Configurations and trades in Steiner triple systems

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Abstract

The main result of this paper is the determination of all pairwise non-isomorphic trade sets of volume at most 10 which can appear in Steiner triple systems. We also enumerate partial Steiner triple systems having at most 10 blocks as well as configurations with no points of degree 1 and tradeable configurations having at most 12 blocks.
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1 Introduction

This paper is primarily concerned with trades in Steiner triple systems. The results are mainly enumerative and were obtained by computational means. However, for reasons which will become clear later, it is appropriate to consider configurations more generally. We begin with some definitions.

A *Steiner triple system* of order v, briefly STS(v), is a pair $(V, \mathcal{B})$ where V is a base set of cardinality v of *elements*, or *points*, and $\mathcal{B}$ is a collection of *triples*, also called *blocks* or *lines*, which has the property that every pair of distinct elements of V occurs in precisely one triple. It is well known that an STS(v) exists if and only if $v \equiv 1$ or 3 (mod 6).

An n-line *configuration* is a collection of n triples which has the property that every pair of distinct elements occurs in at most one triple, i.e. a configuration is a partial Steiner triple system. In a configuration the *degree* of a point is the number of triples which contain it.

A *trade set*, or *n-way trade*, $\mathcal{T} = \{T_1, T_2, \ldots, T_n\}$, $n \geq 2$, is a set of pairwise disjoint m-line configurations, T_i, which has the property that every pair of distinct elements occurs in precisely the same number (zero or one) of triples of each T_i. The number of lines, m, is called the *volume* of the trade set and is denoted by $vol(\mathcal{T})$. The *foundation* of the trade set, $found(\mathcal{T})$, is the set of elements covered by each T_i. A 2-way trade $\{T_1, T_2\}$ is simply called a *trade*. As is pointed out in Street [12], the single collection T_i is often referred to as a trade. To our minds this usage is unfortunate and can be confusing. In this paper we make the important distinction between a trade $\mathcal{T}$ and its constituent configurations T_i: these configurations will be called *tradeable configurations*. Clarity on this point is essential for accurate enumeration of such structures. Some authors use the notation (T_1, T_2) for a 2-way trade, suggesting an ordered pair, but we see no reason to depart from standard set notation.

Definition 1.1 Two trade sets $\mathcal{T} = \{T_1, T_2, \ldots, T_{n_1}\}$ and $\mathcal{T}' = \{T_1', T_2', \ldots, T_{n_2}'\}$ are said to be *isomorphic* if

(i) $vol(\mathcal{T}) = vol(\mathcal{T}')$,
(ii) $found(\mathcal{T}) = found(\mathcal{T}')$,
(iii) $n_1 = n_2 = n$, and
(iv) there exists a function $f : found(\mathcal{T}) \to found(\mathcal{T}')$ such that

$$f(\{T_1, T_2, \ldots, T_n\}) = \{T_1', T_2', \ldots, T_n'\}.$$

In Section 3, we enumerate all pairwise non-isomorphic tradeable configurations and trade sets with volume at most 12 that can occur in Steiner triple systems. Note that an n-way trade gives rise to $\binom{n}{l}$ l-way trades for $2 \leq l \leq n$. However, some of these l-way trades may be isomorphic, and our computational results reflect this. The present paper partly replicates some of the results in Khosrovshahi & Maimani [8], where the numbers of such trades of volume at most nine are given. Unfortunately, in that paper no mention is made of trade sets and no distinction is made between trades and tradeable configurations. However, when the volume exceeds eight the distinction between trades and tradeable configurations becomes crucial. As we show in this paper there exist trades of volume nine containing non-isomorphic tradeable configurations. Also some of the numbers reported in [8] are incorrect; the subsequent paper [3] gives different values for foundation size 9. Our results confirm these latter values. Results for trade sets with volume at most 8 are also given in two earlier papers by Lizzio & Milici [9, 10]. Here again there is an error; in the second paper the two trades identified in Theorem 3.3 are in fact isomorphic and correspond to number 10 in our Table 3.4. Trade sets of volume 9 were also enumerated in Gionfriddo, Milici & Vacirca [4] but the list appears to be incomplete.

2 Algorithms

We begin this section with a few remarks about labellings.

A *labelling* of a configuration C with point set V is a function ϕ which maps V onto the set $\{0, 1, \ldots, |V|-1\}$ of *labels*.

Following Colbourn & Rosa [1], we extend the usual ordering $<$ of the integers to pairs of integers, triples and sets of triples. Pairs of integers are given the reverse-lexicographical ordering; for $a < b$ and $c < d$, $\{a, b\} < \{c, d\}$ if $b < d$, or $b = d$ and $a < c$. Triples are ordered by their smallest pairs; if $a < b < c$ and $d < e < f$, then $\{a, b, c\} < \{d, e, f\}$ if $\{a, b\} < \{d, e\}$, or $\{a, b\} = \{d, e\}$ and $c < f$. Two sets of triples, A and B, are ordered by the smallest triple in their symmetric difference; thus $A < B$ if the smallest triple in $A \setminus B$ is less than the smallest triple in $B \setminus A$.

A *canonical labelling* of a configuration C is a labelling ϕ for which $\phi(C)$ is as small as possible. If two configurations have the same canonical labellings, they are isomorphic, and the number of canonical labellings of C is equal to the order of $Aut(C)$, the group of automorphisms of C.

It is relatively straightforward to determine all trade sets $\mathcal{T}$ with $vol(\mathcal{T}) \leq 7$ by elementary arguments, but after that the reasoning becomes rather tedious and the subcases to be considered proliferate. Dealing with $vol(\mathcal{T}) > 7$ is a task for a computer. Our main algorithm uses a simple back-tracking procedure to generate all possible labelled trades $\{C, D\}$ from a given labelled configuration C. It is clear from the following presentation that it has the desired effect.

Algorithm 2.1 Suppose we are given a labelled configuration C with point set V. Set $L = \{\{r, s\} : \{r, s\} \text{ is a pair in } C\}$ and set $D = \{\}$. Then we perform a procedure called ADD BLOCK to add triples one at a time to D.

> ADD BLOCK
>
> Choose a pair $\{r, s\}$ in L.
>
> For each point $t \in V \setminus \{r, s\}$ for which $\{r, s, t\}$ is not a block in C and both $\{r, t\}$ and $\{s, t\}$ are pairs in L:
>
> > Add $\{r, s, t\}$ to D and remove the pairs $\{r, s\}$, $\{r, t\}$, $\{s, t\}$ from L.
> >
> > If L is empty, report a trade, $\{C, D\}$; otherwise perform the procedure ADD BLOCK.
> >
> > Remove $\{r, s, t\}$ from D and restore its pairs to L.
>
> Return.

Clearly, Algorithm 2.1 presupposes that we have at our disposal a list containing every tradeable configuration of n blocks. Therefore we first implemented an algorithm to determine, for $1 \leq n \leq 10$, all pairwise non-isomorphic n-line configurations which can occur as blocks of a Steiner triple system. The method is straightforward. We add a new line in every possible manner to every $(n-1)$-line configuration and reject isomorphs using Miller's algorithm [11]. As described in Section 4.2 of Colbourn & Rosa [1], Miller's algorithm constructs canonical labellings for Steiner triple systems. Although we do not give details in this paper, we found it was necessary to implement a number of elementary enhancements to make the algorithm work efficiently for configurations that are not complete designs.

The numbers, $C(n)$, of n-line configurations are known for $1 \leq n \leq 8$ (Grannell & Griggs [5]) and so provide a check on the correctness of the program. Also we would like to thank Professor C. J. Colbourn, who kindly made available to us a list of all 6-, 7- and 8-line configurations thereby providing independent verification of our computations for $6 \leq n \leq 8$.

From this catalogue of configurations we then constructed all pairwise non-isomorphic n-line configurations in which each point has degree at least 2. For $1 \leq n \leq 10$, this is simply a matter of selecting the appropriate configurations from the catalogue. For $n = 11, 12$ we used the elementary observation that any 10-line subconfiguration of an n-line configuration with the additional property has at most $3(n-10)$ points of degree 1. Hence we can begin with all 10-line configurations containing at most $3(n-10)$ points of degree 1, extend them in every possible way so that every point has degree at least 2, and reject isomorphs. We denote the number of these configurations by $B(n)$.

There are two good reasons why it is relevant to identify configurations all of whose points have degree at least 2. First, it is elementary that every point of a tradeable configuration must have degree greater than 1, and a subcatalogue of configurations with this property is a much smaller database to consider than the catalogue of all configurations. Secondly, configurations in which every point has degree at least 2 are interesting in their own right. This is due to a theorem, proved in Horák, Phillips, Wallis & Yucas [7], that the number of occurrences of any n-line configuration in an STS(v) can be expressed as a linear combination of the number of occurrences of a single block and all m-line configurations, $1 \leq m \leq n$, having all points of degree at least 2, where the coefficients are polynomials in v.

Thus for $1 \leq n \leq 12$ we were able to construct a list containing every tradeable configuration of n blocks. By giving the points of these configurations a canonical labelling and then applying Algorithm 2.1 we were therefore able to create a list of the different labelled trades $\{C, D\}$ that originate from each canonically labelled tradeable configuration C.

3 Results

It is inappropriate, and indeed infeasible, to record all of our detailed results in this paper. However, it is appropriate to summarize the results, give details of trades of small volume and point to references where other information can be obtained.

First, we give in Table 3.1, below, the number, $C(n)$, of n-line configurations, $1 \leq n \leq 10$, the number, $B(n)$, of n-line configurations in which each point has degree at least 2, $1 \leq n \leq 12$ and the number, $A(n)$, of n-line tradeable configurations, $1 \leq n \leq 12$.

Listings of all n-line configurations, together with formulae for their numbers of occurrence in terms of v (the order of the Steiner triple system), p (the number of Pasch configurations), and m (the number of mitres) are given in Grannell, Griggs & Mendelsohn [6] for $1 \leq n \leq 4$ and Danziger, Mendelsohn, Grannell & Griggs [2] for $n = 5$.

Table 3.1. Configuration counts

n	1	2	3	4	5	6	7	8
$C(n)$	1	2	5	16	56	282	1865	17100
$B(n)$	0	0	0	1	1	5	19	153
$A(n)$	0	0	0	1	0	2	2	10

n	9	10	11	12
$C(n)$	207697	3180571	-	-
$B(n)$	1615	25180	479238	10695820
$A(n)$	17	102	436	3822

The n-line configurations in which each point has degree at least 2, $1 \leq n \leq 7$, are listed in Table 3.2. Set brackets and delimiting commas are omitted for clarity. The configurations have canonical labellings with the blocks presented in lexicographical order.

Table 3.2. Configurations with no points of degree 1

	Lines	Points		
1	4	6	012 034 135 245	Pasch
2	5	7	012 034 135 236 456	mitre
3	6	7	012 034 135 146 236 245	semihead
4	6	8	012 034 135 147 236 567	
5	6	8	012 034 135 246 257 367	6-cycle
6	6	9	012 034 135 267 468 578	
7	6	9	012 034 156 278 357 468	
8	7	7	012 034 056 135 146 236 245	STS(7)
9	7	8	012 034 057 135 146 236 247	
10	7	8	012 034 135 147 236 257 456	
11	7	9	012 034 058 135 147 236 678	
12	7	9	012 034 135 147 168 236 578	
13	7	9	012 034 135 147 236 258 678	
14	7	9	012 034 135 147 236 468 578	
15	7	9	012 034 078 135 236 457 468	
16	7	9	012 034 135 178 236 457 468	
17	7	9	012 034 067 135 168 245 278	
18	7	9	012 034 067 135 168 245 378	
19	7	9	012 034 067 135 168 245 478	
20	7	9	012 034 078 135 168 246 257	
21	7	9	012 034 135 168 246 257 378	
22	7	10	012 034 067 135 268 479 589	
23	7	10	012 034 067 135 268 489 579	
24	7	10	012 034 135 236 478 579 689	
25	7	10	012 034 078 135 246 579 689	
26	7	10	012 034 135 178 246 579 689	

In Table 3.3 we denote by $A(n,m)$ the number of n-line, m-point tradeable configurations. Let $L(n,m)$ denote the number of *labelled* (2-way) trades of volume n and foundation m. In determining $L(n,m)$ we take each tradeable configuration C of n lines and m points, assign a fixed set of labels to the points of C and count every possible (2-way) trade between C and labelled configurations D. These are precisely the trades that are generated by Algorithm 2.1 and for given n and m they are not necessarily pairwise non-isomorphic.

Table 3.3. Counts of tradeable configurations and trade sets

Lines	Points			Non-isomorphic trades			
n	m	$A(n,m)$	$L(n,m)$	2-way	3-way	4-way	
4	6	1	1	1			Pasch
6	7	1	2	1	1		semihead
6	8	1	1	1			6-cycle
7	7	1	8	1			STS(7)
7	9	1	1	1			
8	8	1	3	1	1	1	(1)
8	9	3	3	3			
8	10	4	4	4			
8	11	1	1	1			(2)
8	12	1	1	1			(3)
9	9	7	11	7	3		
9	10	7	8	5			
9	11	3	3	2			
10	9	3	12	3	1		
10	10	37	51	29			
10	11	39	43	34			
10	12	19	21	18			
10	13	3	4	3			
10	14	1	1	1			(4)
		134	179	117	6	1	

(1) point-deleted STS(9)
(2) two Pasch configurations with a common point
(3) two disjoint Pasch configurations
(4) disjoint Pasch configuration and 6-cycle

In the Appendix, we present a table of the 35 pairwise non-isomorphic trade sets of volume up to and including 9 (Table 3.4). The trades are arranged by volume and then by foundation. The points are labelled with non-negative integers; set brackets and commas have been omitted and labels 10 and 11 are represented by lower case Roman letters a and b, respectively. Each non-isomorphic tradeable configuration is assigned a unique number in this table, thus making it easy to distinguish between trades both of whose configurations are isomorphic and trades where the configurations are non-isomorphic.

The 35 trade sets consist of 29 2-way trade sets, five 3-way trade sets and one 4-way trade set. There is a small amount of duplication; for $k > 2$ the pairwise nonisomorphic $(k-1)$-way subsets of a k-way trade set appear as separate entries in the list. But this seems the clearest way to present the results. Thus the (2-way) trades numbered 2, 22, 24 and 27 in the list are subtrades of the 3-way trade sets numbered 3, 23, 25 and 28, respectively, and the (2-way) trade number 7 is a subtrade of the 3-way trade number 8 which in turn is a subtrade of the 4-way trade number 9. In Table 3.4 the first configuration in each trade set has a canonical labelling with the blocks presented in lexicographical order.

There are 89 trade sets of volume 10, all but one of which are 2-way. Of these, 72 trade sets are between isomorphic configurations and 16 between non-isomorphic configurations. But the 3-way trade set is of particular interest. The three tradeable configurations are:

(i)	012 034 057 068 135 146 178 236 247 258
(ii)	013 026 047 058 124 157 168 235 278 346
(iii)	018 027 035 046 125 136 147 234 268 578

Configurations (ii) and (iii) are isomorphic but are not isomorphic to configuration (i). The 3-way trade set gives rise to two 2-way trade sets, one between isomorphic tradeable configurations ((ii) and (iii)) and one between non-isomorphic tradeable configurations ((i) and (ii)).

Another interesting situation occurs with the three following non-isomorphic tradeable configurations.

(iv)	012 034 057 068 135 146 236 245 569 789
(v)	013 025 046 078 126 145 234 356 579 689
(vi)	014 023 056 078 125 136 246 345 579 689

There are 2-way trade sets between configurations (iv) and (v) and between (iv) and (vi) but not between configurations (v) and (vi). Hence the 2-way trade sets do not extend to a 3-way trade set. A full listing of the trade sets of volume 10 is available from the authors, and will appear in the first author's Ph.D. thesis.

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Table 3.4. Pairwise non-isomorphic trade sets				
	Lines	Points	Config.	
1	4	6	1	012 034 135 245
			1	013 024 125 345
2	6	7	2	012 034 135 146 236 245
			2	013 024 126 145 235 346
3	6	7	2	012 034 135 146 236 245
			2	013 024 126 145 235 346
			2	014 023 125 136 246 345
4	6	8	3	012 034 135 246 257 367
			3	013 024 125 267 346 357
5	7	7	4	012 034 056 135 146 236 245
			4	013 025 046 126 145 234 356
6	7	9	5	012 034 067 135 168 245 378
			5	016 024 037 125 138 345 678
7	8	8	6	012 034 067 135 147 236 257 456
			6	013 026 047 127 145 235 346 567
8	8	8	6	012 034 067 135 147 236 257 456
			6	013 026 047 127 145 235 346 567
			6	014 027 036 123 157 256 345 467
9	8	8	6	012 034 067 135 147 236 257 456
			6	013 026 047 127 145 235 346 567
			6	014 027 036 123 157 256 345 467
			6	017 023 046 125 134 267 356 457
10	8	9	7	012 034 057 135 146 236 278 568
			7	014 027 035 123 156 268 346 578
11	8	9	8	012 034 135 146 178 236 247 258
			8	013 024 126 147 158 235 278 346
12	8	9	9	012 034 135 147 236 258 378 468
			9	014 023 125 137 268 346 358 478
13	8	10	10	012 034 135 146 178 236 379 589
			10	014 023 126 137 158 346 359 789
14	8	10	11	012 034 067 089 135 245 568 579
			11	013 024 068 079 125 345 567 589
15	8	10	12	012 034 135 246 257 289 368 379
			12	013 024 125 268 279 346 357 389
16	8	10	13	012 034 135 246 257 368 589 679
			13	013 024 125 267 346 358 579 689
17	8	11	14	012 034 067 089 135 245 68a 79a
			14	013 024 068 079 125 345 67a 89a
18	8	12	15	012 034 135 245 678 69a 79b 8ab
			15	013 024 125 345 679 68a 78b 9ab

	Lines	Points	Config.	
19	9	9	16	012 034 057 068 135 146 178 236 245
			16	013 026 045 078 124 157 168 235 346
20	9	9	17	012 034 057 135 146 236 247 258 378
			17	014 027 035 125 136 238 246 347 578
21	9	9	18	012 034 057 135 146 236 247 258 678
			18	014 027 035 125 136 234 268 467 578
22	9	9	19	012 034 058 135 147 236 278 468 567
			19	014 028 035 123 157 267 346 478 568
23	9	9	19	012 034 058 135 147 236 278 468 567
			19	014 028 035 123 157 267 346 478 568
			19	015 023 048 127 134 268 356 467 578
24	9	9	20	012 034 058 135 147 236 248 257 456
			20	014 028 035 123 157 247 256 346 458
25	9	9	20	012 034 058 135 147 236 248 257 456
			20	014 028 035 123 157 247 256 346 458
			20	015 023 048 127 134 246 258 356 457
26	9	9	21	012 034 078 135 147 236 258 468 567
			21	017 023 048 125 134 268 356 467 578
27	9	9	22	012 034 078 135 168 246 257 367 458
			22	013 027 048 126 158 245 346 357 678
28	9	9	22	012 034 078 135 168 246 257 367 458
			22	013 027 048 126 158 245 346 357 678
			22	018 024 037 125 136 267 345 468 578
29	9	10	23	012 034 057 135 146 178 236 279 389
			24	014 027 035 126 138 157 239 346 789
30	9	10	25	012 034 057 135 146 178 236 279 689
			25	014 027 035 123 157 168 269 346 789
31	9	10	26	012 034 078 135 146 179 236 245 389
			27	017 024 038 126 139 145 235 346 789
32	9	10	28	012 034 058 069 135 147 189 236 379
			28	014 026 035 089 123 158 179 347 369
33	9	10	29	012 034 058 135 147 236 289 469 579
			29	014 028 035 123 157 269 346 479 589
34	9	11	30	012 034 067 135 168 245 379 38a 69a
			31	016 024 037 125 138 345 39a 679 68a
35	9	11	32	012 034 135 246 257 39a 489 58a 678
			32	013 024 125 267 349 35a 468 578 89a