A lower bound for the number of triangular embeddings of some complete graphs and complete regular tripartite graphs

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Abstract

We prove that, for a certain positive constant a and for an infinite set of values of n, the number of nonisomorphic triangular embeddings of the complete graph K_n is at least n^{an^2}. A similar lower bound is also given, for an infinite set of values of n, on the number of nonisomorphic triangular embeddings of the complete regular tripartite graph $K_{n,n,n}$.

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1 Introduction

The necessary conditions, obtained from Euler's formula, for the existence of a triangular surface embedding of the complete graph K_n were shown to be sufficient for $n \neq 7$ in both the orientable and nonorientable cases in the course of the solution of the famous Heawood map colouring problem for surfaces of positive genus. Ringel's 1974 book [14] gives one such embedding in each case. However, until comparatively recently the maximum number of known nonisomorphic triangular embeddings of K_n for any particular n in either an orientable or nonorientable surface was a mere three [13]. In 2000, the present authors together with C. P. Bonnington and J. Širáň established that, for n lying in certain residue classes, there are at least $2^{n^2/54-o(n^2)}$ such embeddings [3]. The proof relies on a recursive construction that applies in certain cases when the embeddings are face 2-colourable. In these cases, the faces in each colour class determine a Steiner triple system of order n, and the two systems together form a twofold triple system of order n. It is easily shown, for example by using the method of Lemma 5.2 of [4], that an upper bound for the number of distinct twofold triple systems of order n is $n^{n^2/3}$. As observed in [1], there is a one-to-one correspondence between twofold triple systems of order n and triangular embeddings of K_n in generalized pseudosurfaces. Consequently the number of nonisomorphic triangular surface embeddings of K_n cannot exceed $n^{n^2/3}$.

Subsequent to [3], further recursive constructions by the present authors and J. Širáň have extended the range of residue classes for which a lower bound having the form 2^{an^2} (where $a > 0$ is a constant) on the number of isomorphism classes of such embeddings has been established [6]. Furthermore, V. P. Korzhik and H.-J. Voss have established a lower bound of the form 2^{an} (where $a > 0$ is a constant) for all sufficiently large n [9, 10, 11, 12]. The work of these latter authors extends to cases of minimum genus embeddings of K_n that are not triangulations; in such cases there are a small number of non-triangular faces. However it is still possible to establish an upper bound of $n^{n^2/3}$ on the number of possible isomorphism classes of these embeddings.

The current paper substantially narrows the gap between these best known upper and lower bounds for an infinite set of values of n. It is shown that for these values of n and for a certain positive constant a, there are at least n^{an^2} nonisomorphic triangular embeddings of K_n in a nonorientable surface. A major component of the proof is the establishment of a similar lower bound on the number of nonisomorphic face 2-colourable triangular embeddings of the complete regular tripartite graph $K_{n,n,n}$ in an orientable surface. This bound is itself a considerable improvement on the previous best known lower bound. Before proceeding to the proofs, we first review terminology, definitions and some basic results.

A *Steiner triple system of order* n is a pair $(V, \mathcal{B})$ where V is an n-element set (the *points*) and $\mathcal{B}$ is a collection of 3-element subsets (the *blocks*) of V such that each 2-element subset of V is contained in exactly one block of $\mathcal{B}$. It is well known that a Steiner triple system of order n (briefly STS(n)) exists if and only if $n \equiv 1$ or 3 (mod 6) [8]. If, in the definition, the words "exactly one block" are replaced by "exactly two blocks", then we have a *twofold triple system of order* n, TTS(n) for short. A twofold triple system of order n exists if and only if $n \equiv 0$ or 1 (mod 3) [2]. An STS(n) can be considered as a decomposition of the complete graph K_n into triangles (copies of K_3); likewise a TTS(n) can be considered as a decomposition of the twofold complete graph $2K_n$ (in which there are two edges between each pair of vertices) into triangles.

A *transversal design of order* n *and block size* 3 is a triple $(V, \mathcal{G}, \mathcal{B})$ where V is a $3n$-element set (the *points*), $\mathcal{G}$ is a partition of V into 3 parts (the *groups*) each of size n, and $\mathcal{B}$ is a collection of 3-element subsets (the *blocks*) of V such that each 2-element subset of V is either contained in exactly one block of $\mathcal{B}$ or in exactly one group of $\mathcal{G}$, but not both. A transversal design of order n and block size 3 is denoted by TD$(3, n)$; such a design may be considered as a decomposition of the complete tripartite graph $K_{n,n,n}$ into triangles with the tripartition defining the groups of the design. A TD$(3, n)$ is equivalent to a Latin square of side n in which the triples are given by (row, column, entry).

A *parallel class* in an STS$(n) = (V, \mathcal{B})$ is a subset of $\mathcal{B}$ that covers each point of V precisely once. Obviously this requires that $n \equiv 3$ (mod 6). A similar definition may be given for a parallel class in a TD$(3, n)$. Such a parallel class corresponds to a *transversal* in the associated Latin square of side n, that is to say a set of n entries from the square that contains every entry symbol, and covers every row and every column.

We have already described the relationship between triangular surface embeddings of complete graphs and twofold and Steiner triple systems. In a similar way, the colour classes of a face 2-colourable triangular embedding of the complete tripartite graph $K_{n,n,n}$ give two TD$(3, n)$s or, equivalently, two Latin squares of side n. We always take the two colour classes of a face

2-colourable embedding to be black and white. We will say that two STS(n)s, say B and W, are *biembeddable* in a surface S if there is a face 2-colourable triangular embedding of K_n in S such that the black (respectively white) faces form a system isomorphic with B (respectively W). We similarly speak of biembeddings of Latin squares when discussing face 2-colourable triangular embeddings of $K_{n,n,n}$. It was shown in [5], by a very easy argument, that a face 2-colourable triangular embedding of $K_{n,n,n}$, i.e. a biembedding of two Latin squares, is necessarily in an orientable surface.

We note here that all the surfaces we consider will be, unless otherwise stated, closed, connected 2-manifolds, without a boundary: that is, in the orientable case, S_g the sphere with g handles and, in the nonorientable case, N_γ the sphere with γ crosscaps. A generalized pseudosurface is obtained from a finite collection of such surfaces by making a finite number of identifications, each of finitely many points, so that the resulting topological space is connected. Given a surface embedding of some simple graph G with vertex set $V(G)$, the *rotation* at a vertex $v \in V(G)$ is the cyclically ordered permutation of vertices adjacent to v, with the ordering determined by the embedding. The set of rotations at all the vertices of G is called the *rotation scheme* for the embedding.

A necessary condition, derived from Euler's formula, for the existence of a face 2-colourable triangular embedding of K_n in an orientable surface is that $n \equiv 3$ or $7 \pmod{12}$. It follows from embeddings given by Ringel [14] and by Youngs [15] that this necessary condition is also sufficient. In the nonorientable case the condition is $n \equiv 1$ or $3 \pmod 6$, and embeddings given in [14] and in [7] establish the sufficiency of this condition for $n \geq 9$. In each case, when n is divisible by 3, these embeddings have a parallel class of faces in each colour. Whenever n is odd, there is a face 2-colourable triangular embedding of $K_{n,n,n}$ with a parallel class of triangular faces in each colour [6].

We now proceed to establish our lower bound for the number of biembeddings of certain Latin squares.

2 Latin squares and complete tripartite graphs

Unless otherwise stated, a Latin squares of side n will have its rows and columns indexed by the elements of $\mathbb{Z}_n$

Definition 2.1 *Suppose that $A = (a_{i,j})$ is a Latin square of side n. If, for* *each* $i \in \mathbb{Z}_n$*, the permutation*

$$\begin{pmatrix} a_{i,0} & a_{i,1} & \cdots & a_{i,n-1} \\ a_{i+1,0} & a_{i+1,1} & \cdots & a_{i+1,n-1} \end{pmatrix}$$

is a single cycle of length n*, including the case* $i = n-1$ *when* $i+1$ *is taken as* 0*, then we will say that* A *is* consecutively row Hamiltonian*, cr-Hamiltonian for short.*

The Latin square with entries in $\mathbb{Z}_n$ given by $a_{i,j} \equiv i + j \pmod n$ is cr-Hamiltonian, so such squares certainly exist. It seems reasonable to speculate that there are many such squares. The proportion of permutations on $\mathbb{Z}_n$ that are cycles of length n is $\frac{1}{n}$. Setting aside the fact that rows cannot be chosen independently, this suggests that for large n the proportion of Latin squares of side n that are cr-Hamiltonian might be around $\frac{1}{n^n}$. However, we cannot yet prove any result in this direction. Instead we will construct a particular class of cr-Hamiltonian squares with properties desirable for our purposes. Our interest in such squares derives from the following lemma.

Lemma 2.1 *Suppose that* $A = (a_{i,j})$ *is a cr-Hamiltonian Latin square of side* n*. Then* A *has a biembedding with a copy of itself.*

Proof. Define $B = (b_{i,j})$, where $b_{i,j} = a_{i+1,j}$ with subscript arithmetic modulo n. Then B is itself a Latin square of side n and it is isomorphic with A. The square A corresponds to a set $\mathcal{A}$ of ordered triples (r_i, c_j, e_k) where $k = a_{i,j}$ denoting that row i, column j contains entry k. Similarly B corresponds to a set $\mathcal{B}$ of ordered triples of the same form where $k = a_{i+1,j}$. Take the triples from both $\mathcal{A}$ and $\mathcal{B}$ as oriented triangles and sew them together along common edges. The result is certainly a triangulated generalized pseudosurface. We prove that it is actually a triangulated surface. This is done by exhibiting the rotation at each vertex and showing that it is a cycle of length $2n$.

So, consider first the rotation at a "row vertex" r_i. Take $j_0 = 0$. There exists a triple $(r_i, c_{j_0}, e_{k_0}) \in \mathcal{A}$. But there is a triple $(r_i, c_{j_1}, e_{k_0}) \in \mathcal{B}$ for some j_1. Then there are triples $(r_i, c_{j_1}, e_{k_1}) \in \mathcal{A}$ and $(r_i, c_{j_2}, e_{k_1}) \in \mathcal{B}$, and so on. We have $k_0 = a_{i,j_0}$ and $k_0 = a_{i+1,j_1}$, $k_1 = a_{i,j_1}$ and $k_1 = a_{i+1,j_2}$, ... It follows that the permutation $(k_0, k_1, k_2, \ldots)$ is the inverse of the permutation

$$\begin{pmatrix} a_{i,0} & a_{i,1} & \cdots & a_{i,n-1} \\ a_{i+1,0} & a_{i+1,1} & \cdots & a_{i+1,n-1} \end{pmatrix}$$

and is therefore a cycle of length n. Thus $k_0, k_1, k_2, \ldots$ are distinct and so, therefore, are $j_0, j_1, j_2, \ldots$. The rotation at r_i is the single cycle $(c_{j_0}, e_{k_0}, c_{j_1}, e_{k_1}, \ldots, c_{j_{n-1}}, e_{k_{n-1}})$ of length $2n$.

Next consider the case of a "column vertex" c_j. Take $k_0 = 0$. There exists a triple $(r_{i_0}, c_j, e_{k_0}) \in \mathcal{A}$. But there is a triple $(r_{i_0}, c_j, e_{k_1}) \in \mathcal{B}$ for some k_1. Then there are triples $(r_{i_1}, c_j, e_{k_1}) \in \mathcal{A}$ and $(r_{i_1}, c_j, e_{k_2}) \in \mathcal{B}$, and so on. We have $k_0 = a_{i_0,j}$, $k_1 = a_{i_0+1,j} = a_{i_1,j}$, $k_2 = a_{i_1+1,j} = a_{i_2,j}$, ... Thus $i_1 = i_0 + 1, i_2 = i_0+2, \ldots$, and it follows that the permutation $(i_0, i_1, i_2, \ldots)$ is a cycle of length n. The rotation at c_j is the single cycle $(e_{k_0}, r_{i_0}, e_{k_1}, r_{i_1}, \ldots, e_{k_{n-1}}, r_{i_{n-1}})$ of length $2n$.

Finally consider the case of an "entry vertex" e_k. Take $i_0 = 0$. There exists a triple $(r_{i_0}, c_{j_0}, e_k) \in \mathcal{A}$. But there is a triple $(r_{i_1}, c_{j_0}, e_k) \in \mathcal{B}$ for some i_1. Then there are triples $(r_{i_1}, c_{j_1}, e_k) \in \mathcal{A}$ and $(r_{i_2}, c_{j_1}, e_k) \in \mathcal{B}$, and so on. We have $k = a_{i_0,j_0} = a_{i_1+1,j_0} = a_{i_1,j_1} = a_{i_2+1,j_1}$, ... Thus $i_1 = i_0 - 1, i_2 = i_0 - 2, \ldots$, and it follows that the permutation $(i_0, i_1, i_2, \ldots)$ is a cycle of length n. The rotation at e_k is the single cycle $(r_{i_0}, c_{j_0} r_{i_1}, c_{j_1}, \ldots, r_{i_{n-1}}, c_{j_{n-1}})$ of length $2n$. □

As noted in the Introduction, any biembedding of a Latin square of side n is necessarily in an orientable surface and represents a face 2-colourable triangular embedding of the complete tripartite graph $K_{n,n,n}$.

Lemma 2.2 *For each $m = 2^n$ with $n \geq 2$ there is a cr-Hamiltonian Latin square of side $2m$ that has $m^2(m-1)$ subsquares of side 2 and a transversal.*

Proof. We start with the Latin square $A = (a_{i,j})$ formed from the Cayley table of the additive group $\mathbb{Z}_2^n$. It is convenient in our proof of this lemma to represent the elements of the group as the elements of $\mathbb{Z}_m$. We distinguish carefully between addition in $\mathbb{Z}_m$, denoted by $+$, and addition in $\mathbb{Z}_2^n$, denoted by $\oplus$. For example, with $n = 3$, $m = 8$ and $5 + 3 = 0$, but $5 \oplus 3 = 6$ because $101 \oplus 011 = 110$ in $\mathbb{Z}_2^3$. To clarify further, when we write $2i$ for some $i \in \mathbb{Z}_m$ we mean $i + i$ reduced modulo m, and not $i \oplus i = 0$. Then with the rows and columns of A indexed as usual by $\mathbb{Z}_m$, we have $a_{i,j} = i \oplus j$.

Next take a copy A' of A with rows, columns and entries indexed by a copy $\mathbb{Z}'_m$ of $\mathbb{Z}_m$ having elements $0', 1', \ldots, (m-1)'$. Denote by B' the Latin square formed from A' by moving the first column of A' to the last column, that is by applying the permutation $((m-1)', (m-2)', \ldots, 2', 1', 0')$ to the columns of A'. From A, A' and B' form a further Latin square C of side $2m$

by setting

$$C = \left(\begin{array}{c|c} A & A' \\ \hline B' & A \end{array} \right).$$

Now permute the rows of C to form a Latin square D by taking in turn the first row of C, followed by the m-th row of C, then the second row, followed by the $(m+1)$-th row, and so on. We will prove that D has the required properties needed to establish the lemma.

First we give the Latin square D in the case $n = 3$ as an example. This is shown in Table 1. It is easy to check that in this case D is cr-Hamiltonian, and it is not too difficult to see that it has precisely 448 subsquares of side 2. A transversal is indicated by the underlined elements.

$$D = \begin{pmatrix}
0 & \underline{1} & 2 & 3 & 4 & 5 & 6 & 7 & 0' & 1' & 2' & 3' & 4' & 5' & 6' & 7' \\
\underline{1'} & 2' & 3' & 4' & 5' & 6' & 7' & 0' & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & 0 & 3 & \underline{2} & 5 & 4 & 7 & 6 & 1' & 0' & 3' & 2' & 5' & 4' & 7' & 6' \\
0' & 3' & \underline{2'} & 5' & 4' & 7' & 6' & 1' & 1 & 0 & 3 & 2 & 5 & 4 & 7 & 6 \\
2 & 3 & 0 & 1 & 6 & \underline{7} & 4 & 5 & 2' & 3' & 0' & 1' & 6' & 7' & 4' & 5' \\
3' & 0' & 1' & 6' & \underline{7'} & 4' & 5' & 2' & 2 & 3 & 0 & 1 & 6 & 7 & 4 & 5 \\
3 & 2 & 1 & 0 & 7 & 6 & 5 & \underline{4} & 3' & 2' & 1' & 0' & 7' & 6' & 5' & 4' \\
2' & 1' & 0' & 7' & 6' & 5' & \underline{4'} & 3' & 3 & 2 & 1 & 0 & 7 & 6 & 5 & 4 \\
4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 & 4' & \underline{5'} & 6' & 7' & 0' & 1' & 2' & 3' \\
5' & 6' & 7' & 0' & 1' & 2' & 3' & 4' & 4 & 5 & \underline{6} & 7 & 0 & 1 & 2 & 3 \\
5 & 4 & 7 & 6 & 1 & 0 & 3 & 2 & 5' & 4' & 7' & \underline{6'} & 1' & 0' & 3' & 2' \\
4' & 7' & 6' & 1' & 0' & 3' & 2' & 5' & \underline{5} & 4 & 7 & 6 & 1 & 0 & 3 & 2 \\
6 & 7 & 4 & 5 & 2 & 3 & 0 & 1 & 6' & 7' & 4' & 5' & 2' & \underline{3'} & 0' & 1' \\
7' & 4' & 5' & 2' & 3' & 0' & 1' & 6' & 6 & 7 & 4 & 5 & 2 & 3 & \underline{0} & 1 \\
7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & 7' & 6' & 5' & 4' & 3' & 2' & 1' & \underline{0'} \\
6' & 5' & 4' & 3' & 2' & 1' & 0' & 7' & 7 & 6 & 5 & 4 & \underline{3} & 2 & 1 & 0
\end{pmatrix}$$

Table 1: The Latin square D in the case $n = 3$.

We show in general that D is cr-Hamiltonian. There are two cases to consider. With rows and columns of C indexed by $0, 1, \ldots, m-1, 0'$, $1', \ldots, (m-1)'$ in that order, the two cases are:

(a) row i of C followed by row i' of C, and

(b) row i' of C followed by row $i+1$ of C.

In case (a) the permutation is

$$\begin{pmatrix} i\oplus 0 & i\oplus 1 & i\oplus 2 & \ldots & i\oplus(m-1) & (i\oplus 0)' & (i\oplus 1)' & (i\oplus 2)' & \ldots & (i\oplus(m-1))' \\ (i\oplus 1)' & (i\oplus 2)' & (i\oplus 3)' & \ldots & (i\oplus 0)' & i\oplus 0 & i\oplus 1 & i\oplus 2 & \ldots & i\oplus(m-1) \end{pmatrix}.$$

Written in cycle form this is

$$(i\oplus 0, (i\oplus 1)', i\oplus 1, (i\oplus 2)', \ldots, i\oplus(m-1), (i\oplus 0)')$$

which is clearly a cycle of length $2m$.

In case (b) the permutation is

$$\begin{pmatrix} (i\oplus 1)' & (i\oplus 2)' & (i\oplus 3)' & \ldots & (i\oplus 0)' \\ (i+1)\oplus 0 & (i+1)\oplus 1 & (i+1)\oplus 2 & \ldots & (i+1)\oplus(m-1) \end{pmatrix.$$
$$\begin{matrix} i\oplus 0 & i\oplus 1 & i\oplus 2 & \ldots & i\oplus(m-1) \\ ((i+1)\oplus 0)' & ((i+1)\oplus 1)' & ((i+1)\oplus 2)' & \ldots & ((i+1)\oplus(m-1))' \end{pmatrix}.$$

In cycle form this permutation starts with $((i\oplus 1)', (i+1)\oplus 0, \ldots)$. Now suppose that $(i+1)\oplus 0 = i\oplus k$. Then $(i+1)\oplus k = (i\oplus k)\oplus k = i\oplus 0$. So we have the permutation as $((i\oplus 1)', (i+1)\oplus 0, (i\oplus 0)', (i+1)\oplus(m-1), \ldots)$. But then if $(i+1)\oplus(m-1) = i\oplus l$, we have $(i+1)\oplus l = i\oplus(m-1)$. Proceeding in this fashion we see that the permutation in cycle form is

$$((i\oplus 1)', (i+1)\oplus 0, (i\oplus 0)', (i+1)\oplus(m-1), \ldots, (i\oplus 2)', (i+1)\oplus 1)$$

which is clearly a cycle of length $2m$. Taking case (a) with case (b), it follows that D is cr-Hamiltonian.

To show that D has $m^2(m-1)$ subsquares of side 2, consider first the Latin square A. In this square choose any row, say i_1 and any two distinct columns, say j_1 and j_2. The number of such choices is $m^2(m-1)/2$. Suppose that the entry in row i_1, column j_1 is k_1, while that in the same row and column j_2 is k_2. Then $i_1\oplus j_1 = k_1$ and $i_1\oplus j_2 = k_2$. Now locate the entry k_2 in column j_1; suppose this occurs in row i_2. Then $i_2 \neq i_1$ and $i_2\oplus j_1 = k_2$. It follows from these three equations that $i_2\oplus j_2 = k_1$. Thus there is a subsquare of side 2 on the positions $(i_1, j_1), (i_1, j_2), (i_2, j_1), (i_2, j_2)$. Since every such subsquare will be counted twice in this enumeration, it follows that A contains precisely $m^2(m-1)/4$ subsquares of side 2. It then follows that the same applies to the squares A' and B'. Hence C contains at least $m^2(m-1)$ subsquares of side 2. Any additional such subsquares in C must come from four positions covering all four quadrants of this array. However the column permutation of A' to form B' precludes such subsquares. So we may conclude that C, and hence also D, contain precisely $m^2(m-1)$ subsquares of order 2.

It remains to prove that D has a transversal. We prove first that A has a special transversal by specifying a set T of (row, column, entry) triples that cover all rows, all columns and all entries, each precisely once. We take $T = T_1 \cup T_2$ where T_1 and T_2 are as follows.

$$\begin{aligned} T_1 &= \{(0,1,0 \oplus 1), \quad (1,3,1 \oplus 3), \quad (2,5,2 \oplus 5), \quad \ldots, \\ &\qquad \ldots \quad (\tfrac{m}{2}-1, m-1, (\tfrac{m}{2}-1) \oplus (m-1))\}, \\ T_2 &= \{(\tfrac{m}{2}+1, 0, (\tfrac{m}{2}+1) \oplus 0), \quad (\tfrac{m}{2}, 2, (\tfrac{m}{2}) \oplus 2), \\ &\quad (\tfrac{m}{2}+3, 4, (\tfrac{m}{2}+3) \oplus 4), \quad (\tfrac{m}{2}+2, 6, (\tfrac{m}{2}+2) \oplus 6), \\ &\quad \ldots, \quad (m-1, m-4, (m-1) \oplus (m-4)), \\ &\qquad (m-2, m-2, (m-2) \oplus (m-2))\}. \end{aligned}$$

The triples from T_1 have the form $(i, 2i+1, i \oplus (2i+1))$, and those from T_2 have the form $(\frac{m}{2}+i+\delta_i, 2i, (\frac{m}{2}+i+\delta_i) \oplus (2i))$ for $i = 0, 1, \ldots, \frac{m}{2}-1$, where $\delta_i = 1$ if i is even and -1 if i is odd. Clearly the triples from T cover every row and every column precisely once. We must now prove that the corresponding entries are distinct.

Note that if x is even then $x \oplus 1 = x+1$, while if x is odd $x \oplus 1 = x-1$. Hence, if i is even, $\frac{m}{2}+i$ is even and so $(\frac{m}{2}+i+\delta_i) \oplus (2i) = (\frac{m}{2}+i) \oplus (2i) \oplus 1$, while if i is odd, $\frac{m}{2}+i$ is odd and so again $(\frac{m}{2}+i+\delta_i) \oplus (2i) = (\frac{m}{2}+i) \oplus (2i) \oplus 1$. Thus the complete set of entries arising from $T_1 \cup T_2$ is $\{i \oplus (2i) \oplus 1 : i \in \mathbb{Z}_m\}$. We therefore have to show that if $i, j \in \mathbb{Z}_m$ with $i \neq j$, then $i \oplus (2i) \oplus 1 \neq j \oplus (2j) \oplus 1$ or, equivalently, that $i \oplus j \neq (2i) \oplus (2j)$. If i and j are distinct then $i \oplus j \neq 0$ and so the corresponding element of $\mathbb{Z}_2^n$ has a rightmost non-zero digit. Either $(2i) \oplus (2j) = 0$, in which case $i \oplus j \neq (2i) \oplus (2j)$, or $(2i) \oplus (2j)$ has its rightmost non-zero digit one place to the left of that of $i \oplus j$, and so again $i \oplus j \neq (2i) \oplus (2j)$. It follows that T is indeed a transversal of the square A.

Further transversals in A are now easily generated. In particular, if $\frac{m}{2}$ is added to all the row numbers in T_1 and T_2, with corresponding changes to the entries, to form T_3 and T_4 respectively, then $T_3 \cup T_4$ will be a transversal in A. Now denote the corresponding transversals in A' by $T_1' \cup T_2'$ and $T_3' \cup T_4'$ in the obvious way. Similarly define a transversal $U_1' \cup U_2'$ in B' by applying to T_1' and T_2' the permutation of columns that takes A' to B', thereby forming U_1' and U_2'. A transversal $\bar{T}$ in C may now be identified by the following pattern.

$$\bar{T} = \left(\begin{array}{c|c} T_1 & \\ & T'_3 \\ \hline U'_1 & \\ & T_2 \end{array} \right)$$

Here T_1 covers rows numbered 0 to $\frac{m}{2} - 1$ and odd-numbered columns from 1 to $m-1$; T'_3 covers rows $\frac{m}{2}$ to $m-1$ and odd-numbered columns from $1'$ to $(m-1)'$; U'_1 covers rows $0'$ to $(\frac{m}{2}-1)'$ and even-numbered columns from 0 to $m-2$; T_2 covers rows $(\frac{m}{2})'$ to $(m-1)'$ and even-numbered columns from $0'$ to $(m-2)'$. Entries in T_1 and T_2 cover all of $\mathbb{Z}_m$. Entries in U'_1 and T'_3 are the same as those in T'_1 and T'_2 respectively and so cover all of $\mathbb{Z}'_m$. It follows that $\bar{T}$ is a transversal in C, and hence that there is a transversal in D. □

Applying the result of Lemma 2.1 to the square of side $2m = 2^{n+1}$ constructed in Lemma 2.2 gives a biembedding of that Latin square. The biembedding is actually a face 2-colourable triangular embedding of $K_{2m,2m,2m}$ in an orientable surface. In this embedding, each subsquare of order 2 is realized as a Pasch configuration: that is, as a set of four triangles with vertex sets of the form $\{a,b,c\},\{a,d,e\},\{f,b,e\},\{f,d,c\}$. Using the terminology of Lemma 2.1, we may take a and f to be row vertices, b and d to be column vertices, and c and e to be entry vertices. The orientations of the four triangles can then be taken as $(a,b,c),(a,d,e),(f,b,e),(f,d,c)$. There will be $m^2(m-1)$ Pasch configurations in each colour class. The transversal in D will be realized as a parallel class of $2m$ triangles in each of the two colour classes. We now apply a recursive construction for biembeddings of Latin squares with this biembedding as one of the ingredients. The result is a large number of nonisomorphic biembeddings. Our basic construction is a modification of one given in [6]. We start by giving an informal description of that construction.

Take a face 2-colourable triangular embedding P of $K_{p,p,p}$ and another Q of $K_{q,q,q}$, and assume that the latter has a parallel class of triangular faces in one of the two colour classes, say black. Now take q copies of P, say P_i for $i = 1,2,\ldots,q$. For each oriented white triangular face $W = (a,b,c)$ of P, we "bridge" the corresponding white triangles $W_i = (a_i,b_i,c_i)$ of the embeddings P_i. To do this we glue a copy of Q to these triangles in the following manner. We take a copy of Q and label the vertex parts with $\{a_i\},\{b_i\},\{c_i\}$ in such a way that the parallel class has oriented black triangles labelled (a_i,c_i,b_i). We then glue the black triangle (a_i,c_i,b_i) on Q onto the white triangle W_i

on P_i so that corresponding vertices and edges are identified. Repeating this process for every white triangle of P results in a face 2-colourable triangular embedding of $K_{pq,pq,pq}$ in an orientable surface.

In fact, by varying the bridges (that is, by using differently labelled bridges for each of the p^2 white faces of P) it is possible to generate a large number of nonisomorphic biembeddings of Latin squares of side pq using this construction. Details are given in [6]. But we now show how it is possible to carry out the bridging operation in a different manner that leads to a substantially greater number of nonisomorphic biembeddings.

In the construction just described, the bridging operation provides all the "missing" adjacencies between the q copies P_i. The bridge across the q triangles $W_i = (a_i, b_i, c_i)$ yields the adjacencies a_ib_j, a_ic_j, b_ic_j for $i, j = 1, 2, \ldots, q,\ i \neq j$. Now suppose that P contains a Pasch configuration $(a, b, c), (a, d, e), (f, b, e), (f, d, c)$. The four corresponding bridges provide the missing adjacencies $a_ib_j, a_ic_j, b_ic_j, a_id_j, a_ie_j, d_ie_j, f_ib_j, f_ie_j, b_ie_j, f_id_j, f_ic_j, d_ic_j$ for $i \neq j$. It is, however, possible to provide these adjacencies by an alternative arrangement of bridges. Concentrating for a moment on the adjacencies between P_1 and P_2, we may bridge (a_1, b_1, c_1) to (a_2, d_2, e_2), (a_1, d_1, e_1) to (a_2, b_2, c_2), (f_1, b_1, e_1) to (f_2, d_2, c_2), and (f_1, d_1, c_1) to (f_2, b_2, e_2) by suitable renaming of the vertices of the four bridges involved. The first bridge then provides the adjacencies $a_1d_2, a_1e_2, b_1a_2, b_1e_2, c_1a_2, c_1d_2$, the second provides $a_1b_2, a_1c_2, d_1a_2, d_1c_2, e_1a_2, e_1b_2$, the third provides $f_1d_2, f_1c_2, b_1f_2, b_1c_2, e_1f_2, e_1d_2$, and the fourth provides $f_1b_2, f_1e_2, d_1f_2, d_1e_2, c_1f_2, c_1b_2$. It will be seen that these 24 adjacencies are the same as the 24 adjacencies arising for the original bridging arrangement across the Pasch configuration between P_1 and P_2.

We will call the original arrangement of bridges *standard* and an arrangement of the type just described *non-standard*. We slightly tighten this definition of non-standard below, but before doing so we give an example to help clarify the situation.

So consider the case $q = 3$ where the bridges are face 2-colourable triangular embeddings of $K_{3,3,3}$ in the torus. In the standard arrangement we may take the four bridges on a Pasch configuration as shown in Figure 2.1. The lightly shaded triangles are glued to the corresponding triangles on the surfaces defined by P_1, P_2 and P_3.

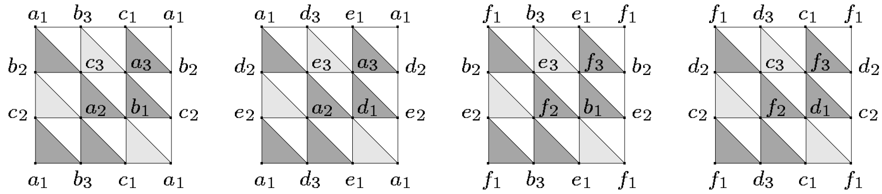

Figure 2.1: Standard bridging arrangement.

In the non-standard arrangement the four bridges are relabelled as shown in Figure 2.2, and the gluing operation is carried out in the same manner.

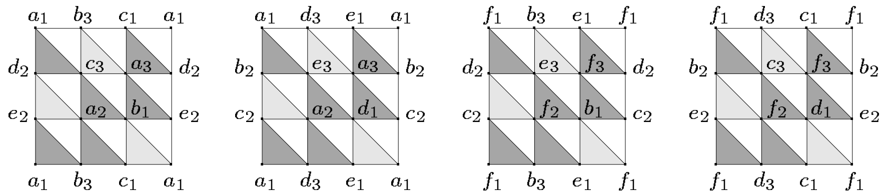

Figure 2.2: Non-standard bridging arrangement.

The embedding that results when a standard arrangement of four bridges on a Pasch configuration is replaced by a non-standard arrangement will always be differently labelled from the original. For example, in this specific case, the triangle (a_1, b_1, c_2) appears in Figure 2.1, but not in Figure 2.2. It is also worth remarking that there are four ways of getting the adjacencies between P_1 and P_2, one standard and three non-standard. The non-standard arrangement shown effectively corresponds to the permutation $(b_2, d_2)(c_2, e_2)$. The other two non-standard arrangements correspond to the permutations $(a_2, f_2)(c_2, e_2)$ and $(a_2, f_2)(b_2, d_2)$. Similar permutations may be applied to $\{a_3, b_3, c_3, d_3, e_3, f_3\}$. This gives $4^2 = 16$ bridging arrangements across the three levels, of which only one is standard. In addition, one may relabel the bridges by applying a permutation to the vertices in a similar manner to that described in [6]. This gives a choice of two differently labelled bridges in each case and hence a total of $2^4 = 16$ standard arrangements across the Pasch configuration and $16.2^4 = 256$ arrangements altogether.

Returning to the general case where the bridges are biembeddings of $K_{q,q,q}$, we now tighten our definition of a non-standard bridging arrangement on a Pasch configuration $\{\{a, b, c\}, \{a, d, e\}, \{f, b, e\}, \{f, d, c\}\}$ by requiring that for some $i, j,\ i \neq j$, the four bridges contain white triangles of the form $(a_i, b_i, e_j), (a_i, d_i, c_j), (f_i, b_i, c_j)$ and (f_i, d_i, e_j). Since a standard bridge on (a, b, c) must contain a white triangle $(a_i, b_i, c_j),\ i \neq j$ (in fact one such triangle for each value of i), this requirement is easily satisfied by relabelling the black triangle (a_j, b_j, c_j) as (a_j, d_j, e_j), and then doing likewise for the other three bridges, when forming the non-standard bridge arrangement.

Although we have only described standard and non-standard arrangements of bridges in the context of orientable embeddings of $K_{p,p,p}$, the concept is easily extended to both orientable and nonorientable embeddings of other graphs with faces that form Pasch configurations. In this more general setting we make the following definition.

Definition 2.2 *Given a face 2-colourable triangular embedding M of a graph G, a collection $\mathcal{C}$ of Pasch configurations in one of the colour classes will be called* independent *if no two of the Pasch configurations have a common block.*

We now establish a result that will assist us in estimating the number of embeddings of some complete tripartite graphs.

Lemma 2.3 *Suppose that M is a face 2-colourable triangular embedding of $K_{p,p,p}$ and that $\mathcal{C}^{(1)}$ and $\mathcal{C}^{(2)}$ are two different independent collections of Pasch configurations in the same colour class, say white. For $i = 1, 2$, let $M^{(i)}$ be the embedding that results when we apply standard $K_{q,q,q}$ bridges to each white face not in $\mathcal{C}^{(i)}$ and non-standard bridge arrangements to each Pasch configuration in $\mathcal{C}^{(i)}$. Then the embeddings $M^{(1)}$ and $M^{(2)}$ will be differently labelled.*

Proof. If the Pasch configurations in $\mathcal{C}^{(1)}$ and $\mathcal{C}^{(2)}$ do not cover the same triples, then there is a Pasch configuration $\{\{a, b, c\}, \{a, d, e\}, \{f, b, e\}, \{f, d, c\}\}$ in $\mathcal{C}^{(1)}$, and without loss of generality we may assume that $\{a, b, c\}$ does not lie in any Pasch configuration in $\mathcal{C}^{(2)}$ and that $M^{(1)}$ has a triangle (a_i, b_i, e_j) with $i \neq j$. However, in $M^{(2)}$ every edge $a_i b_i$ lies either in a triangle (a_i, b_i, c_k) on a bridge or in a triangle (a_i, b_i, x_i) on one copy of the original embedding M, so in this case $M^{(1)}$ and $M^{(2)}$ are differently labelled.

Now suppose that $\mathcal{C}^{(1)}$ and $\mathcal{C}^{(2)}$ cover the same triples and that $M^{(1)}$ and $M^{(2)}$ are identically labelled. We derive a contradiction as follows. If $\mathcal{C}^{(1)}$ contains the Pasch configuration $R = \{\{a, b, c\}, \{a, d, e\}, \{f, b, e\}, \{f, d, c\}\}$ then we may suppose without loss of generality that the contributing blocks do not lie in the same Pasch configuration in $\mathcal{C}^{(2)}$ and that the non-standard bridging arrangement for $\mathcal{C}^{(1)}$ contains a white triangle (a_i, b_i, e_j) with $i \neq j$.

Since the same triangle arises in the non-standard bridging arrangement for $\mathcal{C}^{(2)}$, the triple $\{a, b, c\}$ and either $\{a, d, e\}$ or $\{f, b, e\}$ must lie together in a Pasch configuration in $\mathcal{C}^{(2)}$. So $\mathcal{C}^{(2)}$ contains either the Pasch configuration $S = \{\{a, b, c\}, \{a, d, e\}, \{g, b, d\}, \{g, c, e\}\}$ or $T = \{\{a, b, c\}, \{f, b, e\}, \{h, a, f\}, \{h, c, e\}\}$ for some suitable g or $h \in V(K_{p,p,p})$. However, there is no edge ce in the tripartite graph $K_{p,p,p}$, and so neither S nor T can exist. It follows that $M^{(1)}$ and $M^{(2)}$ must be differently labelled. □

Lemma 2.4 *Suppose that M is a face 2-colourable triangular embedding of $K_{p,p,p}$ having r Pasch configurations in the same colour class, say white. Then the number of different independent collections of Pasch configurations in the white colour class is at least*

$$(4p-3)^{\left(\frac{r}{4(p-1)}-1\right)}.$$

Proof. Denote by I_k the number of distinct independent collections of Pasch configurations in the white colour class that contain precisely k Pasch configurations. Each triangle in the white colour class can lie in at most $(p-1)$ Pasch configurations in that colour class, and there are four triangles to each Pasch configuration. So, for $k-1 < \frac{r}{4(p-1)}$ we have

$$\begin{aligned} I_k &\geq r(r-4(p-1))(r-8(p-1))\cdots(r-4(k-1)(p-1))/k! \\ &\geq (4(p-1))^k N(N-1)(N-2)\cdots(N-(k-1))/k! \end{aligned}$$

where $N = \lfloor \frac{r}{4(p-1)} \rfloor$. Then, summing over $k = 0, 1, \ldots, N$ gives the number of distinct independent collections of Pasch configurations in the white colour class as at least

$$(1+4(p-1))^N \geq (4p-3)^{\left(\frac{r}{4(p-1)}-1\right)}.$$

□

Theorem 2.1 *Let $p = 2^s$ where $s \geq 3$ and suppose there is a face 2-colourable triangular embedding of $K_{q,q,q}$ having a parallel class in one of the colour classes, say black. Then there are at least*

$$(4p-3)^{\left(\frac{p^2(p-2)}{32(p-1)}-1\right)}$$

differently labelled face 2-colourable triangular embeddings of $K_{pq,pq,pq}$ all of which have a common parallel class of black triangular faces. Furthermore, there are at least

$$\frac{(4p-3)^{\left(\frac{p^2(p-2)}{32(p-1)}-1\right)}}{6((pq)!)^3}$$

nonisomorphic face 2-colourable triangular embeddings of $K_{pq,pq,pq}$.

Proof. Take a face 2-colourable triangular embedding of $K_{p,p,p}$ that has $p^2(p-2)/8$ Pasch configurations in each colour class and a parallel class of triangles in each colour class, as guaranteed by Lemmas 2.1 and 2.2. The number of independent collections of Pasch configurations in each colour class is, by Lemma 2.4, at least

$$(4p-3)^{(\frac{p^2(p-2)}{32(p-1)}-1)}.$$

By applying Lemma 2.3 and bridging the white triangles , we see that there is at least this number of differently labelled face 2-colourable triangular embeddings of $K_{pq,pq,pq}$. Since the black faces of the q copies of the $K_{p,p,p}$ embedding are unaltered by bridging the white triangles, all of the resulting embeddings of $K_{pq,pq,pq}$ have a common parallel class of black triangles.

The maximum possible size of an isomorphism class of such an embedding is $6((pq)!)^3$, and so the number of isomorphism classes is at least

$$\frac{(4p-3)^{(\frac{p^2(p-2)}{32(p-1)}-1)}}{6((pq)!)^3}.$$

□

Corollary 2.1.1 *For* $n = 3 \cdot 2^s$ *and* s *sufficiently large, there are at least* $n^{n^2/288}$ *nonisomorphic face 2-colourable triangular embeddings of* $K_{n,n,n}$*, each of which has a parallel class in one colour.*

Proof. In Theorem 2.1 take $q = 3$ and replace p by $n/3$. The number of isomorphism classes is then at least

$$\frac{(\frac{4n}{3}-3)^{(\frac{n^2(n-6)}{288(n-3)}-1)}}{6(n!)^3} > n^{\frac{n^2}{288}},$$

provided that n is sufficiently large. □

Re-phrasing the corollary in the language of Latin squares, it follows that for $n = 3 \cdot 2^s$ and s sufficiently large, there are at least $n^{n^2/288}$ biembeddings of Latin squares of side n, each of which has a parallel class in one colour. Note that these biembeddings are necessarily in an orientable surface.

3 From complete tripartite graphs to complete graphs

A recursive construction given in [6] forms the basis for the results of this section and we start by describing it informally.

The construction takes a face 2-colourable triangular embedding M of K_m, a face 2-colourable triangular embedding R of $K_{r,r,r}$ having a parallel class of black triangular faces, and a face 2-colourable triangular embedding S of K_{2r+1}. The embeddings M and S are biembeddings of Steiner triple systems, and the embedding R is a biembedding of Latin squares. Choose any single vertex ∞ from the embedding M and delete the cap at this point; that is to say, remove all the triangular faces incident with ∞ from M to leave a triangular embedding $\bar{M}$ of K_{m-1} in a bordered surface. Next take r copies of $\bar{M}$, say $\bar{M}_i$ for $i = 1, 2, \ldots, r$. For each white triangular face $W = (a, b, c)$ of $\bar{M}$, bridge the corresponding white triangles $W_i = (a_i, b_i, c_i)$ of the embeddings $\bar{M}_i$ in the manner described in the previous section, using a copy of the embedding R. When this operation is complete, the resulting embedding is "close" to that of $K_{r(m-1)}$ but with a small number of missing adjacencies corresponding to the deleted triangles, and the supporting surface has r disjoint boundaries. If $(m-1)/2$ is odd, these boundaries can then be capped by means of an auxiliary embedding A, which provides one extra vertex and all missing adjacencies, to give a face 2-colourable triangular embedding of $K_{r(m-1)+1}$. The embedding A itself is constructed using copies of the embedding S and an appropriate voltage graph. Full details are given in [6]. If the embeddings M and S are in orientable surfaces, then the resulting embedding of $K_{r(m-1)+1}$ is also orientable. As observed in [6], it is possible to use differently labelled embeddings R to bridge different white triangular faces W of $\bar{M}$. The following result is an extension of Construction 5 of that paper to the nonorientable case, as alluded to in the concluding remarks of [6].

Theorem 3.1 *Suppose that* $m \equiv 3$ *or* 7 (mod 12) *and that* $r \equiv 0$ *or* 4 (mod 6). *Suppose also that there are* k *differently labelled face 2-colourable triangular embeddings of* $K_{r,r,r}$*, all of which have a common parallel class of black triangular faces. Then we may construct* $k^{(m-1)(m-3)/6}$ *differently labelled face 2-colourable triangular embeddings of* $K_{r(m-1)+1}$*, all of which are nonorientable.*

Proof. The condition $m \equiv 3$ or $7 \pmod{12}$ is sufficient to ensure that $(m-1)/2$ is odd and that there exists a face 2-colourable triangular embedding of K_m in some surface, which may be orientable or nonorientable. Likewise the condition $r \equiv 0$ or $4 \pmod 6$ gives $2r+1 \equiv 1$ or $9 \pmod{12}$ which is sufficient to ensure the existence of a face 2-colourable triangular embedding of K_{2r+1} in a nonorientable surface. The remainder of the proof is as given for Construction 5 in [6]. Note that $r(m-1)+1 \equiv 1$ or 9 (mod 12), and so the embeddings of $K_{r(m-1)+1}$ are necessarily nonorientable. $\square$

Corollary 3.1.1 *Suppose that* $m \equiv 3$ *or* $7 \pmod{12}$*, that* $r = 2^s q$*, where* $s \geq 3$*, and that there is a face 2-colourable triangular embedding of* $K_{q,q,q}$ *having a parallel class of black triangular faces. Then there are at least*

$$\left[\frac{4r}{q} - 3\right]^{\left(\frac{r^2(r-2q)}{32q^2(r-q)} - 1\right)\left(\frac{(m-1)(m-3)}{6}\right)}$$

differently labelled face 2-colourable triangular embeddings of $K_{r(m-1)+1}$*, each of which is nonorientable.*

Proof. The result follows immediately from Theorems 2.1 and 3.1. $\square$

Corollary 3.1.1 enables us to state the following result.

Corollary 3.1.2 *Suppose that* $n = 2^s q(m-1)+1$*, where* $m \equiv 3$ *or* 7 $\pmod{12}$*,* $m \geq 7$*, and where* q *is such that there exists a face 2-colourable triangular embedding of* $K_{q,q,q}$ *having a parallel class of black triangular faces. Put* $a = \dfrac{m-3}{192q^2(m-1)}$*. Then, as* $s \to \infty$*, there are at least* $n^{n^2(a-o(1))}$ *nonisomorphic face 2-colourable triangular embeddings of* K_n *in a nonorientable surface.*

Proof. In the estimate given by Corollary 3.1.1, write $r = \frac{n-1}{m-1}$. This gives the lower bound for the number of differently labelled face 2-colourable triangular embeddings of K_n as

$$\left[\frac{4(n-1)}{q(m-1)} - 3\right]^{\left(a(n-1)^2\left(\frac{n-1-2q(m-1)}{n-1-q(m-1)}\right) - \frac{(m-1)(m-3)}{6}\right)}$$

Dividing this expression by $n!$ (the maximum possible size of an isomorphism class), and making the usual estimates, gives the result. $\square$

As noted in the Introduction, for odd q there exists a face 2-colourable triangular embedding of $K_{q,q,q}$ having a parallel class of black faces. If, for example, we take $m = 7$ and $q = 3$ in Corollary 3.1.2, then we can deduce that for n of the form $9 \cdot 2^{s+1} + 1$, as $s \to \infty$, there are at least $n^{n^2\left(\frac{1}{2592} - o(1)\right)}$ nonisomorphic face 2-colourable triangular embeddings of K_n in a nonorientable surface.

4 Concluding remarks

By taking m large and q to have its minimum value 3 in Corollary 3.1.2, our value for the constant a approaches $\frac{1}{1728}$, but the resulting bound applies to a heavily restricted range of values of n. We conjecture that all pairs of Steiner triple systems of order $n \geq 9$ are biembeddable in a nonorientable surface, and that the "correct" value of a, for all n, is in fact $a = \frac{1}{3}$.

The method described in this paper for obtaining estimates is based on biembeddings of cr-Hamiltonian Latin squares. If, as suggested in the Introduction, the proportion of Latin squares of side n that are cr-Hamiltonian is around $\frac{1}{n^n}$, then this would imply that the number of nonisomorphic face 2-colourable triangular embeddings of $K_{n,n,n}$ is at least $n^{n^2(1-o(1))}$. Any such result would improve our results for embeddings of complete graphs by greatly increasing the value of the constant a in Corollary 3.1.2.

The results obtained in Section 3 for biembeddings of Steiner triple systems all relate to nonorientable surfaces. By constructing an infinite class of cr-Hamiltonian Latin squares of odd side n each having a transversal and at least An^3 subsquares of side 2 for some constant $A > 0$, it should be possible to extend the results to orientable biembeddings. We hope to make this the subject of a future paper.

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References

[1] S. R. Alpert, *Twofold triple systems and graph imbeddings,* J. Combin. Theory Ser. A **18** (1975), 101–107.

[2] K. N. Bhattacharya, *A note on two-fold triple systems,* Sankhyā **6** (1943), 313–314.

[3] C. P. Bonnington, M. J. Grannell, T. S. Griggs and J. Širáň, *Exponential families of nonisomorphic triangulations of complete graphs,* J. Combin. Theory Ser. B **78** (2000), 169–184.

[4] C. J. Colbourn and A. Rosa, *Triple Systems,* Clarendon Press, New York, 1999.

[5] M. J. Grannell, T. S. Griggs and M. Knor, *Biembeddings of Latin squares and Hamiltonian decompositions,* Glasgow Math. J. **46** (2004), 443–457.

[6] M. J. Grannell, T. S. Griggs and J. Širáň, *Recursive constructions for triangulations,* J. Graph Theory **39** (2002), 87–107.

[7] M. J. Grannell and V. P. Korzhik, *Nonorientable biembeddings of Steiner triple systems,* Discrete Math. **285** (2004), 121–126.

[8] T. P. Kirkman, *On a problem in combinations,* Cambridge and Dublin Math. J. **2** (1847), 191–204.

[9] V. P. Korzhik, *Exponentially many nonisomorphic orientable triangular embeddings of K_{12s} and K_{12s+3},* in preparation.

[10] V. P. Korzhik and H.-J. Voss, *On the number of nonisomorphic orientable regular embeddings of complete graphs,* J. Combin. Theory Ser. B **81** (2001), 58–76.

[11] V. P. Korzhik and H.-J. Voss, *Exponential families of nonisomorphic non-triangular orientable genus embeddings of complete graphs,* J. Combin. Theory Ser. B **86** (2002), 186–211.

[12] V. P. Korzhik and H.-J. Voss, *Exponential families of nonisomorphic nonorientable genus embeddings of complete graphs,* J. Combin. Theory Ser. B **91** (2004), 253–287.

[13] S. Lawrencenko, S. Negami and A. T. White, *Three nonisomorphic triangulations of an orientable surface with the same complete graph,* Discrete Math. **135** (1994), 367–369.

[14] G. Ringel, *Map Color Theorem,* Springer-Verlag, New York, 1974.

[15] J. W. T. Youngs, *The mystery of the Heawood conjecture* in *Graph Theory and its Applications,* Academic Press, New York, 1970, 17–50.