A constraint on the biembedding of Latin squares

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Abstract

We give a necessary condition for the biembedding of two Latin squares in an orientable surface. As a consequence, it is shown that for $n \geq 2$, there is no biembedding of two Latin squares both lying in the same main class as the square obtained from the Cayley table of the Abelian 2-group C_2^n.

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1 Introduction

Many recent papers have dealt with biembeddings of combinatorial designs in orientable and nonorientable surfaces, that is to say closed connected 2-manifolds. For a recent survey and comprehensive list of references see [2]. The principal designs studied have been Steiner triple systems and Latin squares. A fundamental question underlying all these results has been that of existence of biembeddings for specified designs. In the current paper a necessary condition is obtained relating to biembeddings of Latin squares. This constraint is the first generally applicable result of its type. It enables us to prove that for infinitely many orders n there are pairs of Latin squares of order n which cannot be biembedded, even allowing any relabelling of the squares.

To see the connection between topological embeddings and combinatorial designs, consider a face 2-colourable triangular embedding of a complete graph K_n in a surface. Such an embedding gives rise to two Steiner triple systems. The vertices of the graph form the points of the systems and the triangular faces in each of the two colour classes respectively form the triples of each system. We here recall that an STS(n) may be formally defined as an ordered pair $(V, \mathcal{B})$, where V is an n-element set (the *points*) and $\mathcal{B}$ is a set of 3-element subsets of V (the *triples*), such that every 2-element subset of V appears in precisely one triple. Such systems are known to exist if and only if $n \equiv 1$ or 3 (mod 6) [7]. We say that two STS(n)s are *biembeddable* in a surface if there is a face 2-colourable triangular embedding of K_n in which the face sets forming the two colour classes give copies of the two systems.

Embeddings of complete regular tripartite graphs are discussed in [5, 10] and form a useful tool in recursive constructions for biembeddings of Steiner triple systems, as well as being of considerable interest in their own right. A face 2-colourable triangular embedding of the complete tripartite graph $K_{n,n,n}$ may be considered as a biembedding of a pair of transversal designs $\mathrm{TD}(3,n)$; such a design comprises an ordered triple $(V, \mathcal{G}, \mathcal{B})$, where V is a $3n$-element set (the *points*), $\mathcal{G}$ is a partition of V into three disjoint sets (the *groups*) each of cardinality n, and $\mathcal{B}$ is a set of 3-element subsets of V (the *triples*), such that every unordered pair of elements from V is either contained in precisely one triple or one group, but not both. As with Steiner triple systems, the vertices of the embedded graph $K_{n,n,n}$ form the points of the designs, the tripartition determines the groups, and the faces in each colour class form the triples of each design. The connection with Latin squares is that a $\mathrm{TD}(3,n)$ determines a Latin square by assigning the three groups of the design to label the rows, columns and entries (in any one of the six possible orders) of the Latin square. Thus a face 2-colourable triangulation of $K_{n,n,n}$ may be considered as a biembedding of two Latin squares and we say that two Latin squares of order n are *biembeddable* in a surface if there is a face 2-colourable triangular embedding of $K_{n,n,n}$ in which the face sets forming the two colour classes give copies of the two squares.

In this paper, a Latin square of order n, $\mathrm{LS}(n)$, will be taken as an $n \times n$ array with entries from $E = \{e_1, e_2, \ldots, e_n\}$ and rows and columns indexed by $R = \{r_1, r_2, \ldots, r_n\}$ and $C = \{c_1, c_2, \ldots, c_n\}$ respectively. The defining property is that each $e_i \in E$ appears precisely once in each row and once in each column. We will identify the Latin square L with its set of n^2 {row, column, entry} triples so that, for example writing $\{r_i, c_j, e_k\} \in L$ means that L has entry e_k in row r_i, column c_j. We refer to the three sets R, C, E as the row, column and entry identifiers of the square and a transversal design may be formed from L by assigning R, C, E as the three groups of the design.

Two Latin squares of the same order n are said to belong to the same *main class* if the two corresponding transversal designs are isomorphic. In other words, there exist three bijections mapping the row, column and entry identifiers of the first square to those of the second (not necessarily in the same order) which also map the first square to the second. If there are such bijections that map rows to rows, columns to columns and entries to entries, then the squares are said to be *isotopic*.

We assume that the reader is familiar with the description of topological embeddings by means of rotation schemes as described in [6, 9]. Our embeddings will always be in surfaces rather than pseudosurfaces (the latter result from surfaces by repeating a finite number of times the operation of identifying a finite number of points on a surface). Equivalently, in the description of an embedding by means of a rotation scheme, the rotation about each vertex comprises a single cycle. In fact our surfaces will always be orientable because we shall only be considering biembeddings of Latin squares and it was shown in [3] that such biembeddings are necessarily orientable, with the triangles of one Latin square being oriented (row, column, entry) and those of the other being oriented (column, row, entry).

The paper [3] contains a number of computational results of which perhaps the most striking is that given in Table 5 of that paper. It is shown that the 147 main classes of Latin squares of order 7 partition into 16 subsets containing 1, 1, 1, 2, 3, 3, 3, 6, 6, 8, 8, 9, 18, 19, 26 and 33 classes such that the biembeddings of Latin squares exist only when both squares belong to the same subset of the partition. Moreover, within each of these 16 subsets, most pairs of classes admit a biembedding.

In a more recent paper [4], attention was turned to Latin squares of order 8, where there are 283 657 main classes [8]. It was computationally infeasible to determine all possible biembeddings of these squares, so attention was restricted to seeking biembeddings that contain at least one square that arises from the Cayley table of a group of order 8. Another reason for considering these particular squares is as follows. Latin squares which arise from the Cayley tables of cyclic groups can always be biembedded. However, those from the groups $C_2 \times C_2$ and D_3, the only non-cyclic groups of orders less than 8, can not [3]. There are five groups of order 8, usually denoted by $C_2^3 = C_2 \times C_2 \times C_2$, $C_4 \times C_2$, C_8, D_4 and Q. Here C_n denotes the cyclic group of order n, D_n is the dihedral group of order $2n$, and Q is the quaternion group. Table 1 summarizes all the resulting biembeddings where <u>both</u> squares are group-based. The entries give the numbers of nonisomorphic biembeddings of squares from the corresponding main classes.

	C_2^3	$C_4 \times C_2$	C_8	D_4	Q
C_2^3	–	–	–	1	–
$C_4 \times C_2$	–	–	–	4	5
C_8	–	–	13	–	–
D_4	1	4	–	–	–
Q	–	5	–	–	–

Table 1. Numbers of mutual biembeddings of group-based squares.

It can be seen that there are, for example, no biembeddings of two squares both derived from C_2^3.

In the subsequent section of the current paper we obtain an explanation for these results by establishing a necessary condition for the biembedding of two Latin squares. This condition also enables us to show that for $n \geq 2$ there is no biembedding of two Latin squares both derived from the Abelian 2-group C_2^n. This gives the first known infinite class of pairs of Latin squares that admit no biembeddings.

2 Biembeddings of Latin squares

Suppose L is an arbitrary Latin square of order n. For any fixed $i \in N = \{1, 2, \ldots, n\}$, consider the subset of L consisting of the n triples that contain r_i. By the definition of a Latin square each column and each entry occurs in exactly one of these triples; thus we can regard row i of L as defining a bijection $\beta^L_{r,i} : C \to E$, with $\beta^L_{r,i}(c_j) = e_k$ if and only if $\{r_i, c_j, e_k\} \in L$.

Next let A and B be a pair of Latin squares of order n, with common row, column and entry identifiers. Then the composite mapping $\beta^A_{r,i}(\beta^B_{r,i})^{-1}$ is a permutation on the entry identifier set E. Now suppose that there exists a biembedding of A and B, with their current labelling, in some orientable surface. In other words, if the squares are represented by means of triples, and the triples are taken as triangular faces and then sewn together along common edges, a surface (rather than a pseudosurface) is obtained.

Consider the rotation at r_i as shown in Figure 1. This will be a cycle of length $2n$ of the form

$$(c_{\rho(1)}, e_{\tau(1)}, c_{\rho(2)}, e_{\tau(2)}, \dots, c_{\rho(n)}, e_{\tau(n)})$$

for some permutations ρ and τ of N, where $\{r_i, c_{\rho(j)}, e_{\tau(j)}\} \in A$ and $\{r_i, c_{\rho(j+1)}, e_{\tau(j)}\} \in B$ for each $j \in N$. Here and elsewhere, for $j = n$ we take $j + 1$ to be 1. This implies that the permutation $\beta^A_{r,i}(\beta^B_{r,i})^{-1}$ maps $e_{\tau(j)}$ to $e_{\tau(j+1)}$ for $j = 1, 2, \dots, n$, and hence consists of a single permutation cycle of length n.

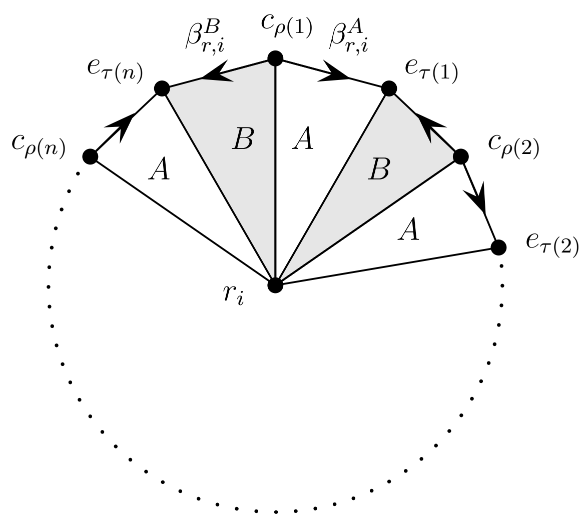

Figure 1. The rotation at r_i.

In a similar way, for any Latin square L of order n and for each $i \in N$ we can define bijections $\beta^L_{c,i} : E \to R$ and $\beta^L_{e,i} : R \to C$ by $\beta^L_{c,i}(e_j) = r_k$ if and only if $\{r_k, c_i, e_j\} \in L$ and $\beta^L_{e,i}(r_j) = c_k$ if and only if $\{r_j, c_k, e_i\} \in L$. Reasoning as above, we see that if Latin squares A and B can be biembedded with their current labelling, then the permutation $\beta^A_{\alpha,i}(\beta^B_{\alpha,i})^{-1}$ must consist of a single n-cycle for each $\alpha \in \{r, c, e\}$ and $i \in N$. In fact (although it is not necessary for our main result), the converse also holds; if each of these permutations consists of a single cycle then A and B with their current labelling can be biembedded in some orientable surface. Thus we have the following result.

Lemma 2.1 *Let A and B be Latin squares of order n. Then A and B can be biembedded with their current labelling in an orientable surface if and only if the permutation $\beta^A_{\alpha,i}(\beta^B_{\alpha,i})^{-1}$ consists of a single n-cycle for each $\alpha \in \{r, c, e\}$ and $i \in N$.*

This condition is easily checked for a given pair of Latin squares. However, the fundamental question is whether there is any relabelling of either Latin square which allows them to be biembedded. We now address this question.

Plainly, if n is odd (respectively, even) then we require $\beta^A_{\alpha,i}(\beta^B_{\alpha,i})^{-1}$ to be an even (respectively, odd) permutation. Since $\beta^A_{\alpha,i}$ and $\beta^B_{\alpha,i}$ are not themselves permutations, there is a technical difficulty about assigning them individual parities. However, they can be replaced by equivalent permutations in the following manner.

Take arbitrary but fixed orderings of R, C and E; we will use $(r_1\ r_2\ \cdots\ r_n)$, $(c_1\ c_2\ \cdots\ c_n)$ and $(e_1\ e_2\ \cdots\ e_n)$. For $L = A$ or B, if $\beta^L_{r,i}(c_k) = e_{j_k}$ for $k \in N$, then writing this in 'two-line' form we have

$$\beta^L_{r,i} = \begin{pmatrix} c_1 & c_2 & \cdots & c_n \\ e_{j_1} & e_{j_2} & \cdots & e_{j_n} \end{pmatrix}.$$

Replace the first line by the chosen fixed ordering of E and so obtain a permutation $\gamma^L_{r,i}$ of E:

$$\gamma^L_{r,i} = \begin{pmatrix} e_1 & e_2 & \cdots & e_n \\ e_{j_1} & e_{j_2} & \cdots & e_{j_n} \end{pmatrix}.$$

In other words, $\gamma^L_{r,i}(e_k) = \beta^L_{r,i}(c_k)$.

In the same way define the permutations $\gamma^L_{c,i}$ of R and $\gamma^L_{e,i}$ of C for $L = A$ and $L = B$. Then note that for $\alpha \in \{r, c, e\}$ we have $\beta^A_{\alpha,i}(\beta^B_{\alpha,i})^{-1} = \gamma^A_{\alpha,i}(\gamma^B_{\alpha,i})^{-1}$. Thus, if this product is an even permutation then the permutations $\gamma^A_{\alpha,i}$ and $\gamma^B_{\alpha,i}$ must have the same parities, and if the product is an odd permutation then $\gamma^A_{\alpha,i}$ and $\gamma^B_{\alpha,i}$ must have opposite parities.

We can now state the following key result.

Theorem 2.1 *Suppose that A and B are Latin squares of order n which can be biembedded with their current labelling in an orientable surface. Then for each $\alpha \in \{r, c, e\}$ and $i \in N$, $\gamma^A_{\alpha,i}$ and $\gamma^B_{\alpha,i}$ have:*

- *the same parity if n is odd;*
- *different parity if n is even.*

Prior to discussing the issue of relabelling the rows, columns and entries of Latin squares, we make the following definition.

Definition 2.1 *For a Latin square L of order n, let*

$$\begin{aligned} x_L &= |\{i \in N \mid \gamma^L_{r,i} \text{ has odd parity}\}|, \\ y_L &= |\{j \in N \mid \gamma^L_{c,j} \text{ has odd parity}\}|, \\ z_L &= |\{k \in N \mid \gamma^L_{e,k} \text{ has odd parity}\}|. \end{aligned}$$

Then the vector (x_L, y_L, z_L) will be called the parity vector *of L.*

The following is an immediate consequence of Theorem 2.1.

Corollary 2.1.1 *Suppose that A and B are Latin squares of order n with parity vectors (x_A, y_A, z_A) and (x_B, y_B, z_B) respectively. If A and B can be biembedded with their current labelling in an orientable surface then*

- *$(x_A, y_A, z_A) = (x_B, y_B, z_B)$ if n is odd;*
- *$(x_A, y_A, z_A) = (n - x_B, n - y_B, n - z_B)$ if n is even.*

We now consider the effect on the parity vector of relabelling the rows, columns and entries of a Latin square or, equivalently, the effect of applying permutations to R, C and E.

Suppose that σ is a permutation of R, which we apply to the rows of a Latin square L to obtain L'. Let the parity vectors of L and L' be (x, y, z) and (x', y', z') respectively. For each i, the bijection $\beta^L_{r,i}$ maps C to E and is thus unaffected except for a change in the row identifier. That is, $\beta^L_{r,i} = \beta^{L'}_{r,\sigma(i)}$ and so $\gamma^L_{r,i} = \gamma^{L'}_{r,\sigma(i)}$. Hence $x' = x$.

However the bijections $\beta^L_{c,i}$ and $\beta^L_{e,i}$ are effectively composed with σ; $\beta^{L'}_{c,i} = \sigma\beta^L_{c,i}$ and $\beta^{L'}_{e,i} = \beta^L_{e,i}\sigma^{-1}$. This will not change the parity of $\gamma^L_{c,i}$ and $\gamma^L_{e,i}$ if σ is even, but if σ is odd then their parities will be reversed. Hence in the former case $y' = y$ and $z' = z$, and in the latter case $y' = n - y$ and $z' = n - z$.

Similar results apply to permutations of C and R. It follows that every Latin square which is isotopic to L has parity vector (x, y, z), $(x, n-y, n-z)$, $(n-x, y, n-z)$, or $(n-x, n-y, z)$.

This result is readily extended to main classes. In addition to isotopisms we allow the sets R, C and E to be swapped. This corresponds to a reordering of the parity vectors. Of the 24 parity vectors formed from (x, y, z), there are at most four distinct ones that have a common minimal first entry. Of these, there is precisely one distinct vector that has a minimal second entry. We call this vector the *main class parity vector* and denote it by $[p, q, r]$, using square brackets to distinguish it from the original parity vectors. If $[p, q, r]$ is a main class parity vector, then $p \le q \le \min\{r, n-r\}$. The following theorem is now an immediate consequence of Corollary 2.1.1.

Theorem 2.2 *Let A and B be two Latin squares of order n, with main class partity vectors $[x_A, y_A, z_A]$ and $[x_B, y_B, z_B]$ respectively. If there exist Latin squares A' and B' which are in the same main class as A and B respectively and which can be biembedded, then*

- $[x_A, y_A, z_A] = [x_B, y_B, z_B]$ *if n is odd;*
- $[x_A, y_A, z_A] = [x_B, y_B, n - z_B]$ *if n is even.*

3 Consequences of Theorem 2.2

Throughout this section, the main classes of Latin squares of order n for $n = 4, 5, 6$ and 7 are numbered as in [1].

We start by giving an example of two main classes of Latin squares of order 6 that admit no biembedding of two squares, one from each class. The classes involved are those numbered 1 and 3 and the squares, represented on $E = \{1, 2, 3, 4, 5, 6\}$ (so that $e_i = i$), are as shown in Table 2.

1	2	3	4	5	6		1	2	3	4	5	6
2	1	4	3	6	5		2	1	4	5	6	3
3	4	5	6	1	2		3	4	1	6	2	5
4	3	6	5	2	1		4	5	6	1	3	2
5	6	1	2	3	4		5	6	2	3	1	4
6	5	2	1	4	3		6	3	5	2	4	1
		A							B			

Table 2. Squares of order 6 from main classes 1 and 3.

As examples of the row, column and entry permutations we give

$$\gamma^A_{r,2} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \end{pmatrix} : \text{odd}$$

$$\gamma^A_{c,3} = \begin{pmatrix} r_1 & r_2 & r_3 & r_4 & r_5 & r_6 \\ r_5 & r_6 & r_1 & r_2 & r_3 & r_4 \end{pmatrix} : \text{even}$$

$$\gamma^A_{e,4} = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ c_4 & c_3 & c_2 & c_1 & c_6 & c_5 \end{pmatrix} : \text{odd}$$

We find that A has main class parity vector $[3, 3, 3]$ and B has [0,0,5]. So, by Theorem 2.2 there is no biembedding of any representative from main class 1 with any from main class 3. This is in agreement with the computational results of [3].

We list in Table 3 the main class parity vectors for squares of orders $n = 4, 5, 6$ and 7, all obtained by computer.

For $n = 4$, Theorem 2.2 establishes that there is no biembedding involving the first main class which corresponds to the Cayley table of C_2^2. For $n = 5$ it establishes that there is no biembedding of a square from class 1 with a square from class 2. For $n = 6$ we see that there can be no biembedding involving the main classes numbered 3, 4 or 10. All of these results are consistent with the computational results of [3]. For $n = 7$ the partitioning of the main classes by their parity vectors into the 16 subsets shown in Table 3 is identical with that of [3] to which we referred in the Introduction.

Order	Vector	Main Class numbers
4	[0, 0, 0]	1
	[2, 2, 2]	2
5	[0, 0, 0]	1
	[1, 1, 4]	2
6	[0, 0, 3]	6
	[0, 0, 5]	3
	[1, 1, 1]	10
	[1, 2, 4]	4
	[1, 3, 3]	9
	[2, 2, 3]	5
	[3, 3, 3]	1, 2, 7, 8, 11, 12
7	[0, 0, 7]	1, 3, 7
	[0, 1, 4]	87
	[0, 1, 6]	6
	[0, 2, 3]	90, 124, 125
	[0, 2, 5]	105, 136
	[0, 3, 4]	2, 4, 5
	[1, 1, 1]	52, 76, 112, 141, 143, 147
	[1, 1, 3]	71, 81, 108, 109, 121, 140

Order	Vector	Main Class numbers
7	[1, 1, 5]	78
	[1, 2, 2]	12, 15, 51, 65, 68, 79, 97, 130
	[1, 2, 4]	57, 63, 66, 82, 86, 92, 119, 120, 122
	[1, 3, 3]	29, 36, 38, 43, 45, 50, 55, 60, 91, 93, 103, 104, 107, 113, 116, 123, 126, 142
	[2, 2, 3]	16, 17, 32, 41, 42, 48, 49, 56, 83, 85, 89, 94, 101, 106, 117, 118, 127, 131, 133
	[2, 2, 5]	8, 10, 46, 77, 84, 129, 135, 146
	[2, 3, 4]	9, 14, 19, 20, 26, 27, 31, 34, 35, 37, 40, 47, 54, 61, 62, 67, 70, 72, 80, 88, 95, 99, 100, 128, 132, 134
	[3, 3, 3]	11, 13, 18, 21, 22, 23, 24, 25, 28, 30, 33, 39, 44, 53, 58, 59, 64, 69, 73, 74, 75, 96, 98, 102, 110, 111, 114, 115, 137, 138, 139, 144, 145

Table 3. Main classes and their parity vectors.

For Latin squares of order 8 obtained from the group tables, the main class parity vectors are shown in Table 4.

Vector	Main Classes
[0, 0, 0]	$C_2^3, C_4 \times C_2$
[0, 0, 8]	D_4, Q
[4, 4, 4]	C_8

Table 4. Group squares of order 8 and their parity vectors

These parity vectors and Theorem 2.2 explain all but one of the blank entries in Table 1; the exception being the non-existence of a biembedding involving C_2^3 and Q.

Our final result is the following theorem.

Theorem 3.1 *For $n \geq 2$ there is no biembedding of two Latin squares both lying in the same main class as the square obtained from the Cayley table of the Abelian 2-group C_2^n.*

Proof. If $x, y \in C_2^n$ then $(x * y) * y = x$. It follows that for $\alpha \in \{r, c, e\}$, the permutation $\gamma_{\alpha,i}^{C_2^n}$ is either the identity or it consists of

2^{n-1} transpositions. In either case it is an even permutation. Hence the main class parity vector is $[0, 0, 0]$, and the result follows. $\square$

Theorem 3.1 proves that for infinitely many orders n there are pairs of Latin squares of order n which cannot be biembedded, even allowing any relabelling of the squares.

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