More on exact bicoverings of 12 points

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Abstract

The minimum number of incomplete blocks required to cover, exactly λ times, all t-element subsets from a set V of cardinality v ($v > t$) is denoted by $g(\lambda, t; v)$. The value of $g(2, 2; v)$ is known for $v = 3, 4, \ldots, 11$. It was previously known that $14 \leq g(2, 2; 12) \leq 16$. We prove that $g(2, 2; 12) \geq 15$.

1 Introduction

A *pairwise balanced design* of *index* λ and *order* v, a PBD$(v; \lambda)$, is a pair $(V, \mathcal{B})$ where V is a set of cardinality v (referred to as the points) and $\mathcal{B}$ is a family of subsets of V (called the blocks) with the property that every pair of elements of V occurs in exactly λ blocks of $\mathcal{B}$. We are concerned with the case $\lambda = 2$ and the PBD is then referred to as an (exact) *bicovering* of V. In particular, we are interested in finding the minimum size of $\mathcal{B}$ in the case $v = 12$ and $\lambda = 2$ with the additional constraint that each block $B \in \mathcal{B}$ satisfies $|B| < v$, that is, $\mathcal{B}$ contains only incomplete blocks.

Following notation introduced by Woodall [5], the notation $g(\lambda, t; v)$ will be used to denote the minimum number of incomplete blocks required to cover, exactly λ times, all t-element subsets of a v-set, where $v > t$. The values of $g(2, 2; v)$ have been determined for $v = 3, 4, \ldots, 11$ [4]. For $v = 12$, it is known that $14 \leq g(2, 2, 12) \leq 16$. The upper bound follows from the existence of a symmetric balanced incomplete block design with parameters $v = 16$, $k = 6$ and $\lambda = 2$ by the deletion of four points. That $g(2, 2; 12) \geq 14$ was determined by Allston et al. in [1]. This paper builds upon the results of [1] to show that $15 \leq g(2, 2; 12) \leq 16$.

In proving our results, we make use of the concept of a *Steiner triple system* of order v, an STS(v). A Steiner triple system of order v is a PBD(v, 1) in which all the blocks are of size three. Since the blocks are of size three, they are often referred to as triples. Such a system is said to be *resolvable* if the triples can be grouped into *parallel classes*, the triples of each parallel class collectively covering all v points precisely once.

We also make use of results from [3] which concern the values of $g^{(k)}(v)$, the minimum number of blocks required to cover, exactly once, each pair of elements from a v-set, subject to the restriction that the maximum block size is precisely k, where $k < v$. A complete tabulation of the values of $g^{(k)}(v)$ for $v \leq 13$ is given in [3], together with an enumeration of all corresponding non-isomorphic solutions to this problem. A set of blocks which cover, exactly once, each pair of elements from a v-set will be called a *single cover* of the v-set.

2 Preliminaries

For the remainder of this paper $g(2, 2; 12)$ will be denoted by g. The proof that $g \neq 14$ will be presented in cases, and these cases are determined by two further parameters associated with a minimal exact bicovering, namely the length ℓ of the longest block and the cardinality d of the largest intersection of distinct blocks. Throughout, we take our set of 12 points to be $\{1, 2, \ldots, 9, t, e, w\}$. We often omit brackets and commas, for example, writing the triple $\{1, 2, 3\}$ as 123. A block written as $B = \{1, 2, 3, 4, \ldots\}$ or as $B = 1234\ldots$ indicates that the points 1, 2, 3 and 4 definitely lie in B, and that B may or may not contain additional points.

The following three lemmas were proven in [1].

Lemma 2.1 *If* $13 \leq g \leq 15$ *then* $l \geq 6$.

Lemma 2.2 *If* $l \geq 6$ *and* $d = 2$ *then* $g \geq 16$.

Lemma 2.3 *If* $d \geq 5$ *then* $g \geq 16$.

Thus, to show that $g \geq 15$, we need only consider minimal exact bicoverings in which the length of the longest block is at least six and in which the largest intersection of any two distinct blocks is three or four. The proof will consist primarily of two lemmas: Lemma 3.1 in which we consider $d = 4$ and Lemma 3.2 in which we consider $d = 3$. However, we first construct two near minimal single covers which are needed later in the paper.

Lemma 2.4 *Let $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, t\}$. Up to isomorphism, there is a unique single cover of X with thirteen blocks and maximum block size four.*

Proof Consider any single cover of X with thirteen blocks and maximum block size four. Let n_i be the number of blocks of size i. Since we require each pair of points of X to occur in precisely one block, we have

$$n_2 + 3n_3 + 6n_4 = 45.$$

We require exactly thirteen blocks, so we also have

$$n_2 + n_3 + n_4 = 13.$$

Thus $2n_3 + 5n_4 = 32$. The only solutions to these equations are

$$(n_2, n_3, n_4) = (6, 1, 6), (3, 6, 4) \text{ and } (0, 11, 2).$$

Consider the case $(n_2, n_3, n_4) = (0, 11, 2)$. Since $n_4 = 2$ there will be an element which does not occur in any block of size four. However, this element must occur exactly once with each of nine elements, which is impossible to do in blocks of size three. Thus, such a single cover does not exist.

Consider the case $(n_2, n_3, n_4) = (6, 1, 6)$. Each element of X must occur exactly once with each of nine elements, so an element can appear in at most three blocks of size four. Since there are six blocks of size four, there are 24 occurrences of elements in the blocks of size four, so at least four elements occur in three blocks of size four. Suppose that an element, 1 say, appears in three blocks of size four. Then, without loss of generality, the blocks are

$$1234 \quad 1567 \quad 189t.$$

Then, without loss of generality, the next block of size four would contain the elements 258. However there is no element available to complete this block. Hence, no element can occur in three blocks of size four and so such a single cover does not exist.

Consider the case $(n_2, n_3, n_4) = (3, 6, 4)$. As above, no element can occur in four blocks of size four, and no element can occur in three blocks of size four. Thus, each element can appear in at most two blocks of size four. Consequently exactly four elements of X must each appear precisely once in the blocks of since four, and it is easily shown that each of these must lie in a different block of size four. It follows that, without loss of generality, the four blocks of size four are

$$1234 \quad 2567 \quad 3689 \quad 479t.$$

Now the elements $1, 5, 8, t$ must each occur with six further elements and the elements $2, 3, 4, 6, 7, 9$ must each occur with three further elements. Thus, each of $2, 3, 4, 6, 7, 9$ must appear in one block of size three and one block of size two, and each of $1, 5, 8, t$ must appear in three blocks of size three. Without loss of generality, the unique single cover of X with thirteen blocks and maximum block size four is as follows.

1234	3689	29	28t	35t
2567	479t	37	458	61t
		46	718	915

□

Lemma 2.5 *Let $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Up to isomorphism, there is a unique single cover of X with thirteen blocks and maximum block size four.*

Proof Consider any single cover of X with thirteen blocks and maximum block size four. As in Lemma 2.4, we have

$$n_2 + 3n_3 + 6n_4 = 36 \quad \text{and} \quad n_2 + n_3 + n_4 = 13.$$

Thus $2n_3 + 5n_4 = 23$. The only solutions to these equations are

$$(n_2, n_3, n_4) = (6, 4, 3) \text{ and } (3, 9, 1).$$

Consider the case $(n_2, n_3, n_4) = (3, 9, 1)$. Let the block of size four be 1234. Now each of the elements $1, 2, 3, 4$ must occur with each of the five elements $5, 6, 7, 8, 9$. Every occurrence of an element in a block of size three gives two pairs involving that element, so each of $1, 2, 3, 4$ must appear in a distinct block of size two. However, there are only three blocks of size two. Thus, no such single cover exists.

Consider the case $(n_2, n_3, n_4) = (6, 4, 3)$. Since there are nine elements, no element can occur in all three blocks of size four. Since $n_4 = 3$, there are twelve occurrences of elements in blocks of size four. So three elements occur in two blocks of size four and the remaining six elements occur once in the blocks of size four. Thus, without loss of generality, the three blocks of size four are

$$1234 \quad 1567 \quad 2589.$$

Now the elements $1, 2, 5$ must each occur with two further elements and the elements $3, 4, 6, 7, 8, 9$ must each occur with five further elements. Thus, each of $1, 2, 5$ must appear in two blocks of size two, and each of $3, 4, 6, 7, 8, 9$ must appear in one block of size two and two blocks of size three. Without loss of generality, the unique single cover of X with thirteen blocks and maximum block size four is as follows.

1234	18	27	468	369
1567	19	35	479	378
2589	26	45		

□

We now return to our discussion of bicoverings of a set of twelve points.

Suppose that an exact bicovering exists with a longest block of length ℓ, where $\ell \geq 6$. Let the points in the longest block be called A-points; without loss of generality, these will be the points $1, 2, 3, \ldots, \ell$. Let the remainder of the points be called B-points, so there are ℓ A-points and $12 - \ell$ B-points. Let a_i be the number of blocks containing i A-points and let b_i be the number of blocks containing i B-points ($i \geq 0$). Count the pairs of A-points (which must each occur once outside the longest block), keeping in mind that $d \leq 4$. Then we have

$$\begin{aligned} a_2 + 3a_3 + 6a_4 &= \tbinom{\ell}{2} \quad \text{if } d = 4, \\ a_2 + 3a_3 &= \tbinom{\ell}{2} \quad \text{if } d = 3. \end{aligned}$$

Since we require the length of the longest block of our bicovering to be at least six, we know that there are at most six B-points. Counting pairs of B-points (which must each occur twice) we have

$$b_2 + 3b_3 + 6b_4 + 10b_5 + 15b_6 = 2\binom{12-\ell}{2}.$$

Now let $n_{i,j}$ be the number of blocks having i A-points and j B-points. Counting the mixed pairs, that is, the pairs of an A-point with a B-point we have

$$\sum_{\substack{0 \leq i \leq d \\ 0 \leq j \leq 12-\ell}} ij n_{i,j} = 2\ell(12-\ell).$$

One method we have used in the proof of the case $d = 3$ is to set up an integer linear program based on these equations. The linear program was set up as follows. We first determined the solutions to $a_2 + 3a_3 = \binom{\ell}{2}$. For this illustration, assume that $a_2 = p$ and $a_3 = q$ is a solution to $a_2 + 3a_3 = \binom{\ell}{2}$. The variables in the linear program are $n_{i,j}$, for $0 \leq i \leq d$, $0 \leq j \leq 12 - \ell$.

$$
\begin{array}{ll}
\text{MAX:} & \displaystyle\sum_{\substack{0\le i\le 3\\ 0\le j\le 12-\ell}} ijn_{i,j} \\
\text{SUBJECT TO:} & n_{3,0}+n_{3,1}+n_{3,2}+n_{3,3}+\ldots+n_{3,\min(\ell-3,12-\ell)}=q \\
& n_{2,0}+n_{2,1}+n_{2,2}+n_{2,3}+\ldots+n_{2,\min(\ell-2,12-\ell)}=p \\
& n_{0,2}+n_{0,3}+n_{0,4}+\ldots+n_{0,12-\ell}+n_{1,1}+n_{1,2}+n_{1,3}+ \\
& \qquad\qquad \ldots+n_{1,\min(\ell-1,12-\ell)}=13-p-q \\
& n_{0,2}+n_{1,2}+n_{2,2}+n_{3,2}+3(n_{0,3}+n_{1,3}+n_{2,3}+n_{3,3}) \\
& \qquad\qquad +6(n_{0,4}+n_{1,4}+\ldots+n_{\min(3,\ell-4),4}) \\
& \qquad\qquad +10(n_{0,5}+n_{1,5}+\ldots+n_{\min(3,\ell-5),5}) \\
& \qquad +15(n_{0,6}+n_{1,6}+\ldots+n_{\min(3,\ell-6),6})=2\binom{12-\ell}{2} \\
& n_{i,j}\ge 0 \\
& n_{i,j} \text{ an integer for all } 0\le i\le 3,\ 0\le j\le 12-\ell
\end{array}
$$

This integer linear program was then solved using the free software lpsolve [2]. In many cases the maximum value found for $\sum ijn_{i,j}$ was less than $2\ell(12-\ell)$, thus allowing us to eliminate that case immediately. This method can be implemented by hand and the computer results have been checked by hand, but the details will not be included in this paper.

3 Proof that $g \neq 14$

Throughout the following we assume that the length of the longest block in the bicovering is $\ell \geq 6$. The proof that $g \geq 15$ is split into two main cases, namely $d = 4$ and $d = 3$ and each case then splits into subcases.

Lemma 3.1 *If $d = 4$ then $g \geq 15$.*

Proof Following the method of [1], suppose that $d = 4$ and so there exist blocks of the bicovering, $B_1 = 1234\ldots$ and $B_2 = 1234\ldots$ both of cardinality at least four. Consider the number of points not in either B_1 or B_2.

Case A Suppose there are two distinct points, say e, w, such that $e, w \notin B_1 \cup B_2$. Then it was shown in [1] that $g \geq 16$.

Case B Again, we use a similar argument to that used in [1]. Suppose there is a unique point, say w, such that $w \notin B_1 \cup B_2$. Then we may assume that $B_1 = 12345678\ldots$ and $B_2 = 1234\ldots$. Consider the pairs $1w, 1w, 2w, 2w, 3w, 3w, 4w, 4w$. These must lie in eight blocks distinct from one another and from B_1 and B_2. Denote these eight blocks by $C_1, C_2, \ldots, C_8$, where the blocks C_{2i-1} and C_{2i} contain the pair iw. Now consider the pairs $15, 25, 35, 45$. At most two of these can lie in $C_1, C_2, \ldots, C_8$. So the remaining blocks, say $D_1, D_2, \ldots$, contain at least two occurrences of the point 5. Similarly, $D_1, D_2, \ldots$ contain at least two occurrences of each of the points 6, 7 and 8. Now consider packing the points 5, 6, 7 and 8 into $D_1, D_2, \ldots$. Without loss of generality, there are three possibilities

(1) $5, 6, 7, 8 \in D_1$,

(2) $5, 6, 7 \in D_1$ but $8 \notin D_1$,

(3) each of $D_1, D_2, \ldots$ contains at most a pair from $\{5, 6, 7, 8\}$.

For case **B(1)** it was shown in [1] that $g \geq 15$.

For case **B(2)** there must be blocks D_2, D_3, D_4 containing the points 5, 6 and 7, respectively, and if $g = 14$ then no more blocks are allowed. Since each of D_1, D_2, D_3, D_4 can contain at most one of the points from $\{1, 2, 3, 4\}$ and since each of the points $\{5, 6, 7\}$ can be adjoined to at most two C-blocks, we have that each of D_1, D_2, D_3, D_4 must contain exactly one of $\{1, 2, 3, 4\}$.Thus we have, without loss of generality:

$$\begin{array}{llll} B_1 = 12345678\ldots & C_1 = 1w\ldots & C_5 = 3w5\ldots & D_1 = 5671\ldots \\ B_2 = 1234\ldots & C_2 = 1w\ldots & C_6 = 3w7\ldots & D_2 = 52\ldots \\ & C_3 = 2w6\ldots & C_7 = 4w5\ldots & D_3 = 63\ldots \\ & C_4 = 2w7\ldots & C_8 = 4w6\ldots & D_4 = 74\ldots \end{array}$$

Now consider pairs involving the point 8. By the earlier argument the point 8 must occur in at least two of D_2, D_3, D_4. Without loss of generality, let $D_2 = 528\ldots$ and $D_3 = 638\ldots$. However, it is now impossible for 8 to be adjoined to two C-blocks in order to obtain two occurrences of the pair $w8$. Thus no such bicovering exists on 14 blocks, so $g \geq 15$ in this case.

For case **B(3)** there must be blocks D_1, D_2, D_3, D_4, and without loss of generality $D_1 = 56\ldots, D_2 = 57\ldots, D_3 = 68\ldots, D_4 = 78\ldots$. Now if $g = 14$ then there are no other blocks. As shown above, the blocks $C_1, \ldots, C_8$ must contain exactly two occurrences of each of $5, 6, 7, 8$. Without loss of generality, let $5 \in C_1$ and $5 \in C_3$. This implies, without loss of generality, that $3 \in D_1$ and $4 \in D_2$, and also that $8 \in C_1$ and $67 \in C_2$. Now none of the points $6, 7, 8$ can be adjoined to C_3 but the pairs $26, 27, 28$ must still occur, so, without loss of generality, let $2 \in D_3$ and $7 \in C_4$. This implies $3 \in D_4$, $6 \in C_7$ and $8 \in C_8$. Now consider the occurrences of the element 9. The element 9 occurs in exactly one of B_1 or B_2, and so, by considering the pairs $19, 29, 39, 49, 9w$, must occur in exactly two of $C_1, \ldots, C_8$ and exactly two of $D_1, \ldots, D_4$. If $9 \in B_1$ and if 9 occurs in two of $D_1, \ldots, D_4$, then, by considering the pairs $59, 69, 79, 89$, it is clearly impossible to adjoin 9 to two blocks in $C_1, \ldots, C_8$. If $9 \in B_2$, then we require two occurrences of each of the pairs $59, 69, 79, 89$ to occur in the C and D blocks, which is impossible. Thus no such bicovering exists on 14 blocks and so $g \geq 15$ in this case.

Case C Suppose all twelve points occur in $B_1 \cup B_2$. As in [1] we split this case into subcases depending on the value of $|B_1|$. Clearly we may assume that $|B_1| \geq 8$. In each subcase we consider the intersections of the remaining blocks of the bicovering with B_1. These yield an exact single cover of the set of points in B_1. Because $d = 4$ and $|B_1 \cap B_2| = 4$, this single covering has largest block length 4. Thus, we can use results from [3] on the values of $g^{(4)}(|B_1|)$ to deal with each subcase.

C(1) Let $|B_1| = 11$. We take $B_1 = 123456789te$ so $B_2 = 1234w$. If there exists an exact bicovering with 14 blocks, then we require a single cover of the points in B_1 on at most 13 blocks. It was shown in [3] that $g^{(4)}(11) = 13$ and that this single cover is unique. Thus if there exists an exact bicovering with 14 blocks, then for this case, without loss of generality, the 14 blocks would be as shown.

$123456789te$	$1234w$	$17te\ldots$	$159\ldots$	$168\ldots$	$4e\ldots$
	$456t\ldots$	$258e\ldots$	$267\ldots$	$29t\ldots$	
	$4789\ldots$	$369e\ldots$	$38t\ldots$	$357\ldots$	

Call the points in B_1 the A-points, and consider the pairs xw where x is an A-point. Letting a_i denote the number of blocks containing w that have i A-points, we require

$$2a_2 + 3a_3 + 4a_4 = 22.$$

The only solutions of this equation which satisfy $1 \le a_4 \le 6$, $0 \le a_3 \le 6$ and $0 \le a_2 \le 1$ are

$$(a_2, a_3, a_4) = (1, 0, 5), (0, 2, 4), (1, 4, 2) \text{ and } (0, 6, 1).$$

The first and last of these four cases are ruled out immediately since it is clear that some pairs xw would be covered more than twice.

Consider the case $(a_2, a_3, a_4) = (0, 2, 4)$. Since we have the block $1234w$, if we adjoin w to two triples of A-points, then we have two occurrences of two of the pairs $1w$, $2w$ and $3w$. Therefore we can adjoin w to at most one of the blocks $17te\ldots$, $258e\ldots$ and $369e\ldots$. To then have $a_4 = 4$ implies that we must adjoin w to both of $456t\ldots$ and $4789\ldots$, giving three occurrences of the pair $4w$.

Consider the case $(a_2, a_3, a_4) = (1, 4, 2)$. Since we have the block $1234w$, letting $a_3 = 4$ would give too many occurrences of one of the pairs $1w$, $2w$ or $3w$.

Therefore no such bicovering with $|B_1| = 11$ exists.

(C)(2) Let $|B_1| = 10$. We take $B_1 = 123456789t$ so $B_2 = 1234ew$. If there exists an exact bicovering with 14 blocks, then we require a single cover of the points in B_1 on at most 13 blocks. It was shown in [3] that $g^{(4)}(10) = 12$ and that this single cover is unique. So if the single cover is on 12 blocks, then, without loss of generality, we can take the first 13 blocks of our bicovering to be

$123456789t$	$1234ew$	$158\ldots$	$269\ldots$	$359\ldots$
	$4567\ldots$	$16t\ldots$	$278\ldots$	$37t\ldots$
	$489t\ldots$	$179\ldots$	$25t\ldots$	$368\ldots$

Call the points in B_1 the A-points, and consider the pairs xe where x is an A-point. By the proof of the corresponding case in [1], the element e must appear in the 14th block and it must appear with exactly one A-point. Letting a_i denote the number of blocks containing e that have i A-points, we require

$$3a_3 + 4a_4 = 20 - 1.$$

The only solution of this equation which satisfies $1 \leq a_4 \leq 3$ and $0 \leq a_3 \leq 9$ is

$$(a_3, a_4) = (5, 1).$$

Since we have the block $1234ew$, the element e can be adjoined to at most one of the blocks $158\ldots, 16t\ldots, 179\ldots$, at most one of the blocks $269\ldots, 278\ldots, 25t\ldots$, and at most one of the blocks $37t\ldots, 359\ldots, 368\ldots$. Therefore it is impossible to have $(a_3, a_4) = (5, 1)$.

Now suppose that the single cover of the points in B_1 is on 13 blocks. It was shown in Lemma 2.4 that such a single cover is unique. So if the single cover is on 13 blocks, then, without loss of generality, we can take the 14 blocks of our bicovering to be

$123456789t$	$2567\ldots$	$28t\ldots$	$16t\ldots$	$29\ldots$
$1234ew$	$3689\ldots$	$35t\ldots$	$178\ldots$	$37\ldots$
	$479t\ldots$	$458\ldots$	$159\ldots$	$46\ldots$

Call the points in B_1 the A-points, and consider the pairs xe where x is an A-point. Letting a_i denote the number of blocks containing e that have i A-points, we require

$$2a_2 + 3a_3 + 4a_4 = 20.$$

The only solutions of this equation which satisfy $1 \leq a_4 \leq 4$, $0 \leq a_3 \leq 6$ and $0 \leq a_2 \leq 3$ are

$$(a_2, a_3, a_4) = (2, 0, 4), (1, 2, 3), (0, 4, 2), (3, 2, 2) \text{ and } (2, 4, 1).$$

Consider the case $(a_2, a_3, a_4) = (2, 0, 4)$. If we adjoin e to all four blocks containing four A-points, then the pairs we have yet to cover are $1e, 5e, 8e, te$. This is not possible given the blocks containing two A-points which we have.

Consider the case $(a_2, a_3, a_4) = (1, 2, 3)$. If we adjoin e to three of the blocks with four A-points, then we will have two occurrences of the pairs $2e, 3e, 6e$ or $2e, 4e, 7e$ or $3e, 4e, 9e$. So it is not possible to adjoin e to any of the blocks with two A-points.

Consider the cases $(a_2, a_3, a_4) = (0, 4, 2)$ and $(2, 4, 1)$. Since we have the block $1234ew$, we can adjoin e to at most one of $16t\ldots, 178\ldots, 159\ldots$. So in order to have $a_3 = 4$, we must adjoin e to each of $28t\ldots, 35t\ldots, 458\ldots$. Then it is impossible to adjoin e to another block with four A-points, and also impossible to adjoin e to any block with two A-points.

Consider the case $(a_2, a_3, a_4) = (3, 2, 2)$. ¿From $a_2 = 3$ we would have the blocks $29e\ldots, 37e\ldots, 46e\ldots$. These, along with the block $1234ew$ mean that e cannot be adjoined to any other block with four A-points.

Therefore no such bicovering with $|B_1| = 10$ exists.

(C)(3) Let $|B_1| = 9$. We take $B_1 = 123456789$ so $B_2 = 1234tew$. If there exists an exact bicovering with 14 blocks, then we require a single cover of the points in B_1 on at most 13 blocks. It was shown in [3] that $g^{(4)}(9) = 12$ and that this single cover is unique. So if the single cover is on 12 blocks, then, without loss of generality, we can take the first 13 blocks of our bicovering to be

123456789	$1234tew$	$258\ldots$	$269\ldots$	$27\ldots$
$1567\ldots$	$368\ldots$	$379\ldots$	$35\ldots$	
$189\ldots$	$478\ldots$	$459\ldots$	$46\ldots$	

Call the points in B_1 the A-points, and consider the pairs xt where x is an A-point. By the proof of the corresponding case in [1], the element t must appear in the 14th block. Similarly, the elements e and w must appear in the 14th block and the 14th block must contain tew and one A-point, say α. Consider the pairs xy where x is an A-point and $y \in \{t, e, w\}$. Letting a_i denote the number of blocks containing y that have i A-points, we require

$$2a_2 + 3a_3 + 4a_4 = 18 - 1.$$

The only solutions of this equation which satisfy $1 \leq a_4 \leq 2$, $0 \leq a_3 \leq 7$ and $0 \leq a_2 \leq 3$ are

$$(a_2, a_3, a_4) = (0, 3, 2), (3, 1, 2) \text{ and } (2, 3, 1).$$

Since we have the blocks $1234tew$ and $tew\alpha$, we cannot adjoin any pair of t, e, w to the same block. Therefore at least two of these elements must use the solution $(a_2, a_3, a_4) = (2, 3, 1)$. If we adjoin t to two blocks containing two A-points, then by considering the pairs $1t$, $2t$, $3t$ and $4t$, we see that $a_3 \leq 2$. Thus it is impossible to obtain a bicovering in this case.

Now suppose that the single cover of the points in B_1 occurs on 13 blocks. It was shown in Lemma 2.5 that such a single cover is unique. So if the single cover is on 13 blocks, then, without loss of generality, we can take the 14 blocks of our bicovering to be

123456789	$1567\ldots$	$479\ldots$	$18\ldots$	$27\ldots$
$1234tew$	$2589\ldots$	$369\ldots$	$19\ldots$	$35\ldots$
	$468\ldots$	$378\ldots$	$26\ldots$	$45\ldots$

Call the points in B_1 the A-points, and consider the pairs xt where x is an A-point. Letting a_i denote the number of blocks containing t that have i A-points, we require

$$2a_2 + 3a_3 + 4a_4 = 18.$$

The only solutions of this equation which satisfy $1 \leq a_4 \leq 3$, $0 \leq a_3 \leq 4$ and $0 \leq a_2 \leq 6$ are

$$(a_2, a_3, a_4) = (0, 2, 3), (3, 0, 3), (2, 2, 2), (5, 0, 2), (1, 4, 1) \text{ and } (4, 2, 1).$$

Consider the cases in which $a_4 = 3$. If we adjoin t to all three blocks with four A-points, then the pairs we have yet to cover are $3t, 4t, 6t, 7t, 8t, 9t$. This is not possible to do using two of the blocks containing three A-points, or by using three of the blocks containing two A-points.

Consider the case $(a_2, a_3, a_4) = (2, 2, 2)$. We have the block $1234tew$, and suppose we adjoin t to the block $1567\ldots$. Then by considering the pairs $3t$ and $4t$, and $a_3 = 2$, we must adjoin t to either the blocks $468\ldots, 378\ldots$ or to the blocks $369\ldots, 479\ldots$. But then it is impossible to adjoin t to two blocks of two A-points. Similarly, if we adjoined t to the block $2589\ldots$, then we must adjoin t to either the blocks $468\ldots, 369\ldots$ or to the blocks $479\ldots, 378\ldots$. Again it would then be impossible to adjoin t to two blocks of two A-points.

Consider the case $(a_2, a_3, a_4) = (5, 0, 2)$. Since we have the block $1234tew$ and either $1567t\ldots$ or $2589t\ldots$, we can adjoin t to at most two blocks containing two A-points.

Consider the cases in which $a_4 = 1$. Since we have the block $1234tew$, consider the pairs $3t$ and $4t$ to see that $a_3 \leq 2$. If $a_3 = 2$, then by considering the pairs $1t$, $2t$, $3t$ and $4t$, we see that $a_2 \leq 2$. Thus both the cases $(a_2, a_3, a_4) = (1, 4, 1)$ and $(4, 2, 1)$ are impossible.

Therefore no such bicovering with $|B_1| = 9$ exists.

(C)(4) Let $|B_1| = 8$. It was shown in [1] that in this case $g \geq 15$.

$\square$

Lemma 3.2 *If $d = 3$ then $g \geq 15$.*

Proof We use a similar method to that of [1]. Let B_1 be the longest block, of length ℓ, and call the points in B_1 the A-points. Let a_i be the number of blocks that contain i A-points. Since $d = 3$, we have $a_2 + a_3 \leq g - 1$ and $a_2 + 3a_3 = \binom{\ell}{2}$. Consider the different possible values of ℓ.

Case A If $\ell = 11$ then $a_2 + 3a_3 = 55$ and the minimum value of $a_2 + a_3 = 1 + 18 = 19$, giving $g \geq 20$.

Case B If $\ell = 10$ then $a_2 + 3a_3 = 45$ and the minimum value of $a_2 + a_3 = 0 + 15 = 15$, giving $g \geq 16$.

Case C If $\ell = 9$ then $a_2 + 3a_3 = 36$. Solutions of this immediately give $g \geq 15$ apart from the case $(a_2, a_3) = (0, 12)$. This case corresponds to the twelve triples of an STS(9) on the nine points of B_1. The associated bicovering has at least 13 distinct blocks which we may take as:

123456789	$123\ldots$	$147\ldots$	$159\ldots$	$168\ldots$
	$456\ldots$	$258\ldots$	$267\ldots$	$249\ldots$
	$789\ldots$	$369\ldots$	$348\ldots$	$357\ldots$

where undeclared entries are from $\{t, e, w\}$. Call the 12 blocks other than B_1 the C-blocks. Suppose a solution exists on 14 blocks. Then there is precisely one extra block and this contains at most one point from B_1. There are 18 pairs xt with $x \in B_1$ to be covered and it is not possible to achieve this number if one of the pairs comes from the extra block. The same argument applies to the points e and w, and so we deduce that the extra block contains no points from B_1 and that each of t, e, w appear six times in the C-blocks. The only way of achieving this is that two C-blocks contain the pair te but not w, two contain tw but not e, two contain ew but not t, and the remaining six C-blocks each contain exactly one of t, e, w. But then the extra block cannot contain any pair from $\{t, e, w\}$ and so cannot exist. Thus there is no solution on 14 blocks and we conclude that if $l = 9$, then $g \geq 15$.

Case D If $\ell = 8$ then $a_2 + 3a_3 = 28$. Solutions of this immediately give $g \geq 15$ apart from the cases D1 where $(a_2, a_3) = (1, 9)$ and D2 where $(a_2, a_3) = (4, 8)$. It was shown in [1] that the configuration in case D1 does not exist. It was also shown that the unique configuration for the case D2 is given by deleting one point from the twelve triples of an STS(9), thus the exact bicovering would have the form:

12345678	$123\ldots$	$147\ldots$	$15\ldots$	$168\ldots$
	$456\ldots$	$258\ldots$	$267\ldots$	$24\ldots$
	$78\ldots$	$36\ldots$	$348\ldots$	$357\ldots$

where undeclared entries are from $\{9, t, e, w\}$. Call the 12 blocks other than B_1 the C-blocks. Suppose a solution exists on 14 blocks; then there is one extra block. The one extra block may contain at most one point from $\{1, 2, \ldots, 8\}$.

(a) Suppose no point from $\{1, 2, \ldots, 8\}$ occurs in the extra block. Each of the points $\{9, t, e, w\}$ must occur twice with each of $\{1, 2, \ldots, 8\}$ giving 16 pairs for each of $9, t, e, w$. For the point 9, let m be the number of C-blocks containing two A-points in which it occurs and let n be the number of C-blocks containing three A-points in which it occurs. Thus $2m + 3n = 16$. The only solution to this with $0 \leq m \leq 4$ is $(m, n) = (2, 4)$ and this shows that 9 must appear six times amongst the C-blocks. Similar arguments apply to each of t, e, w and, as in case C, this implies that the six pairs $9t, 9e, 9w, te, tw, ew$ each appear twice amongst the C-blocks and so the extra block cannot contain any pair from $\{9, t, e, w\}$ and so cannot exist.

(b) Suppose one point from $\{1, 2, \ldots, 8\}$ occurs in the extra block. Without loss of generality, let 1 occur in the extra block and let 9 occur in the extra block. As in part (a), 9 must now occur in 15 pairs with A-points in the twelve C-blocks. Thus with the same notation, $2m + 3n = 15$. The solutions to this with $0 \leq m \leq 4$ are $(m, n) = (0, 5)$ and $(3, 3)$. Consider the case $(m, n) = (0, 5)$ and look for five triples of A-points amongst the C-blocks to which to adjoin the point 9. Without loss of generality, choose one triple containing the point 1, say 123. Then the triples 147 and 168 cannot be chosen. The point 9 can occur with at most one of 258 and 267, and at most one of 348 and 357, giving a total of three triples so far. There is only one remaining triple 456, so it is impossible for the point 9 to occur with five triples.

Now consider the case $(m, n) = (3, 3)$. Without loss of generality, adjoining 9 to three triples of A-points amongst the C-blocks will miss at least one element, so there will be a pair $x9$ with $x \in \{1, 2, 3, 4, 5, 6, 7, 8\}$ that must still occur twice. If the element $x \neq 1$, then since 9 cannot occur with x twice in C-blocks containing two A-points, this case is impossible. If the element $x = 1$, then we must adjoin 9 to the pair 15. Thus 9 can have been adjoined to only one of the triples 456, 258 and 357 and must have been adjoined to the triples 267 and 348. It is then impossible to choose two more pairs to complete the bicovering.

Thus, if $\ell = 8$, then $g \geq 15$.

Case E If $\ell = 7$ then $a_2 + 3a_3 = 21$. Solutions to this immediately give $g \geq 15$ apart from the cases $(a_2, a_3) = (0,7), (3,6), (6,5), (9,4)$. Call these cases E1, E2, E3 and E4 respectively. We use the linear programming method with the following linear program:

$$\begin{array}{ll}
\text{MAX:} & \displaystyle\sum_{\substack{0\le i\le 3\\ 0\le j\le 5}} ijn_{i,j} \\
\text{SUBJECT TO:} & n_{3,0}+n_{3,1}+n_{3,2}+n_{3,3}+n_{3,4}=a_3 \\
& n_{2,0}+n_{2,1}+n_{2,2}+n_{2,3}+n_{2,4}+n_{2,5}=a_2 \\
& n_{0,2}+n_{0,3}+n_{0,4}+n_{0,5}+n_{1,1}+n_{1,2}+n_{1,3}+n_{1,4}+n_{1,5} \\
& \qquad\qquad = 13-a_2-a_3 \\
& n_{0,2}+n_{1,2}+n_{2,2}+n_{3,2}+3(n_{0,3}+n_{1,3}+n_{2,3}+n_{3,3}) \\
& +6(n_{0,4}+n_{1,4}+n_{2,4}+n_{3,4})+10(n_{0,5}+n_{1,5}+n_{2,5})=20 \\
& n_{i,j}\ge 0 \\
& n_{i,j} \text{ to be an integer for all } 0\le i\le 3,\ 0\le j\le 5.
\end{array}$$

Since $\ell = 7$ we require that $\sum ijn_{i,j} = 70$ for an exact bicovering on 14 blocks to exist. For the cases E1, E2 and E3, the largest values for $\sum ijn_{i,j}$ were 67, 68 and 69, respectively. For the case E4, the largest value for $\sum ijn_{i,j}$ was 70, but this value was obtained in a unique way.

Thus the only case left to be eliminated for $\ell = 7$ is where

$$(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7) = (0,0,9,4,0,0,0,1), \quad (b_0, b_1, b_2, b_3, b_4, b_5) = (1,1,8,4,0,0),$$

the points being arranged so that there is one block of length seven consisting of the seven A-points, four blocks of length six each containing three A-points and three B-points, eight blocks of length four each containing two A-points and two B-points, and one final block of length three containing two A-points and one B-points.

By considering possible arrangements of four triples on seven points, we see that the A-points must be arranged in one of the following three configurations.

Case E(4a)	**Case E(4b)**	**Case E(4c)**
1234567	1234567	1234567
123	123	123
145	145	456
246	167	147
356	246	257
17	25	15
27	27	16
37	34	24
47	35	26
57	36	34
67	37	35
25	47	36
34	56	37
16	57	67

Suppose a bicovering on 14 blocks exists; then there are no extra blocks, and we require the combinations of A-points and B-points to be as outlined above. Consider the element 8 and suppose it occurs m times in the blocks containing three A-points and n times in the blocks containing two A-points. Then $3m + 2n = 14$. Solutions to this are:

$$(m, n) = (0, 7), (2, 4) \text{ and } (4, 1).$$

Examining the sets of pairs that can be chosen out of the nine blocks containing two A-points, it is clear that the case $(m, n) = (0, 7)$ will not work for any of the three configurations above, and the same result holds for the elements $9, t, e$ and w.

E(4a) Consider the configuration listed as E(4a), and consider adjoining the points $8, 9, t, e, w$. At most one of these points can use the solution $(m, n) = (4, 1)$ calculated above, so at least four of these points must use the solution $(m, n) = (2, 4)$. To adjoin the element 8 say, to four blocks containing two A-points, at least two of the blocks must be from the set $\{25\ldots, 34\ldots, 16\ldots\}$ since all other such blocks contain the element 7. Adjoining 8 to any two blocks containing three A-points forces the choice of the two blocks from $\{25\ldots, 34\ldots, 16\ldots\}$ and it is easily verified that it is then impossible to adjoin 8 to two more blocks from the blocks containing two A-points. Thus no bicovering containing the configuration E(4a) exists.

E(4b) Consider the configuration listed as E(4b), and consider adjoining the points $8, 9, t, e, w$. Since the element 1 occurs in three blocks containing three A-points, it is impossible for any of $8, 9, t, e, w$ to use the solution $(m, n) = (4, 1)$. Hence the adjoining of each of these points must use the solution $(m, n) = (2, 4)$. By considering the pairs containing 1, each of $8, 9, t, e, w$ must be adjoined to two of the blocks in the set $\{123\ldots, 145\ldots, 167\ldots\}$, and thus at least two of $8, 9, t, e, w$, say 8 and 9, are adjoined to the same two blocks from this set. However, the choice of two blocks from $\{123\ldots, 145\ldots, 167\ldots\}$ forces the choice of the blocks containing two

A-points to which an element is adjoined and so the pair 89 would occur more than twice. Thus no bicovering containing the configuration E(4b) exists.

E(4c) Consider the configuration listed as E(4c), and consider adjoining the points $8, 9, t, e, w$. At most one of these points can use the solution $(m, n) = (4, 1)$, so at least four of these points must use the solution $(m, n) = (2, 4)$. It is easily verified that there are precisely four ways in which to do this, namely by adjoining the B-point to one of the following four sets of six blocks

123...	147...	147...	147...
456...	257...	257...	257...
15...	16...	16...	26...
24...	26...	36...	36...
37...	34...	24...	15...
67...	35...	35...	34...

We can adjoin at most one B-point to the first set of six blocks. We can adjoin at most one B-point to at most one of the last three sets of six blocks (since each set of six blocks intersects in three blocks), so clearly not all the B-points can be adjoined. Thus no bicovering containing the configuration E(4c) exists.

Thus, if $\ell = 7$, then $g \geq 15$.

Case F If $\ell = 6$ then $a_2 + 3a_3 = 15$. Solutions to this immediately give $g \geq 15$ apart from the cases $(a_2, a_3) = (0, 5), (3, 4), (6, 3), (9, 2), (12, 1)$. Call these cases F1, F2, F3, F4 and F5 respectively. We use the linear programming method with the following linear program:

MAX: $\displaystyle\sum_{\substack{0 \leq i \leq 3 \\ 0 \leq j \leq 6}} ijn_{i,j}$

SUBJECT TO:

$$n_{3,0} + n_{3,1} + n_{3,2} + n_{3,3} = a_3$$
$$n_{2,0} + n_{2,1} + n_{2,2} + n_{2,3} + n_{2,4} = a_2$$
$$n_{0,2} + n_{0,3} + n_{0,4} + n_{0,5} + n_{0,6} + n_{1,1} + n_{1,2} + n_{1,3} + n_{1,4} + n_{1,5} = 13 - a_2 - a_3$$
$$n_{0,2} + n_{1,2} + n_{2,2} + n_{3,2} + 3(n_{0,3} + n_{1,3} + n_{2,3} + n_{3,3}) + 6(n_{0,4} + n_{1,4} + n_{2,4}) + 10(n_{0,5} + n_{1,5}) + 15(n_{0,6}) = 30$$
$$n_{i,j} \geq 0$$

$n_{i,j}$ to be an integer for all $0 \leq i \leq 3$, $0 \leq j \leq 6$.

Since $\ell = 6$ we require that $\sum ijn_{i,j} = 72$ for an exact bicovering on 14 blocks to exist. However, for the cases F1 to F5, the largest values of $\sum ijn_{i,j}$ were 64, 68, 70, 71 and 71, respectively.

Thus, if $\ell = 6$, then $g \geq 15$. □

We conclude by combining the results of Lemmas 3.1 and 3.2.

Theorem 3.1 $g(2, 2; 12) \geq 15$.

References

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