On the covering number $g_1^{(4)}(18)$

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Abstract

The minimum number of blocks having maximum size precisely four that are required to cover, exactly λ times, all pairs of elements from a set of cardinality v is denoted by $g_\lambda^{(4)}(v)$. The values of $g_\lambda^{(4)}(v)$ are known apart from the cases $(v, \lambda) = (17, 1)$ and $(18, 1)$. We prove that $g_1^{(4)}(18) \geq 32$, thereby reducing this outstanding case to just two possible values, namely 32 and 33.

AMS classifications: Primary: 05B40, Secondary: 05B05, 05B30.

Keywords: pairwise balanced design, covering number.

1 Introduction

The covering number $g_{\lambda}^{(k)}(v)$ is defined as the cardinality of the minimal pairwise balanced design (PBD) on a base set of v points and having largest block size k such that every pair occurs exactly λ times in the design. In this paper we will be concerned with the case $k = 4$, so that the designs may contain pairs, triples and quadruples, with at least one block of the latter type. Because of the requirement for a block of size 4, $g_{\lambda}^{(4)}(v)$ is only defined if $v \geq 4$. The values of $g_{\lambda}^{(4)}(v)$ are already known for all (v, λ) apart from the cases $(v, \lambda) = (17, 1)$ and $(18, 1)$. The values for $\lambda > 1$ are given in [2, 3, 4].

For $\lambda = 1$, the problem of determining $g_1^{(4)}(v)$ was solved by Stanton and Stinson, [11], apart from three exceptional cases $v = 17, 18, 19$. The general results are summarized in Table 1.

v	$g_1^{(4)}(v)$	(Pairs, Triples)	Comments
$12s$	$12s^2 + s$	$(0, 4s)$	
$12s + 1$	$12s^2 + s$	$(0, 0)$	BIBD
$12s + 2$	$12s^2 + 7s + 1$	$(1, 8s)$	
$12s + 3$	$12s^2 + 7s + 1$	$(0, 4s + 1)$	
$12s + 4$	$12s^2 + 7s + 1$	$(0, 0)$	BIBD
$12s + 5$	$12s^2 + 13s + 4$	$(1, 8s + 3)$	$v \neq 5, 17$
$12s + 6$	$12s^2 + 13s + 4$	$(0, 4s + 3)$	$v \neq 6, 18$
$12s + 7$	$12s^2 + 13s + 7$	$(0, 7)$	$v \neq 7, 19$
$12s + 8$	$12s^2 + 19s + 8$	$(1, 8s + 5)$	$v \neq 8$
$12s + 9$	$12s^2 + 19s + 8$	$(0, 4s + 4)$	$v \neq 9$
$12s + 10$	$12s^2 + 19s + 11$	$(0, 7)$	$v \neq 10$
$12s + 11$	$12s^2 + 25s + 13$	$(1, 8s + 6)$	

Table 1: $g_1^{(4)}(v)$.

For $5 \leq v \leq 10$ we have:

$g_1^{(4)}(5) = 5$ (one quadruple and four pairs),
$g_1^{(4)}(6) = 8$ (one quadruple, one triple and six pairs),
$g_1^{(4)}(7) = 10$ (one quadruple, three triples and six pairs),
$g_1^{(4)}(8) = 11$ (one quadruple, six triples and four pairs),
$g_1^{(4)}(9) = 12$ (two quadruples, seven triples and three pairs),
$g_1^{(4)}(10) = 12$ (three quadruples and nine triples).

The results of [11] show that $g_1^{(4)}(v) \geq 29$ for $v = 17, 18, 19$. For $v = 17$, Seah and Stinson, [6], have given a PBD with 31 blocks comprising 17 quadruples, 10 triples and 4 pairs. The design is listed in [12]. Recently, Stanton, [10], has ruled out the value 29. So $30 \leq g_1^{(4)}(17) \leq 31$. For

$v = 18$, Stanton, [9] and [8], has shown that $30 \leq g_1^{(4)}(18) \leq 33$. More recently, Grüttmüller, Roberts and Stanton [5] have improved this to $31 \leq g_1^{(4)}(18) \leq 33$. Finally, Stanton, [7], determined the exact value of $g_1^{(4)}(19)$ as 35 by exhibiting a design with 22 quadruples and 13 triples.

In the course of this paper we will make use of a range of combinatorial terminology, details of which may be found in [1]. Given a PBD, we shall use g_i to denote the number of blocks of size i. Blocks of sizes 2 and 3 will be described as *small* blocks. For a point x of a PBD, we use $r_i(x)$ to denote the number of blocks of size i in which x appears. We will often write blocks with brackets and commas omitted so that, for example, abc means the triple $\{a, b, c\}$. An expression such as $ab\cdot\cdot$ indicates a block containing the points a and b with two unspecified points.

2 Initial arguments

We will investigate the structures of possible PBDs which achieve the value $g_1^{(4)}(18) = 31$, and in each case we will derive a contradiction. So, suppose that $g_1^{(4)}(18) = 31$. Then, in any corresponding PBD, if a point x appears in two or more pairs, by deleting x we obtain a PBD on 17 points with at most 29 blocks, contradicting the result that $g_1^{(4)}(17) \geq 30$. Hence we may assume that, for every point x in a PBD having 31 blocks, $r_2(x) = 0$ or 1.

Counting blocks and then pairs in such a PBD gives $g_2 + g_3 + g_4 = 31$ and $g_2 + 3g_3 + 6g_4 = 153$, and it follows that $5g_2 + 3g_3 = 33$. The possible solutions to this last equation give $(g_2, g_3, g_4) = (0, 11, 20), (3, 6, 22)$ or $(6, 1, 24)$. But then counting pairs containing each point x, we find that $r_2(x)+2r_3(x)+3r_4(x) = 17$, and a consequence of this is that for every point x, $r_2(x) + 2r_3(x) \equiv 2 \pmod 3$. Thus every point x appears in some small block and if $r_2(x) = 1$ then $r_3(x) \geq 2$. However, if $g_3 = 1$, then $r_3(x) \leq 1$ for every x and so we may exclude the solution $(g_2, g_3, g_4) = (6, 1, 24)$.

For the case $(g_2, g_3, g_4) = (3, 6, 22)$ there must exist points a, b, c, d, e, f which each appear in one pair and at least two triples. Altogether in six triples there are 18 occurrences of points, and a, b, c, d, e, f account for at least twelve of these. So at most six more points can occur in the small blocks, leaving at least a further six points which do not occur in any small block. But this contradicts the earlier requirement that every point must appear in some small block, so this case may also be excluded.

Now consider the remaining case $(g_2, g_3, g_4) = (0, 11, 20)$. For every x we have $r_2(x) = 0$ and so $2r_3(x) + 3r_4(x) = 17$. The solutions to this are $(r_3(x), r_4(x)) = (1, 5), (4, 3)$ or $(7, 1)$. Suppose that we have α points of type $(1, 5)$, β points of type $(4, 3)$ and γ points of type $(7, 1)$. Then counting points and then point occurrences in the eleven triples gives $\alpha+\beta+\gamma=18$ and $\alpha+4\beta+7\gamma=33$, and it follows that $\beta+2\gamma=5$. The possible solutions to this last equation give $(\alpha,\beta,\gamma)=(15,1,2),(14,3,1)$ or $(13,5,0)$. However, if $\gamma=2$ and we denote the corresponding points by a and b, it is clearly impossible to arrange for seven occurrences of both a and b in eleven triples without repeating the pair ab. So we may exclude the possibility that $(\alpha,\beta,\gamma)=(15,1,2)$.

Similarly, if $\gamma=1$, so that $\beta=3$, and we denote the corresponding four points by a,b,c and d, then it is easily seen that it is impossible to arrange seven occurrences of a together with four occurrences of each of b,c and d, in eleven triples with no repeated pair. So the case $(\alpha,\beta,\gamma)=(14,3,1)$ may also be excluded.

We are therefore left with the case $(\alpha,\beta,\gamma)=(13,5,0)$. Denote the five points having $(r_3(x),r_4(x))=(4,3)$ by a,b,c,d,e, and the remaining 13 points having $(r_3(x),r_4(x))=(1,5)$ by $0,1,\ldots,9,l,m,n$. Now consider the occurrences of a,b,c,d,e within the eleven triples of the PBD; suppose that there are λ triples formed entirely of these points, μ triples which each contain precisely two of these points, and ν triples which contain precisely one of these points. Counting occurrences of these points in the triples gives $3\lambda+2\mu+\nu=20$, counting triples of the PBD gives $\lambda+\mu+\nu\leq 11$, and counting pairs of these elements gives $3\lambda+\mu\leq 10$. From these it follows that $\lambda=0$ or 1. Furthermore, in the $\lambda=0$ case we must have either $(\mu,\nu)=(10,0)$ or $(9,2)$, and in the $\lambda=1$ case we must have $(\mu,\nu)=(7,3)$. Thus the triples of the PBD may be taken, without loss of generality, to have one of the three following forms.

Case A	Case B	Case C
$ab0$	$ab0$	abc
$ac1$	$ac1$	$ad0$
$ad2$	$ad2$	$ae1$
$ae3$	$ae3$	$bd2$
$bc4$	$bc4$	$be3$
$bd5$	$bd5$	$cd4$
$be6$	$be6$	$ce5$
$cd7$	$cd7$	$de6$
$ce8$	$ce8$	$a78$
$de9$	$d9l$	$b9l$
lmn	emn	cmn

Table 2: Triples for the three cases.

We examine each case in turn and prove that it is impossible. In all three cases we may assume that the design has no pairs, eleven triples and twenty quadruples; that for $\alpha\in\{a,b,c,d,e\}$ the point α appears in four triples and three quadruples, and that the remaining points each appear in one triple and five quadruples.

3 Case A

There are 15 quadruples of the form $\alpha x \cdot\cdot$ where $\alpha \in \{a, b, c, d, e\}$ and $x \in \{l, m, n\}$. The blank entries in these 15 quadruples are taken up by the points $0, 1, \ldots, 9$ and each of these points must appear here three times so that the pairs βx for $\beta \in \{0, 1, \ldots, 9\}$ and $x \in \{l, m, n\}$ may appear in the design. In the remaining five quadruples therefore, each of the points $0, 1, \ldots, 9$ must appear twice and so, for some permutation $0', 1', \ldots, 9'$ of these points, these remaining five quadruples are $\{0', 1', 2', 3'\}$, $\{0', 4', 5', 6'\}$, $\{1', 4', 7', 8'\}$, $\{2', 5', 7', 9'\}$ and $\{3', 6', 8', 9'\}$. The missing pairs of points from these latter five quadruples, which must be appear in the first 15 quadruples, are $0'7', 0'8', 0'9', 1'5', 1'6', 1'9', 2'4', 2'6', 2'8', 3'4', 3'5', 3'7', 4'9', 5'8', 6'7'$. Taking these as graph edges, they form a copy of the Petersen graph. It is well-known (see [1]) that this graph cannot be partitioned into three one-factors and so it is impossible to assign these 15 pairs into the three groups of five quadruples of the forms $\alpha l \cdot\cdot$, $\alpha m \cdot\cdot$ and $\alpha n \cdot\cdot$ for $\alpha \in \{a, b, c, d, e\}$. Thus case A cannot be completed.

4 Case B

Without loss of generality, we may assume that the quadruple containing the pair de is $de01$. We may then list the 20 quadruples as shown in Table 3(a). Referring to Table 3(a), the block $b1 \cdot\cdot$ contains at most two of $9, l, m, n$, and so the three blocks $1 \cdot\cdot\cdot$ must collectively contain at least two of $9, l, m, n$. Similarly, the block $c0 \cdot\cdot$ contains at most two of $9, l, m, n$ and the three blocks $0 \cdot\cdot\cdot$ must collectively contain at least two of $9, l, m, n$. However, the last six blocks listed in Table 3(a) collectively contain precisely one occurrence of each of $9, l, m, n$ and so the "at least" and "at most" in the previous two sentences must actually be "exactly". So, without loss of generality, we may assume that $b1 \cdot\cdot$ is $b19m$. Then two of the three blocks of the form $1 \cdot\cdot\cdot$ are $1l \cdot\cdot$ and $1n \cdot\cdot$, two of the three blocks of the form $0 \cdot\cdot\cdot$ are $09 \cdot\cdot$ and $0m \cdot\cdot$, and $c0 \cdot\cdot$ is $c0ln$. The remaining two blocks containing b must be $bl \cdot\cdot$ and $bn \cdot\cdot$, and the remaining two blocks containing c must be $c9 \cdot\cdot$ and $cm \cdot\cdot$. At this stage, the design covers all pairs from $\{9, l, m, n\}$ apart from $9n$ and lm; these must therefore appear in quadruples containing a as $a9nx$ and $almy$ where $x, y \in \{4, 5, 6, 7, 8\}$ and $x \neq y$. The quadruples at this stage are shown in Table 3(b).

quadruple	missing points
$de01$	–
$dm\cdot\cdot$	$3,4,6,8$
$dn\cdot\cdot$	
$e9\cdot\cdot$	$2,4,5,7$
$el\cdot\cdot$	
$a\cdot\cdot\cdot$	$4,5,6,7,8,9,l,m,n$
$a\cdot\cdot\cdot$	
$a\cdot\cdot\cdot$	
$b1\cdot\cdot$	$2,3,7,8,9,l,m,n$
$b\cdot\cdot\cdot$	
$b\cdot\cdot\cdot$	
$c0\cdot\cdot$	$2,3,5,6,9,l,m,n$
$c\cdot\cdot\cdot$	
$c\cdot\cdot\cdot$	
$0\cdot\cdot\cdot$	$2,2,3,3,4,4,5,5,6,6,7,7,8,8,9,l,m,n$
$0\cdot\cdot\cdot$	
$0\cdot\cdot\cdot$	
$1\cdot\cdot\cdot$	
$1\cdot\cdot\cdot$	
$1\cdot\cdot\cdot$	

3(a)

quadruple	missing points
$de01$	–
$dm\cdot\cdot$	$3,4,6,8$
$dn\cdot\cdot$	
$e9\cdot\cdot$	$2,4,5,7$
$el\cdot\cdot$	
$a9nx$	$4,5,6,7,8$ (including x,y)
$almy$	
$a\cdot\cdot\cdot$	
$b19m$	$2,3,7,8$
$bl\cdot\cdot$	
$bn\cdot\cdot$	
$c0ln$	$2,3,5,6$
$c9\cdot\cdot$	
$cm\cdot\cdot$	
$09\cdot\cdot$	$2,2,3,3,4,4,5,5,6,6,7,7,8,8$
$0m\cdot\cdot$	
$0\cdot\cdot\cdot$	
$1l\cdot\cdot$	
$1n\cdot\cdot$	
$1\cdot\cdot\cdot$	

3(b)

Table 3: Quadruples for case B.

Consider the point x. If $x = 6$ then it must appear in the block $dm6\cdot$, but then it cannot appear in any block with c, and so $x \neq 6$. Similarly, if $x = 7$ then it must appear in the block $el7\cdot$, but then it cannot appear in any block with b, and so $x \neq 7$. If $x = 4$ then it must appear in blocks $dm4\cdot$ and $el4\cdot$; if $x = 5$ then it must appear in blocks $el5\cdot$ and $cm5\cdot$; if $x = 8$ then it must appear in blocks $dm8\cdot$ and $bl8\cdot$. So, whichever point x actually is, the pairs xl and xm appear in blocks containing b, c, d or e. Now consider the location of the pair $0x$. This pair cannot occur in $de01, c0ln, 09\cdot\cdot$ or $0m\cdot\cdot$, and so it must appear in a block $0x\cdot\cdot$ where the two blank entries lie in $\{2,3,4,5,6,7,8\}\setminus\{x\}$. Similarly, the pair $1x$ cannot occur in $de01, b19m, 1l\cdot\cdot$ or $1n\cdot\cdot$, and so it must appear in a block $1x\cdot\cdot$ where the two blank entries lie in $\{2,3,4,5,6,7,8\}\setminus\{x\}$.

If we now consider the point y, applying arguments similar to those used in the previous paragraph, we deduce that the blocks $0x\cdot\cdot$ and $1x\cdot\cdot$ obtained there also contain y. Thus the pair xy is repeated; a contradiction. It follows that case B cannot be completed.

5 Case C

Put $X = \{0, 1, 2, 3, 4, 5, 6\}$ and $Y = \{7, 8, 9, l, m, n\}$. The 20 quadruples may be classified as types A, B, C, D, E and N as shown in Table 4. None

	quadruple	missing points
A	$a\cdots$, $a\cdots$, $a\cdots$	$2, 3, 4, 5, 6,$ $9, l, m, n$
B	$b\cdots$, $b\cdots$, $b\cdots$	$0, 1, 4, 5, 6,$ $7, 8, m, n$
C	$c\cdots$, $c\cdots$, $c\cdots$	$0, 1, 2, 3, 6,$ $7, 8, 9, l$
D	$d\cdots$, $d\cdots$, $d\cdots$	$1, 3, 5,$ $7, 8, 9, l, m, n$
E	$e\cdots$, $e\cdots$, $e\cdots$	$0, 2, 4,$ $7, 8, 9, l, m, n$
N	$\cdots\cdot$, $\cdots\cdot$, $\cdots\cdot$, $\cdots\cdot$, $\cdots\cdot$	2 copies of X, 1 copy of Y

Table 4: Quadruples for case C.

of the A, B, C, D or E-blocks can contain a triple of points from X because this would imply that the two occurrences of these three points in the N-blocks would require six distinct blocks in order to avoid a repeated pair. Hence two of the A-blocks must each contain a pair from X and a point from Y, and the third A-block must contain a point from X and a pair from Y. The same holds for the B and C-blocks. The D-blocks either contain three pairs from Y (one pair per block) or four pairs from Y with one block having a triple from Y, another having a pair from Y, and the third having a single point from Y. The same holds for the E-blocks. Noting that the eleven triples cover three pairs from Y, it can be seen that of the remaining twelve pairs from Y, the A, B, C, D and E-blocks contain nine, ten, or eleven, and the remaining pairs occur in the N-blocks. Without loss of generality, we therefore have the following three possibilities.

1. The N-blocks contain three pairs from Y, while the D-blocks contain three pairs from Y, and the E-blocks contain three pairs from Y.

2. The N-blocks contain two pairs from Y, while the D-blocks contain four pairs from Y, and the E-blocks contain three pairs from Y.

3. The N-blocks contain one pair from Y, while the D-blocks contain four pairs from Y, and the E-blocks contain four pairs from Y.

We now examine each of these possibilities in turn.

In case 1, either

(a) one N-block contains a triple from Y, another three N-blocks each contain a single point from Y, and the remaining N-block contains no points from Y, or

(b) three N-blocks each contain a pair from Y, and the remaining two N-blocks contain no points of Y.

If 1(a) holds then the three pairs from Y in the triples, the three pairs from Y in the D-blocks, and the three pairs from Y in the E-blocks all form 1-factors of K_6 on the vertex set Y, and the triple from Y which lies in an N-block forms a triangle in this graph. It is easy to see that, up to isomorphism, this partial decomposition of K_6 is unique, and that the remaining three edges of the graph form a second disjoint triangle. Thus, without loss of generality, we may assume that the quadruples are as follows.

	D	E	N	A	B	C
Quadruples	$d89\cdot$ $dlm\cdot$ $dn7\cdot$	$e7l\cdot$ $e8m\cdot$ $e9n\cdot$	$79m\cdot$ $8\cdot\cdot\cdot$ $l\cdot\cdot\cdot$ $n\cdot\cdot\cdot$ $\cdot\cdot\cdot\cdot$	$aln\cdot$ $a9\cdot\cdot$ $am\cdot\cdot$	$b8n\cdot$ $b7\cdot\cdot$ $bm\cdot\cdot$	$c8l\cdot$ $c7\cdot\cdot$ $c9\cdot\cdot$
Missing points	$1,3,5$	$0,2,4$	2 copies of X	$2,3,4,5,6$	$0,1,4,5,6$	$0,1,2,3,6$

We cannot have $79mx$ for $x \in \{0,1,2,3,4,5\}$ because x could not then occur in a D or E-block. Thus the block $79m\cdot$ must be $79m6$. But the point 6 appears in the A, B and C-blocks. We cannot have $a96\cdot$ or $am6\cdot$, so we must have $aln6$. Similarly we must have $b8n6$. But then the pair $n6$ is repeated, so 1(a) is impossible.

If 1(b) holds then the three pairs from Y in the triples, the three pairs from Y in the D-blocks, the three pairs from Y in the E-blocks, and the three pairs from Y in the N-blocks all form 1-factors of K_6 on the vertex set Y. The remaining three pairs form a further 1-factor, and the resulting

1-factorization of K_6 is unique up to isomorphism (see [1]). So, without loss of generality, the quadruples can be taken as follows.

	D	E	N	A	B	C
Quadruples	$d89\cdot$ $dlm\cdot$ $dn7\cdot$	$e7l\cdot$ $e8n\cdot$ $e9m\cdot$	$79\cdot\cdot$ $8m\cdot\cdot$ $ln\cdot\cdot$ $\cdot\cdot\cdot\cdot$ $\cdot\cdot\cdot\cdot$	$a9n\cdot$ $al\cdot\cdot$ $am\cdot\cdot$	$b7m\cdot$ $b8\cdot\cdot$ $bn\cdot\cdot$	$c8l\cdot$ $c7\cdot\cdot$ $c9\cdot\cdot$
Missing points	$1,3,5$	$0,2,4$	2 copies of X	$2,3,4,5,6$	$0,1,4,5,6$	$0,1,2,3,6$

The two N-blocks containing only points of X cannot contain a repeated point $x \in \{0,1,2,3,4,5\}$ because x would then appear with two points from Y amongst the D and E-blocks, and in two of the three blocks $a9n\cdot, b7m\cdot$ and $c8l\cdot$. But it is easy to see that this cannot be achieved without a repeated pair. Consequently, both the N-blocks containing only points of X must contain the point 6. Now consider the block $79\alpha\beta$ where $\alpha, \beta \in \{0,1,2,3,4,5\}$. Without loss of generality, we must have the blocks $dlm\alpha$ and $e8n\beta$. But then α can only appear amongst the A, B and C-blocks in $b8\cdot\cdot$ and $bn\cdot\cdot$. Since α must appear twice amongst these blocks, this cannot be achieved without repeating the pair $b\alpha$. Thus 1(b) is also impossible.

In case 2 we may, without loss of generality, take the E-blocks as $\{e89\cdot, elm\cdot, e7n\cdot\}$. One of the possibilities for the D-blocks is then $\{d79m, d8l\cdot, dn\cdot\cdot\}$. The other possibilities for the D-blocks are equivalent to this under combinations of the mappings $\pi_1 = (0\ 2\ 4)(1\ 3\ 5)(7\ 9\ m)(8\ l\ n)(a\ b\ c)$ and $\pi_2 = (2\ 4)(3\ 5)(7\ 8)\ (9\ n)(l\ m)(b\ c)$. So, taking the D and E-blocks as shown, along with the triples from Table 2, the missing pairs from Y are $7l, 8m, 8n, 9n$ and ln. In view of the triple cmn, the last four of these pairs cannot appear in a C-block, and since one pair must appear in a C-block, we have a quadruple $c7l\cdot$. If the pair $8n$ were to appear in an N-block then none of the three remaining pairs could also appear in an N-block. So $8n$ must appear in an A or B-block, and in view of the triple $a78$, we must have $b8n\cdot$. This leaves the pairs $8m, 9n$ and ln unlocated. So we either have

(a) a quadruple $aln\cdot$, with the pairs $8m$ and $9n$ in different N-blocks, or

(b) a quadruple $a9n\cdot$, with the pairs $8m$ and ln in different N-blocks.

In case 2(a) we have the quadruples as follows.

	D	E	N	A	B	C
Quadruples	$d79m$ $d8l\cdot$ $dn\cdot\cdot$	$e89\cdot$ $elm\cdot$ $e7n\cdot$	$8m\cdot\cdot$ $9n\cdot\cdot$ $7\cdot\cdot\cdot$ $l\cdot\cdot\cdot$ $\cdot\cdot\cdot\cdot$	$aln\cdot$ $a9\cdot\cdot$ $am\cdot\cdot$	$b8n\cdot$ $b7\cdot\cdot$ $bm\cdot\cdot$	$c7l\cdot$ $c8\cdot\cdot$ $c9\cdot\cdot$
Missing points	$1,3,5$	$0,2,4$	2 copies of X	$2,3,4,$ $5,6$	$0,1,4,$ $5,6$	$0,1,2,$ $3,6$

Now consider the block $e7nx$ where $x \in \{0,2,4\}$. The point x appears with precisely two of the points $8,9$ and m in the A, B and C-blocks, and so can only appear in the N-blocks $l\cdots$ and $\cdots\cdot$. But then it can only appear with five of the six points of Y. Thus case 2(a) is impossible.

In case 2(b) we have the quadruples as follows.

	D	E	N	A	B	C
Quadruples	$d79m$ $d8l\cdot$ $dn\cdot\cdot$	$e89\cdot$ $elm\cdot$ $e7n\cdot$	$8m\cdot\cdot$ $ln\cdot\cdot$ $7\cdot\cdot\cdot$ $9\cdot\cdot\cdot$ $\cdot\cdot\cdot\cdot$	$a9n\cdot$ $al\cdot\cdot$ $am\cdot\cdot$	$b8n\cdot$ $b7\cdot\cdot$ $bm\cdot\cdot$	$c7l\cdot$ $c8\cdot\cdot$ $c9\cdot\cdot$
Missing points	$1,3,5$	$0,2,4$	2 copies of X	$2,3,4,$ $5,6$	$0,1,4,$ $5,6$	$0,1,2,$ $3,6$

Consider the block $elmx$ where $x \in \{0,2,4\}$ and list the ways in which pairs αx for $\alpha \in Y$ can occur amongst the A, B, C and N-blocks, noting that the pair nx must appear. We find that there are just two possibilities, namely $a9nx$ with $c8x\cdot$, or $b8nx$ with $c9x\cdot$, and in each case the N-blocks necessarily contain the quadruples $7x\cdot\cdot$ and $x\cdots$, where the unspecified points are from X. Similarly, if we consider the block $d8lw$ where $w \in \{1,3,5\}$, we find that there must be blocks $a9nw$ and $bmw\cdot$, and that the N-blocks must contain the quadruples $7w\cdot\cdot$ and $w\cdots$ with the unspecified points from X. Putting together these two deductions, we have the two distinct blocks $7xw\cdot$ and $xw\cdot\cdot$, a contradiction. Thus case 2(b) is impossible.

In case 3, the D and E-blocks each containing three points from Y are either disjoint, or they have one point from Y in common. In the former case it is impossible to have a further D-block (or E-block) containing a pair from Y. So we may assume that these blocks have a point in common and we may take them to be $d79m$ and $e7ln$. The D-block which contains a pair from Y is then either $d8l\cdot$ or $d8n\cdot$, and the E-block which contains a pair from Y is either $e89\cdot$ or $e8m\cdot$. The resulting four possibilities for the D and E-blocks reduce to two under the permutation $\pi = (2\;4)(3\;5)(9\;m)(l\;n)(b\;c)$, and we may take these two cases to be

	$d79m$	$e7ln$			$d79m$	$e7ln$
(a)	$d8l\cdot$	$e89\cdot$	and	(b)	$d8l\cdot$	$e8m\cdot$
	$dn\cdot\cdot$	$em\cdot\cdot$			$dn\cdot\cdot$	$e9\cdot\cdot$

In case (a) the pairs from Y which have to appear in the A, B, C and N-blocks are $8m, 8n, 9n, lm$, and one of these must appear in each of the A, B, C and N collections. However, none can appear in C since the C-blocks only contain the points 7, 8, 9 and l from Y. So case (a) is impossible.

In case (b) the pairs from Y which have to appear in the A, B, C and N-blocks are $89, 8n, 9n, lm$, and one of these must appear in each of the A, B, C and N collections. But only 89 can appear with c and only $8n$ can appear with b. So either $9n$ or lm, but not both, appears with a and the remaining pair appears in the N-blocks. So there are two possibilities for the A, B, C and N-blocks, namely:

(i)

A	B	C	N
			$lm\cdot\cdot$
$a9n\cdot$	$b8n\cdot$	$c89\cdot$	$7\cdot\cdot\cdot$
$al\cdot\cdot$	$b7\cdot\cdot$	$c7\cdot\cdot$	$8\cdot\cdot\cdot$
$am\cdot\cdot$	$bm\cdot\cdot$	$cl\cdot\cdot$	$9\cdot\cdot\cdot$
			$n\cdot\cdot\cdot$

or

(ii)

A	B	C	N
			$9n\cdot\cdot$
$alm\cdot$	$b8n\cdot$	$c89\cdot$	$7\cdot\cdot\cdot$
$a9\cdot\cdot$	$b7\cdot\cdot$	$c7\cdot\cdot$	$8\cdot\cdot\cdot$
$an\cdot\cdot$	$bm\cdot\cdot$	$cl\cdot\cdot$	$l\cdot\cdot\cdot$
			$m\cdot\cdot\cdot$

In case (i) the block $lm\cdot\cdot$ cannot contain the point 6 because if it did then we must have $a9n6$ and this would render it impossible for the point 6 to occur in both the B and C-blocks. Similarly in case (ii) the block $9n\cdot\cdot$ cannot contain the point 6.

So suppose we have case (i) and consider the block $lmwx$ where $w, x \in \{0, 1, 2, 3, 4, 5\}$. Without loss of generality we must have the block $dnx\cdot$, and then x appears with four points from Y including l, m and n amongst the D and N-blocks, and so must appear with a further two points from Y in two occurrences in the A, B and C-blocks. But this is impossible, so case (i) cannot apply.

Finally, suppose we have case (ii) and consider the block $9nwx$ where $w, x \in \{0, 1, 2, 3, 4, 5\}$. Without loss of generality we must have the block

$d8lx$, and then x appears with five points from Y amongst the D and N-blocks, and so cannot appear twice in the A, B and C-blocks without a repeated pair. Thus case (ii) cannot apply.

This completes the elimination of case C.

6 Conclusion

The results of sections 2 to 5 establish that $g_1^{(4)}(18) \geq 32$. We remark that we have also run exhaustive computer searches using a long-established computer program on each of cases A, B and C to provide independent verification that no solution exists. It follows that $g_1^{(4)}(18) = 32$ or 33.

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