 1$.

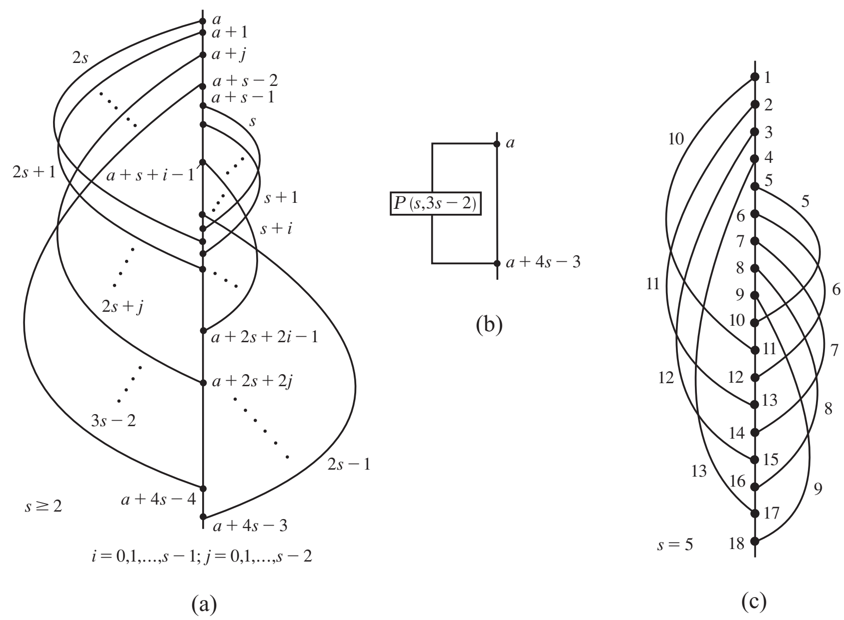

Figure 8: The family of curves $P(s, 3s-2)$.

Claim 1 *Consider the family $P(s, 3s-2)$ covering the subinterval $[a+2s-1, a+6s-4]$ of the interval $[a+1, a+6s-4]$ (see Fig. 9(a)). For every $i = 0, 1, \ldots, 3s-3$ add a new curve joining the vertices $a+1+i$ and $a+6s-4-i$. Then the resulting set of curves is normal and there are exactly $s-1$ paths: one path joins the end vertices $a+2s-3$ and $a+2s-2$, and for every $j = 0, 1, \ldots, s-3$, a path joins the end vertices $a+1+j$ and $a+2s-4-j$ (see Fig. 9(a)).*

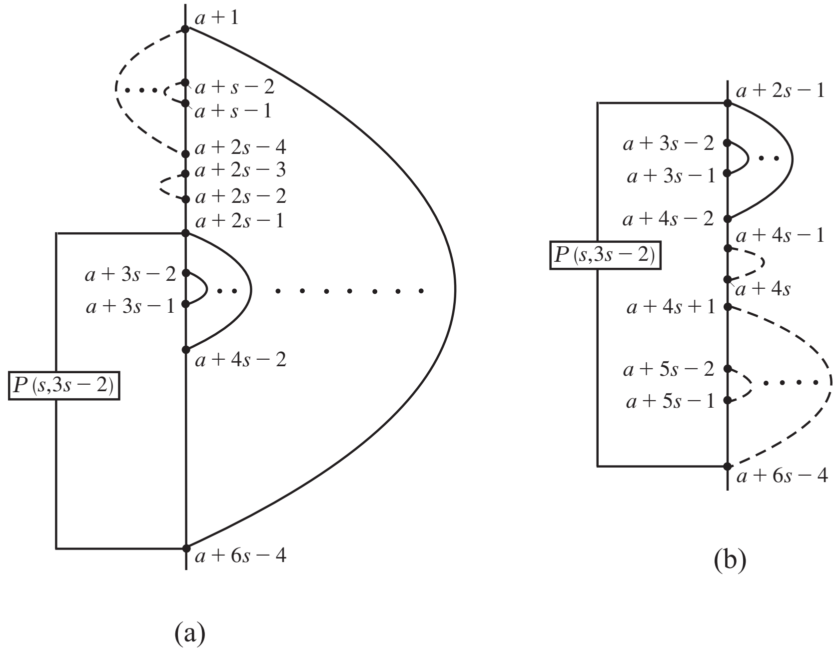

Figure 9: A curve set including the family $P(s, 3s-2)$.

Proof. It suffices to show that the set of curves in Fig. 9(b) is normal and that there are exactly $s-1$ paths of the normal set: one path joins the end vertices $a+4s-1$ and $a+4s$, and for every $j = 0, 1, \ldots, s-3$, a path joins the end vertices $a+4s+1+j$ and $a+6s-4-j$.

Now consider Fig. 10(a). The figure is obtained from Fig. 9(b) if we put $a = 0$ (for convenience), draw the curves of the family $P(s, 3s-2)$, and depict in thick lines all the curves that are not part of the family $P(s, 3s-2)$. We see (Fig. 10(b)) that the thick curve joining vertices $2s-1+t$ and $4s-2-t$, $t = 1, 2, \ldots, s-2$, enters into the path of length three joining the end vertices $4s+2t-1$ and $6s-2t-2$. Taking into account that $(4s+2t-1)+(6s-2t-2) = 10s-3$, it is easy to check that the set of unordered pairs $\{4s+2t-1, 6s-2t-2\}$, $t = 1, 2, \ldots, s-2$, is the set of unordered pairs $\{4s+1+j, 6s-4-j\}$, $j = 0, 1, \ldots, s-3$. All curves not entering into the $s-2$ paths of length three are shown in Fig. 10(c); these curves form a path joining the end vertices $4s-1$ and $4s$. ■

We define $Q(s, 3s-2)$ to be the family of curves shown in Fig. 11(a) that covers the interval $[a, a+4s-3]$. This family has exactly $2s-1$ curves which have labels $s, s+1, \ldots, 3s-2$, respectively. Fig. 11(b) is a schematic representation of the family $Q(s, 3s-2)$, and Fig. 11(c) shows $Q(5, 13)$ for the case $a = 1$.

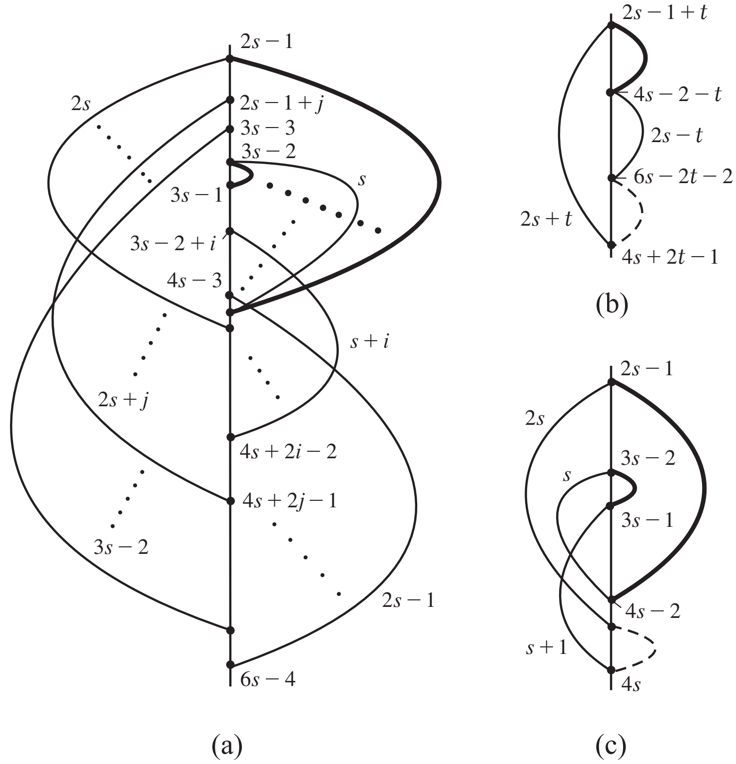

Figure 10: Paths of the curve set in Fig. 9(b).

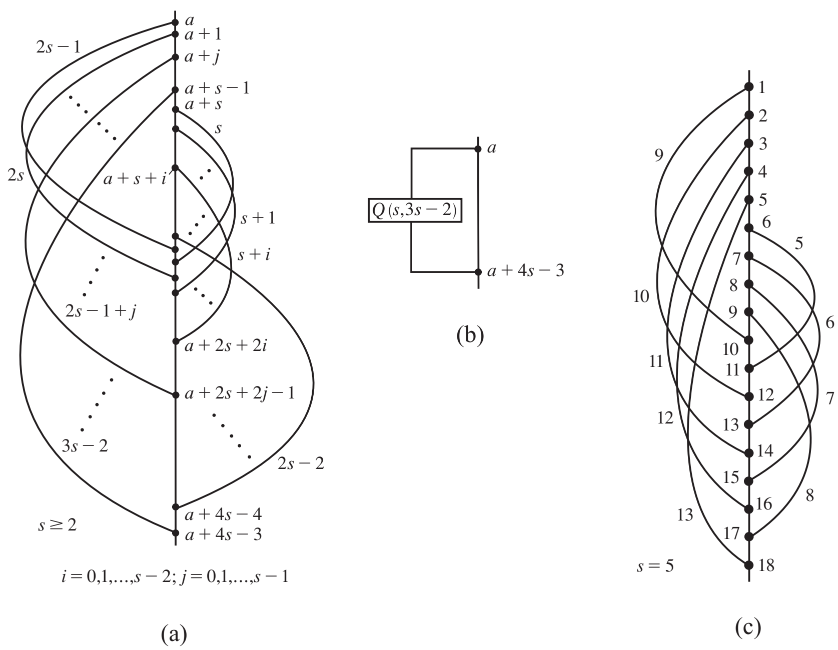

Figure 11: The family $Q(s, 3s-2)$ of curves.

Claim 2 *Consider the family* $Q(s, 3s-2)$ *covering the subinterval* $[a+2s-1, a+6s-4]$ *of the interval* $[a+1, a+6s-4]$ *(see Fig. 12(a)). Add a new curve joining the vertices* $a+6s-5$ *and* $a+6s-4$*, and for every* $i = 0, 1, \dots, 3s-4$ *add a new curve joining the vertices* $a+1+i$ *and* $a+6s-6-i$*. Then the resulting set of curves is normal and there are exactly* $s-1$ *paths: for every* $j = 0, 1, \dots, s-2$*, a path joins the end vertices* $a+1+j$ *and* $a+2s-2-j$ *(see Fig. 12(a)).*

Proof. It suffices to show that the set of curves in Fig. 12(b) is normal and that there are exactly $s-1$ paths of the normal set: for every $j = 0, 1, \dots, s-2$, a path joins the end vertices $a+4s-3+j$ and $a+6s-6-j$.

Now consider Fig. 13(a). The figure is obtained from Fig. 12(b) if we put $a = 0$ (for convenience), draw the curves of the family $Q(s, 3s-2)$, and depict in thick lines all curves that are not part of the family $Q(s, 3s-2)$. We see (Fig. 13(b)) that the thick curve joining vertices $2s-1+t$ and $4s-4-t$, $t = 0, 1, \dots, s-3$, enters into the path of length three joining the end vertices $4s+2t-2$ and $6s-2t-7$. Taking into account that $(4s+2t-2)+(6s-2t-7) = 10s-9$, it is easy to check that the set of unordered pairs $\{4s+2t-2, 6s-2t-7\}$, $t = 0, 1, \dots, s-3$, is the set of unordered pairs $\{4s-3+j, 6s-6-j\}$, $j = 1, 2, \dots, s-2$. All curves not entering into the $s-2$ paths of length three are shown in Fig. 13(c); these curves form a path joining the end vertices $4s-3$ and $6s-6$. ■

The diagram curves of an SK(n) cover the interval $[n+1, 3n]$. Consider an interval $[a, b]$ such that $b-a = 2n-1$. When we say that we cover the interval $[a, b]$ by the curves of the SK(n), we mean the following: for each diagram curve of the SK(n), if the curve has label i and joins vertices $n+1+v(i)$ and $n+1+w(i)$ (where $i+v(i) = w(i)$), then we join by a curve with label i the vertices $a+v(i)$ and $a+w(i)$ of the interval $[a, b]$.

Let $\mathcal{F}$ and $\mathcal{F}'$ be two sets of curves joining vertices of an interval $[a, b]$ of odd order. For each $x \in [a, b]$, denote by x^* the element of the interval such that $x + x^* = a + b$. The sets $\mathcal{F}$ and $\mathcal{F}'$ are *equivalent* if $\mathcal{F}'$ is obtained from $\mathcal{F}$ in the following way: for every curve of $\mathcal{F}$, if the curve joins vertices v and w, replace the curve by a curve joining vertices v^* and w^*. Roughly speaking, $\mathcal{F}'$ is obtained from $\mathcal{F}$ by rotating $\mathcal{F}$ through 180° about a horizontal axis bisecting the interval $[a, b]$. This usage of the term "equivalent" is consistent with the earlier usage of the same term amongst CS($12n+7$)s: two CS($12s+7$)s are equivalent if and only if their sets of diagram curves are equivalent.

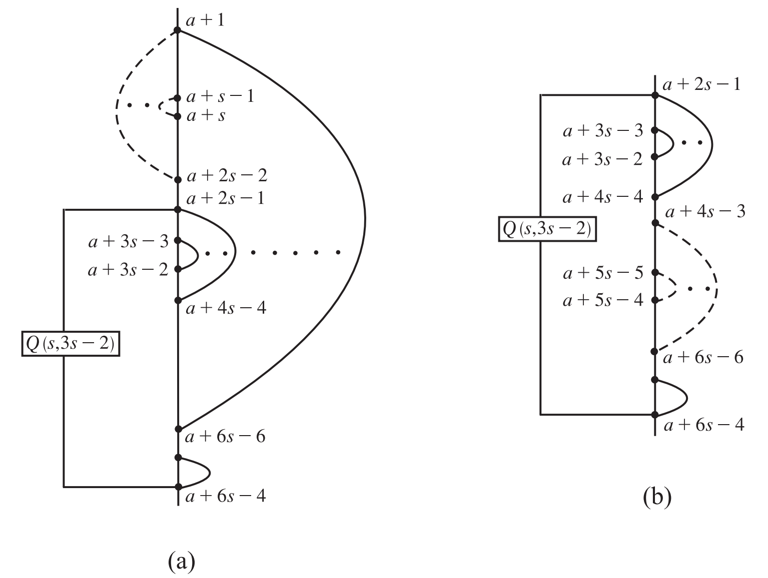

Figure 12: A curve set including the family $Q(s, 3s-2)$.

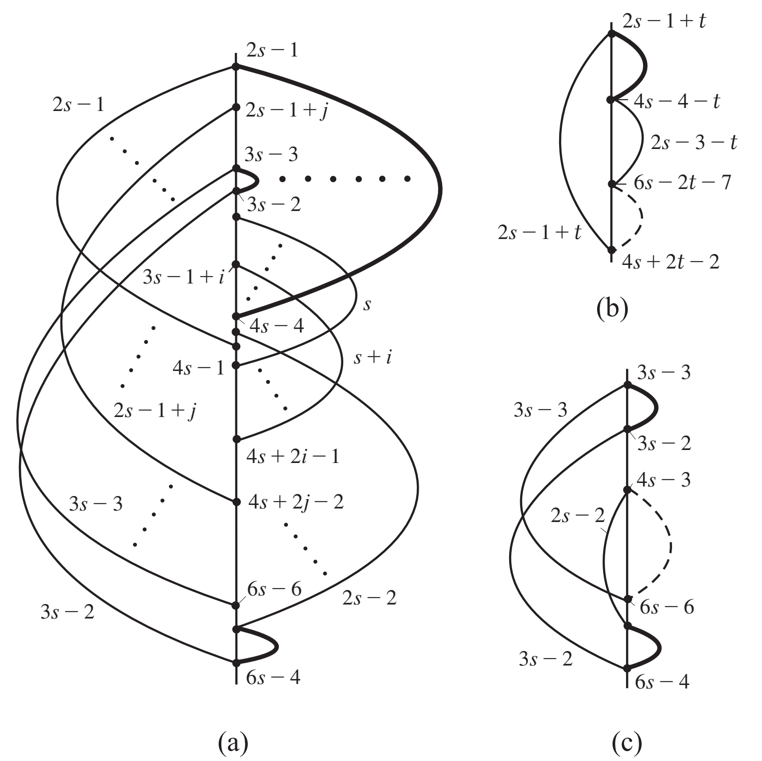

Figure 13: Paths of the curve set in Fig. 12(b).

Lemma 3 *If there are h different skew SK(n)s, then there are $2h$ different skew SK$(9n+4)$s.*

Proof. Consider the set $\mathcal{F}$ of curves shown in Fig. 14 with $a = 9n+4$. Taking into account Claims 1 and 2, we see that the set $\mathcal{F}$ is normal, the end vertices form the subinterval $[9n+5, 11n+4]$ and the paths of the set connect the end vertices as shown in the figure. The set $\mathcal{F}'$ of curves equivalent to $\mathcal{F}$ is also normal, the end vertices form the subinterval $[25n+13, 27n+12]$, and for every $i = 0, 1, \ldots, n-1$, a path of $\mathcal{F}'$ joins the end vertices $25n+13+i$ and $27n+12-i$.

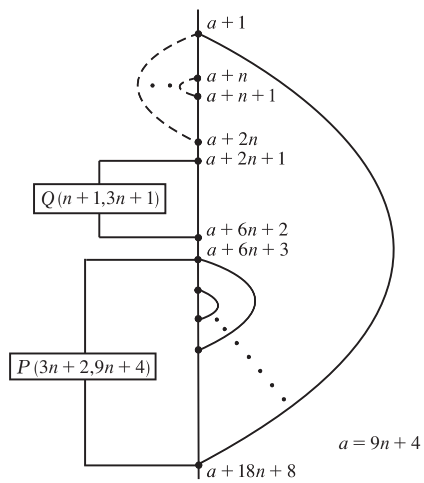

Figure 14: A recursive construction for obtaining a skew SK(t).

Now, for each of the h different skew SK(n)s, cover by the curves of the SK(n) the subinterval $[9n+5, 11n+4]$ in the case of $\mathcal{F}$, or the subinterval $[25n+13, 27n+12]$ in the case of $\mathcal{F}'$. We obtain the auxiliary diagrams of two SK$(9n+4)$s; each of the diagrams has exactly one cycle. It is easy to see that the $2h$ SK$(9n+4)$s obtained are all different. ■

Note that if $n \equiv 1 \pmod 4$, then it is also the case that $9n+4 \equiv 1 \pmod 4$.

Theorem 4 *For $\ell = 1, 2, \ldots$, there are at least $6(\frac{2t+1}{11})^{1/\log_2 9}$ different skew SK(t)s, where $t = \frac{1}{2}(11 \cdot 9^{\ell-1} - 1)$.*

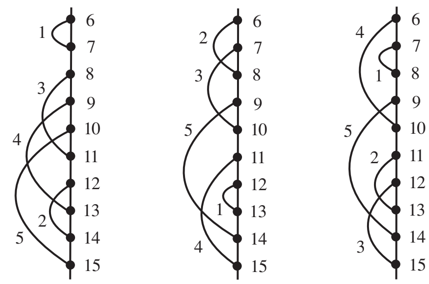

Figure 15: Mutually inequivalent SK(5)s.

Proof. Fig. 15 shows three mutually inequivalent skew SK(5)s. For each of these SK(5)s, take the equivalent SK(5), thereby obtaining six different skew SK(5)s.

Define a function $f(\ell)$, $\ell = 1, 2, \ldots$, as follows:

$$f(\ell + 1) = 9f(\ell) + 4, \qquad f(1) = 5. \tag{4}$$

If we consider $f(\ell)$ as the order of a skew SK then, by Lemma 3, for $\ell = 1, 2, \ldots$, there are at least $6 \cdot 2^{\ell-1} = 3 \cdot 2^{\ell}$ different skew SK($f(\ell)$)s. The recurrence given by equations (4) has solution

$$f(\ell) = 9^{\ell-1} f(1) + \frac{9^{\ell-1} - 1}{2} = \frac{11 \cdot 9^{\ell-1} - 1}{2}$$

for $\ell = 1, 2, \ldots$. Put $t = f(\ell)$ so that $9^{\ell-1} = \frac{2t+1}{11}$. Then $\ell = 1 + \frac{1}{\log_2 9} \log_2 (\frac{2t+1}{11})$ and we obtain $3 \cdot 2^{\ell} = 6 \cdot (\frac{2t+1}{11})^{1/\log_2 9}$. ■

We will use the SK($4n+1$)s constructed above to expand the set of known nonisomorphic orientable biembeddings of cyclic STS($24n + 7$)s. To do this we apply some results from [12]. These results relate to digraphs and when we apply the results to the graph $G = \mathrm{ML}(2n + 1)$, we consider G to be the digraph obtained from ML($2n + 1$) by replacing each edge e by two reverse arcs e^+ and e^-, and any current assignment on the graph G becomes a current assignment on the digraph G.

Considering $G = \mathrm{ML}(2n+1)$ as a digraph, let λ and λ' be current assignments on G determined by two SK($2n + 1$)s. The pairs $\langle G, \lambda\rangle$ and $\langle G, \lambda'\rangle$ are said to be *isomorphic* if there is an automorphism $\omega : V(G) \to V(G)$ of G and an automorphism φ of $\mathbb{Z}_{12n+7}$ such that $\lambda'[\omega(v), \omega(w)] = \varphi\lambda[v, w]$ for every arc $[v, w]$ of G (here $[x, y]$ denotes an arc directed from vertex x

to vertex y). The automorphism ω is called an *isomorphism* of $\langle G, \lambda\rangle$ onto $\langle G, \lambda'\rangle$ and the automorphism φ of $\mathbb{Z}_{12n+7}$ is said to be associated with this isomorphism.

Lemma 4 *Let λ and λ' be current assignments on the graph $G = ML(2n+1)$ determined by two inequivalent $SK(2n+1)$s. Let $\langle G, \lambda, D\rangle$ and $\langle G, \lambda', D'\rangle$ be two index one current graphs generating orientable triangular embeddings Φ and Φ' of K_{12n+7}, respectively. Then the embeddings Φ and Φ' are not isomorphic.*

Proof. Theorems 1 and 2 of [12] establish that for the embeddings Φ and Φ' to be isomorphic, the current graphs $\langle G, \lambda, D\rangle$ and $\langle G, \lambda', D'\rangle$ must be isomorphic, and this requires that the pairs $\langle G, \lambda\rangle$ and $\langle G, \lambda'\rangle$ be isomorphic as well. Hence, to prove the lemma, it suffices to show that $\langle G, \lambda\rangle$ and $\langle G, \lambda'\rangle$ are not isomorphic.

Suppose that $\langle G, \lambda\rangle$ and $\langle G, \lambda'\rangle$ are isomorphic with an associated automorphism φ of $\mathbb{Z}_{12n+7}$. In each of the current assignments λ and λ', the currents on the rungs of G form the subset $\mathcal{E} = \{\pm 1, \pm 2, \ldots, \pm(2n+1)\}$ of elements of $\mathbb{Z}_{12n+7}$. Every automorphism of G takes the set of the rungs onto itself, hence φ takes $\mathcal{E}$ onto itself. It is well-known (and easily proved) that the set of all automorphisms of $\mathbb{Z}_{12n+7}$ is the set $\{\varphi_k : k \in \mathbb{Z}_{12n+7}, (k, 12n+7) = 1\}$, where $\varphi_k(x) = k \cdot x$ for all $x \in \mathbb{Z}_{12n+7}$, (k, m) is the greatest common divisor of k and m, and arithmetic is performed in the ring $\mathbb{Z}_{12n+7}$. Hence $\varphi = \varphi_k$ for some $k \in \mathbb{Z}_{12n+7}$. Since $k \cdot 1 \in \mathcal{E}$, we have $k \in \mathcal{E}$. For every $h \in \{2, 3, \ldots, 2n+1\} \subset \mathbb{Z}$ we have $2n+2 \leq h\lceil \frac{2n+2}{h}\rceil \leq 2n+2+h \leq 4n+3 < 10n+5$. Therefore, if $k \neq \pm 1$, there is $x \in \mathcal{E}$, namely $x = \lceil \frac{2n+2}{|k|}\rceil$, such that $k \cdot x \notin \mathcal{E}$, a contradiction. If $\varphi \in \{\varphi_1, \varphi_{-1}\}$, then λ and λ' are the same current assignment (up to reversing all the currents) determined by equivalent SK$(2n+1)$s (see (E)), and again this is a contradiction. ■

Theorem 5 (a) *For every $n \geq 1$, there are at least $\frac{1}{4n+1}2^{4n-2}$ nonisomorphic orientable biembeddings of cyclic $STS(24n+7)$s.*

(b) *For $\ell = 1, 2, \ldots$ there are at least $\frac{1}{4n+1}2^{4n-2} \cdot 3(\frac{8n+3}{11})^{1/\log_2 9}$ nonisomorphic orientable biembeddings of cyclic $STS(24n+7)$s, where $4n+1 = \frac{1}{2}(11 \cdot 9^{\ell-1} - 1)$.*

Proof. Lemma 1 implies the following:

(i) The graph ML($4n+1$) has at least 2^{4n+1} different rotations inducing exactly one circuit.

Suppose there is a ($24n+7$)-current assignment λ on the graph $G =$ ML($4n+1$). Theorem 4 of [12] states that if the automorphism group of G has order m, and if there are h different rotations D of G inducing exactly one circuit, then among the h current graphs $\langle G, \lambda, D\rangle$ there are at least $\frac{h}{2m}$ current graphs generating nonisomorphic embeddings. The graph ML($4n+1$) can be considered as a cycle with $8n+2$ vertices where every two antipodal vertices are joined by an edge, hence the automorphism group of ML($4n+1$) is the automorphism group of a regular ($8n+2$)-gon, that is, the dihedral group D_{8n+2} of order $16n+4$. Now, taking (i) into account, it follows that there are at least $\frac{1}{4n+1}2^{4n-2}$ nonisomorphic orientable biembeddings of cyclic STS($24n+7$)s.

As remarked earlier, a skew SK($4n+1$) was constructed for every $n \geq 1$ by Youngs and is shown in Fig. 6. Hence, for every $n \geq 1$ there is a ($24n+7$)-current assignment λ on the graph $G =$ ML($4n+1$) and we obtain (a).

By putting $t = 4n+1$ in Theorem 4, we see that for $\ell = 1, 2, \ldots$, there are at least $3(\frac{8n+3}{11})^{1/\log_2 9}$ inequivalent skew SK($4n+1$)s, where $4n+1 = \frac{1}{2}(11 \cdot 9^{\ell-1} - 1)$. Each of the skew SK($4n+1$)s determines a ($24n+7$)-current assignment λ on the graph $G =$ ML($4n+1$) and there are at least $\frac{1}{4n+1}2^{4n-2}$ different current graphs $\langle G, \lambda, D\rangle$ generating nonisomorphic embeddings. By applying Lemma 4, we obtain (b). ■

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