On the small covering numbers $g_1^{(5)}(v)$

A.D. Forbes, M.J. Grannell, T.S. Griggs
Department of Mathematics and Statistics
The Open University
Walton Hall, Milton Keynes, MK7 6AA
United Kingdon
{m.j.grannell, t.s.griggs}@open.ac.uk
anthony.d.forbes@gmail.com

R.G. Stanton
Department of Computer Science
University of Manitoba
Winnipeg, MB, Canada R3T 2N2
stanton@cc.umanitoba.ca

Abstract

The minimum number of blocks having maximum size precisely five that is required to cover, exactly once, all pairs of elements from a set of cardinality v is denoted by $g_1^{(5)}(v)$. As a prelude to further work, we give the values of $g_1^{(5)}(v)$ for $v \le 25$, and construct all designs which attain this bound.

1 Introduction

The covering number $g_\lambda^{(k)}(v)$ is defined as the cardinality of the minimum pairwise balanced design (PBD) on a set of v points that has largest block size k and such that every pair occurs exactly λ times in the design. The values of $g_\lambda^{(k)}(v)$ for $k = 4$ and all λ have been determined, in a series of papers [4] to [17].

In the present paper we consider the case $k = 5$, $\lambda = 1$. A lower bound on $g_1^{(5)}(v)$ was given in [1] as

$$\frac{1}{20}\left\{6g_2 + 2g_3 + v(8\left\lceil\frac{v-1}{4}\right\rceil - (v-1))\right\},$$

where g_2 and g_3 are the number of blocks of lengths 2 and 3 respectively in the exact covering. Various applications were also given. However, this bound provides no useful information for small values of v and so we here provide a table of the values of $g_1^{(5)}(v)$ for $v \leq 25$. The values for 18, 19, 20, 21, 22, 23, 24, 25, are already given in [1]. But in the present paper we also determine all designs which attain the lower bound $g_1^{(5)}(v)$ for $v \leq 25$. Of course $g_1^{(5)}(v+1) \geq g_1^{(5)}(v)$, since deletion of a point not in all quintuples converts a design on $v+1$ points into a design on v points.

The values immediately following $v = 25$ will be the most difficult to determine, just as those following $v = 16$ (namely 17, 18, 19) were the most difficult in the case $k = 4$.

2 The cases $v = 5$ through 9

Trivially, $g_1^{(5)}(5) = 1$. For $v = 6, 7, 8$, there can be only one quintuple. Call it ABCDE and use numbers to denote the additional elements.

For $v = 6$, we need ABCDE with A1, B1, C1, D1, E1. So $g_1^{(5)}(6) = 6$. Similarly, for $v = 7$, we need A12, B1, B2, C1, C2, D1, D2, E1, E2, and thus $g_1^{(5)}(7) = 10$. Finally, for $v = 8$, we have a choice of A123, B1, B2, B3, ..., E1, E2, E3; or A12, A3, B13, B2, C23, C1, D1, D2, D3, E1, E2, E3. The first possibility requires 14 blocks, whereas the second requires only 13 blocks. Hence $g_1^{(5)}(8) = 13$.

For $v = 9$, there can be two quintuples ABCDE and A1234. However, this would require 18 blocks (these 2, plus 16 blocks containing pairs B1, B2, B3, B4, ..., E1, E2, E3, E4). We can do better than this by using a single quintuple ABCDE.

If we use ABCDE, we might take 1234 but this will require 22 blocks (these 2, plus 20 blocks containing pairs A1, A2, ..., E3, E4). Alternatively we might take ABCDE with A123, A4 along with B14, B2, B3, C24, C1, C3, D34, D1, D2, E1, E2, E3, E4, which requires 16 blocks.

But if we do not use any quadruple, we can obtain three non-isomorphic designs each having 15 blocks

(i) ABCDE, A12, A34, B13, B24, C14, C23, D1, D2, D3, D4, E1, E2, E3, E4;

(ii) ABCDE, A12, A34, B13, B24, C14, C2, C3, D23, D1, D4, E1,

E2, E3, E4;
(iii) ABCDE, A12, A34, B13, B2, B4, C24, C1, C3, D14, D2, D3, E23, E1, E4.
The only other possibility using a triple is ABCDE with 123 which would require 19 blocks (these 2, plus 17 further blocks). A design using the quintuple ABCDE and only pairs would require 27 blocks. Hence $g_1^{(5)}(9) = 15$.

3 The cases $v = 10$ and 11

We first consider the case $v = 11$. If we take a block ABCDE and attach the five one-factors of a one-factorization of $\{1, 2, 3, 4, 5, 6\}$ to the five letters A, B, C, D, E, we obtain a covering with 16 blocks.

Using two quintuples would require a minimum of 25 other blocks if the quintuples were disjoint. If the quintuples were ABCDE, A1234, there would have to be at least $3 + 4(4) = 19$ blocks. So the minimum number of blocks certainly occurs when there is only one quintuple.

Suppose that there are a_j sub-blocks of length j $(j = 1, 2, 3)$ attached to the 5 letters A, B, C, D, E. Then by counting pairs

$$\begin{aligned} 3a_3 + 2a_2 + a_1 &= 6(5) = 30, \\ 3a_3 + a_2 &= 15 - \epsilon, \end{aligned}$$

where ϵ denotes the number of pairs not contained in the sub-blocks attached to A, B, C, D, E (a priori, $\epsilon > 0$ is unlikely).

Then $a_1 + a_2 = 15 + \epsilon$, and so $\epsilon = 0$ (otherwise there would be more than 16 blocks). But then for the same reason we must have $a_3 = 0$. However the equations then give $a_2 = 15$ and $a_1 = 0$. In short, the design created using the one-factorization of K_6 is the unique minimal design and $g_1^{(5)}(11) = 16$.

For $v = 10$, we can produce a design in 16 blocks by deleting element 6 from the design on 11 elements. We show that no better design is possible.

As for $v = 11$, it is clear that two disjoint quintuples require at least $2 + 5(5) = 27$ blocks. Two intersecting quintuples require at least $3 + 4(4) = 19$ blocks. So we must consider a single quintuple ABCDE. Using the same notation as before for the sub-blocks attached to A, B, C, D, E, we have

$$\begin{aligned} 3a_3 + 2a_2 + a_1 &= 5(5) = 25, \\ 3a_3 + a_2 &= 10 - \epsilon. \end{aligned}$$

Then $a_1 + a_2 = 15 + \epsilon$. Hence $\epsilon = 0$ and $a_3 = 0$, since otherwise there would be more than 16 blocks. It follows that $a_2 = 10$ and $a_1 = 5$ and we have the usual one-factorization for K_5 (obtained as 5 sets, each comprising 2 pairs and a singleton) that results from deleting an element from K_6; hence $g_1^{(5)}(10) = g_1^{(5)}(11) = 16$.

4 The case $v = 12$

We shall prove that $g_1^{(5)}(12) = 18$. Observe first that the presence of two disjoint quintuples requires at least $2 + 5(5) = 27$ blocks. If there are two intersecting quintuples, the design requires at least $3 + 4(4) = 19$ blocks. If there is a single quintuple, we use the previous notation and write

$$\begin{aligned} 3a_3 + 2a_2 + a_1 &= 35, \\ 3a_3 + a_2 &= 21 - \epsilon. \end{aligned}$$

Then $a_1 + a_2 = 14 + \epsilon$, $3a_3 - a_1 = 7 - 2\epsilon$. So

$$a_1 + a_2 + a_3 = \frac{49 + a_1 + \epsilon}{3},$$

and we see that the design must contain at least $1 + 17 = 18$ blocks. Furthermore, if $\epsilon > 0$ and $a_1 + a_2 + a_3 = 17$ then the design has at least 19 blocks. Hence any solution in 18 blocks requires $\epsilon = 0$, $a_1 = 2$, $a_2 = 12$ and $a_3 = 3$.

In such a case, each of A, B, C, D, E appears in blocks with the other seven points $1, 2, \ldots, 7$. Hence without loss of generality A, B, and C must each appear in one quadruple and two triples while D and E must each appear in one pair and three triples.

For each $x \in \{1, 2, \ldots, 7\}$ let $r_i(x)$ denote the number of occurrences of x in blocks of size i. Since each x appears with A, B, C, D, E, $r_2(x) + r_3(x) + r_4(x) = 5$. Since each x appears with six other points from $1, 2, \ldots, 7$, $r_3(x) + 2r_4(x) = 6$. Thus $(r_2(x), r_3(x), r_4(x)) =$

$(0,4,1),(1,2,2)$ or $(2,0,3)$. Suppose there are α,β,γ points of each of the types respectively. Then $\alpha+\beta+\gamma = 7$ and $\beta+2\gamma = 2$, the latter since $a_1 = 2$. This gives $(\alpha,\beta,\gamma) = (5,2,0)$ or $(6,0,1)$. We now show that the former of these does not yield a design in 18 blocks but the latter does.

In the former case, take the two points of type $(1,2,2)$ to be 1 and 2. Without loss of generality this gives the design shown in Table 1,

Table 1

ABCDE	B145	C267	D2*	E1*
A123	B2*	C1*	D**	E**
A**	B**	C**	D**	E**
A**			D1	E2

where the asterisks indicate undetermined entries. It is easy to see that the pairs B6 and B7 cannot both be included in this design.

Finally we show in Table 2 how to construct a design in 18 blocks having one point of type $(2,0,3)$, say 1, and six points of type $(0,4,1)$.

Table 2

ABCDE	B1bb	C1cc	Ddd	Eee
A1aa	Bbb	Ccc	Ddd	Eee
Aaa	Bbb	Ccc	Ddd	Eee
Aaa			D1	E1

The lower case letters a, b, c, d, e in Table 2 are used as place holders for the five one-factors of the one-factorisation of K_6 on the set $\{2,3,4,5,6,7\}$. Moreover, the quadruples, i.e. the blocks containing the pairs A1, B1 and C1, must also contain a one-factor on $\{2,3,4,5,6,7\}$. It is not difficult to complete Table 2 to obtain, to within isomorphism, the unique minimal design shown in Table 3.

Table 3

ABCDE	B145	C167	D25	E27
A123	B26	C24	D36	E34
A46	B37	C35	D47	E56
A57			D1	E1

This establishes that $g_1^{(5)}(12) = 18$.

5 The case $v = 13$

If $v = 13$, we provide a design in 19 blocks and then show that it is minimal.

We can take three quintuples ABCDE, A1234, A5678. We adjoin 16 triples B15, B26, B37, B48, C16, C27, C38, C45, D17, D28, D35, D46, E18, E25, E36, E47. Since all 78 pairs are covered by these 3 quintuples and 16 triples, we have $g_1^{(5)}(13) \leq 19$.

As before, it is clear that any quadruple or quintuple must intersect ABCDE. Using the same notation as in the last section, we have

$$\begin{aligned} 4a_4 + 3a_3 + 2a_2 + a_1 &= 40, \\ 6a_4 + 3a_3 + a_2 &= 28 - \epsilon. \end{aligned}$$

If $a_4 = 0$, then $a_1 + a_2 = 12 + \epsilon$ and $3a_3 - a_1 = 16 - 2\epsilon$. Thus $a_1 + a_2 + a_3 + a_4 = (52 + a_1 + \epsilon)/3$. But if $\epsilon > 0$ then the design has at least $1 + 18 + 1 = 20$ blocks. Hence $\epsilon = 0$, $a_1 = 2$, $a_2 = 10$ and $a_3 = 6$. There are three possibilities for the design. The first of these is shown in Table 4.

Table 4

ABCDE	B147	C***	D**	E**
A123	B258	C***	D**	E**
A456	B36	C**	D**	E**
A78			D*	E**
			D*	

where the numeric entries are without loss of generality and the asterisks indicate undetermined entries. There are now two alternatives for the blocks containing C; namely C168, C357, C24 and C267, C348, C15 but the permutation (12)(45)(78) is an isomorphism between the two partial designs thus constructed. But now the blocks containing E cannot be completed.

The second possibility is shown in Table 5, using the same notation as above.

Table 5

ABCDE	B147	C168	D**	E**
A123	B258	C357	D**	E**
A456	B36	C2	D**	E**
A78		C4	D**	E**

It is easy to see that the pairs 24, 26 and 27 cannot all be included in this design.

The third possibility is shown in Table 6, again using the previous notation.

Table 6

ABCDE	B147	C***	D***	E**
A123	B258	C*	D*	E**
A456	B36	C**	D**	E**
A78		C**	D**	E**

Without loss of generality, there are now two alternatives for the remaining two quadruples; namely C168, D357 and C267, D348 but again the permutation (12)(45)(78) is an isomorphism between the two partial designs thus constructed. The design is now uniquely completable as shown in Table 7.

Table 7

ABCDE	B147	C168	D357	E15
A123	B258	C5	D1	E38
A456	B36	C27	D26	E67
A78		C34	D48	E24

If $a_4 = 1$, then $a_1 + a_2 = 14 + \epsilon$ and $3a_3 - a_1 = 8 - 2\epsilon$. Thus $a_1 + a_2 + a_3 + a_4 = (53 + a_1 + \epsilon)/3$. But if $\epsilon > 0$ then the design has at least $1 + 18 + 1 = 20$ blocks. Hence $\epsilon = 0,\ a_1 = 1,\ a_2 = 13$ and $a_3 = 3$. Without loss of generality, there are two possibilities for the blocks containing A. The first of these is ABCDE, A1234, A567 and A8. The two further quadruples can then be taken to be

B158 and C268. But it is not now possible to complete the design. The second possibility is ABCDE, A1234, A56 and A78. Two of the three quadruples must contain the same letter, say B157 and B268. But we must then have B34; a contradiction.

Finally, noting that we cannot have $a_4 \geq 3$, if $a_4 = 2$ then $a_1 + a_2 = 16+\epsilon$ and $3a_3 - a_1 = -2\epsilon$. Thus $a_1+a_2+a_3+a_4 = (54+a_1+\epsilon)/3$. But if $\epsilon > 0$ then the design has at least $1 + 19 + 1 = 21$ blocks. Hence $\epsilon = 0,\ a_1 = 0,\ a_2 = 16$ and $a_3 = 0$. One solution of this form is the design given at the beginning of this section. However there is a second non-isomorphic solution ABCDE, A1234, A5678, B15, B26, B37, B48, C16, C25, C38, C47, D17, D28, D35, D46, E18, E27, E36, E45 obtained by using the Latin square based on the Cayley table of the Klein group K_4 rather than the cyclic group C_4. Both of these solutions are on the assumption that the three quintuples intersect in a common point. There are two other possibilities where the quintuples are either ABCDE, A1234, B5678 or ABCDE, A1234, B1567 but it is easy to see that neither of these can be completed to a design using only triples.

Thus there are three non-isomorphic designs which attain the bound $g_1^{(5)}(13) = 19$.

6 The cases $v = 14$ through 17

We first consider the case $v = 17$. If we generate PG(2, 4) by developing the initial block $\{0, 1, 4, 14, 16\}$ modulo 21, and then delete the five points of this block, we can write down the affine geometry on 16 points in the form

A: 2,5,15,17	2,3,6,18	5,8,18,20
A: 6,7,10,20	2,7,8,11	5,3,10,11
A: 3,8,9,12	2,9,10,13	5,7,12,13
A: 11,13,18,19	2,12,19,20	5,6,9,19

15,6,11,12	17,3,7,19
15,7,9,18	17,6,8,13
15,3,13,20	17,9,11,20
15,8,10,19	17,10,12,18

A design on 17 points in 20 blocks is obtained by appending a point A to the first four blocks in this array. Hence $g_1^{(5)}(17) \leq 20$. We show that equality holds.

As in earlier cases, there cannot be two disjoint quintuples since this would require at least $2 + 5(5) = 27$ blocks. Two intersecting quintuples would require at least $4+4(4) = 20$ blocks and this could only occur if the intersection point, call it A, was the intersection of four quintuples; these four quintuples would cover 40 pairs and the remaining 96 pairs would then have to be covered in 16 quadruples. Since deletion of A would produce the unique affine geometry on 16 points, we see that this configuration is uniquely minimal among all configurations containing more than one quintuple.

If there is a single quintuple ABCDE, then with the usual notation

$$\begin{aligned} 3a_3 + 2a_2 + a_1 &= 60, \\ 3a_3 + a_2 &= 66 - \epsilon. \end{aligned}$$

Then $a_1 + a_2 = \epsilon - 6$ and $3a_3 - a_1 = 72 - 2\epsilon$. It follows that

$$a_1 + a_2 + a_3 = \frac{54 + a_1 + \epsilon}{3}.$$

Now $\epsilon \geq 6$ and so $a_1 + a_2 + a_3 \geq 20$. Since ABCDE still needs to be counted, this case leads to a design with more that 20 blocks. Thus, we have proved that $g_1^{(5)}(17) = 20$, and the 20 block design is unique.

As an aside, we note that if we take the design on 17 points specified above and delete elements 2, 5, 15, 17 we obtain a design on 13 points comprising the quintuples $\{A, 6, 7, 10, 20\}$, $\{A, 3, 8, 9, 12\}$, $\{A, 11, 13, 18, 19\}$ and 16 triples which we previously obtained as one of the solutions for $g_1^{(5)}(13) = 19$; in fact the one based on the Cayley table of the Klein group K_4.

Now take the $g_1^{(5)}(17)$ design and reduce it to a design on 14 points by deleting elements 3, 6, 18. The resulting design contains one quintuple, 9 quadruples, and 9 triples, and so contains 19 blocks; hence $g_1^{(5)}(14) \leq 19$. But $g_1^{(5)}(14) \geq g_1^{(5)}(13) = 19$. So we have established that $g_1^{(5)}(14) = 19$.

If $v = 15$, and noting that $g_1^{(5)}(15) \leq g_1^{(5)}(17) = 20$, we cannot have two disjoint quintuples, and two intersecting quintuples require at least $4+4(4)=20$ blocks. If there is only one quintuple, we find, using our previous notation, that

$$\begin{aligned} 3a_3+2a_2+a_1 &= 50, \\ 3a_3+a_2 &= 45-\epsilon. \end{aligned}$$

Then $a_1+a_2=5+\epsilon$, and $3a_3-a_1=40-2\epsilon$. Hence

$$a_1+a_2+a_3=\frac{55+a_1+\epsilon}{3}.$$

This number is at least 19 and so the design must contain at least 20 blocks. Thus, whether we allow one or more quintuples, at least 20 blocks are required. So $g_1^{(5)}(15)=20$, and a solution is provided by deleting two points from the solution for $g_1^{(5)}(17)$.

Since $g_1^{(5)}(17)=g_1^{(5)}(15)=20$, it follows that $g_1^{(5)}(16)=20$, and a solution can be found by deleting one point from a solution for $g_1^{(5)}(17)$. We now show that the designs on 14 and 16 points are unique but there exist two non-isomorphic solutions for $v=15$.

If $v=14$, we cannot have two disjoint quintuples, and two intersecting quintuples require at least $4+4(4)=20$ blocks. So there is a single quintuple ABCDE, and with the usual notation,

$$\begin{aligned} a_3+a_2+a_1 &= 18-\delta, \\ 3a_3+2a_2+a_1 &= 45, \\ 3a_3+a_2 &= 36-\epsilon, \end{aligned}$$

where δ is the number of blocks disjoint from ABCDE and ϵ is the number of pairs in these δ blocks. Solving yields

$$a_1=-\epsilon-3\delta, \quad a_2=9+2\epsilon+3\delta, \quad a_3=9-\epsilon-\delta.$$

Therefore $a_1=\epsilon=\delta=0$ and $a_2=a_3=9$.

The design must therefore be as shown in Table 8,

Table 8

ABCDE	B147	C***	D***	E***
A123	B258	C**	D**	E**
A456	B369	C**	D**	E**
A789		C**	D**	E**

where the asterisks indicate undetermined entries. The triples to be assigned to C, D and E must be three of 159, 267, 348, 168, 249 and 357 and further must form a parallel class, for otherwise the design cannot be completed. Without loss of generality we choose C159, D267 and E348. The design is now uniquely completable as shown in Table 9.

Table 9

ABCDE	B147	C159	D267	E348
A123	B258	C68	D18	E16
A456	B369	C24	D49	E29
A789		C37	D35	E57

If $v = 15$, we have already noted that we cannot have two disjoint quintuples. Let ABCDE be a quintuple, then with the usual notation,

$$\begin{aligned} a_4 + a_3 + a_2 + a_1 &= 19 - \delta, \\ 4a_4 + 3a_3 + 2a_2 + a_1 &= 50, \\ 6a_4 + 3a_3 + a_2 &= 45 - \epsilon, \end{aligned}$$

where δ is the number of blocks disjoint from ABCDE and ϵ is the number of pairs in these δ blocks. Solving yields

$$a_1 = 2 - a_4 - \epsilon - 3\delta, \quad a_2 = 3 + 3a_4 + 2\epsilon + 3\delta, \quad a_3 = 14 - 3a_4 - \epsilon - \delta.$$

Since $\delta > 0$ implies $a_1 < 0$, and $\delta = 0$ implies $\epsilon = 0$, the solution reduces to

$$a_1 = 2 - a_4, \quad a_2 = 3 + 3a_4, \quad a_3 = 14 - 3a_4, \quad 0 \le a_4 \le 2.$$

Suppose $a_4 = 0$. Then $a_1 = 2$, $a_2 = 3$ and $a_3 = 14$. For each $x \in \{1, 2, \ldots, 10\}$, let $r_i(x)$ denote the number of occurrences of x in blocks of size i. Then

$$\begin{aligned} r_2(x) + r_3(x) + r_4(x) &= 5, \\ r_3(x) + 2r_4(x) &= 9, \\ r_2(x) \le 2, \quad r_3(x) &\le 3. \end{aligned}$$

But the only solution is $(r_2(x), r_3(x), r_4(x)) = (0, 1, 4)$; consequently the design does not exist.

Suppose $a_4 = 1$. Then $a_1 = 1$, $a_2 = 6$ and $a_3 = 11$. With $r_i(x)$ as before, we have

$$\begin{aligned} r_2(x) + r_3(x) + r_4(x) + r_5(x) &= 5, \\ r_3(x) + 2r_4(x) + 3r_5(x) &= 9, \\ r_2(x) \le 1, \quad r_5(x) &\le 1, \end{aligned}$$

which for α, β, γ has admissible solutions $(r_2(x), r_3(x), r_4(x), r_5(x)) = (0, 1, 4, 0)$ for α values of x, $(0, 2, 2, 1)$ for β values of x and $(1, 0, 3, 1)$ for γ values of x. Since $a_1 = a_4 = 1$, we have $\gamma = 1$, $\beta = 3$, and hence $\alpha = 10 - \beta - \gamma = 6$. Assuming that the pattern $(1, 0, 3, 1)$ applies to 1, pattern $(0, 2, 2, 1)$ applies to 2, 3 and 4, and denoting 10 by X, we can construct the design as follows.

Without loss of generality, we have blocks ABCDE, A1234, B156, C178, D19X and E1. The remaining blocks containing B or C or D must be one of size 4 and two of size 3. So the remaining blocks containing A must be two of size 4 and containing E must be three of size 4. The design must therefore be as shown in Table 10,

Table 10

ABCDE	B156	C178	D19X	E1
A1234	B***	C***	D***	E2**
A579	B**	C**	D**	E3**
A68X	B**	C**	D**	E4**

where the asterisks indicate undetermined entries. There are two alternatives for the missing pairs in the quadruples containing E; namely 58, 7X, 96 and 5X, 76, 98 but the permutation (56)(78)(9X) is an isomorphism between the two partial designs thus constructed, The design is now uniquely completable as shown in Table 11.

Table 11

ABCDE	B156	C178	D19X	E1
A1234	B389	C45X	D267	E258
A579	B2X	C29	D35	E37X
A68X	B47	C36	D48	E469

Suppose $a_4 = 2$. Then $a_1 = 0,\ a_2 = 9$ and $a_3 = 8$. With $r_i(x)$ as before, we have

$$\begin{aligned} r_3(x) + r_4(x) + r_5(x) &= 5, \\ r_3(x) + 2r_4(x) + 3r_5(x) &= 9, \\ r_5(x) &\leq 2, \end{aligned}$$

which has admissible solutions $(r_3(x), r_4(x), r_5(x)) = (1, 4, 0)$ for α values of x, $(2, 2, 1)$ for β values of x and $(3, 0, 2)$ for γ values of x.

Suppose $\gamma > 0$. Then, since $a_4 = 2$, it follows that $\gamma = 1$, i.e. there exists exactly one number, say 1, which occurs in two blocks of length 5, say A1234 and B1567. Also the pairs A5, A6, A7, B2, B3, B4, and $\{\mathrm{C}, \mathrm{D}, \mathrm{E}\} \times \{1, 2, 3, 4\}$ must occur in separate blocks. Together with ABCDE, there are now at least 21 blocks. Therefore we may assume that $\gamma = 0$, and hence $\beta = 8$ and $\alpha = 2$.

Since we cannot have two disjoint blocks of length 5, we may assume without loss of generality that the design contains blocks ABCDE, A1234, A5678 and A9X. The remainder of the design contains 8 blocks of size 4 and 8 blocks of size 3. Elementary counting further shows that each of B, C, D and E are contained in two blocks of each size. Therefore without loss of generality we can further assume that the design contains blocks B159, B26X, B37 and B48. This leaves only the triples 17X, 18X, 279, 289, 35X, 369, 389, 38X, 45X, 469, 479, 47X which can be assigned to C, D and E. Since the point 1 must occur in one quadruple and two triples with C, D and E there are, again without loss of generality, eight possibilities for the quadruples containing C,

(i)	C17X,	C289,	(v)	C18X,	C279,
(ii)	C17X,	C369,	(vi)	C18X,	C469,
(iii)	C17X,	C389,	(vii)	C18X,	C479,
(iv)	C17X,	C469,	(viii)	C18X,	C369.

But the permutation (34)(78) is an isomorphism between the partial designs constructed by adjoining the respective possibilities (i) and (v), (ii) and (vi), (iii) and (vii), (iv) and (viii) to the chosen blocks containing A and B. So there are just four possibilities to consider.

Possibility (i). This has a unique completion as shown in Table 12.

Table 12

ABCDE	B159	C17X	D369	E38X
A1234	B26X	C289	D45X	E479
A5678	B37	C35	D18	E16
A9X	B48	C46	D27	E25

Possibility (ii). The partial design can be extended with blocks D45X, E479 and D289. But now the triple D3* cannot be completed.

Possibility (iii). The partial design can be extended with blocks D279, E35X and E469. But now the other quadruple containing D cannot be completed.

Possibility (iv). The partial design can be extended with blocks D389, E35X and E279. But again the other quadruple containing D cannot be completed.

If $v = 16$, we cannot have two disjoint quintuples. Let ABCDE be a quintuple, then with the usual notation,

$$\begin{aligned} a_4 + a_3 + a_2 + a_1 &= 19 - \delta, \\ 4a_4 + 3a_3 + 2a_2 + a_1 &= 55, \\ 6a_4 + 3a_3 + a_2 &= 55 - \epsilon, \end{aligned}$$

where δ is the number of blocks disjoint from ABCDE and ϵ is the number of pairs in these δ blocks. Solving yields

$$a_1 = 2 - a_4 - \epsilon - 3\delta, \quad a_2 = 3a_4 - 2 + 2\epsilon + 3\delta, \quad a_3 = 19 - 3a_4 - \epsilon - \delta.$$

Since $\delta > 0$ implies $a_1 < 0$, and $\delta = 0$ implies $\epsilon = 0$, the solution reduces to

$$a_1 = 2 - a_4, \quad a_2 = 3a_4 - 2, \quad a_3 = 19 - 3a_4, \quad 1 \le a_4 \le 2.$$

Suppose $a_4 = 1$. Then $a_1 = a_2 = 1$ and $a_3 = 16$. With $r_i(x)$ denoting the number of occurrences of x in blocks of size i, as before, we have

$$\begin{gathered} r_2(x) + r_3(x) + r_4(x) + r_5(x) = 5, \\ r_3(x) + 2r_4(x) + 3r_5(x) = 10, \\ r_2(x) \le 1, \quad r_3(x) \le 1, \quad r_5(x) \le 1, \end{gathered}$$

which for some $\alpha,\ \beta$ has admissible solutions $(r_2(x), r_3(x), r_4(x), r_5(x)) = (0,0,5,0)$ for α values of x and $(0,1,3,1)$ for β values of x. Hence $r_2(x) = 0$ for all x and the design does not exist.

Suppose $a_4 = 2$. Then $a_1 = 0,\ a_2 = 4$ and $a_3 = 13$. Again, we have

$$\begin{aligned} r_3(x) + r_4(x) + r_5(x) &= 5, \\ r_3(x) + 2r_4(x) + 3r_5(x) &= 10, \\ r_3(x) \leq 4, \quad r_5(x) &\leq 2, \end{aligned}$$

which for some $\alpha,\ \beta,\ \gamma$ has admissible solutions $(r_3(x), r_4(x), r_5(x)) = (0,5,0)$ for α values of x, $(1,3,1)$ for β values of x and $(2,1,2)$ for γ values of x. If $r_5(x) = 2$ for some x, we can assume that there exist blocks A1234 and B1567; hence the pairs $\{\mathrm{A}\} \times \{5,6,7\},\ \{\mathrm{B}\} \times \{2,3,4\}$ and $\{\mathrm{C},\mathrm{D},\mathrm{E}\} \times \{1,2,3,4\}$ must occur in distinct blocks, implying a total block count exceeding 20. Therefore $\gamma = 0,\ \beta = 8$ and $\alpha = 3$.

Denote 10 by X and 11 by Y. Elementary counting shows that one of A, B, C, D or E has two quadruples and one triple attached whilst the other four points have three triples and one pair attached. Also the points contained in the two attached quadruples are the same as the points contained in the four attached pairs. To construct the design therefore, without loss of generality we begin with blocks ABCDE, A1234, A5678, A9XY, B15, C26, D37, E48.

The design now has a unique completion to within isomorphism. We can add further blocks as shown in Table 13,

Table 13

ABCDE	B15	C26	D37	E48
A1234	B27*	C35*	D45*	E25*
A5678	B38*	C47*	D28*	E36*
A9XY	B46*	C18*	D16*	E17*

where the asterisks indicate undetermined entries. The points 9, X and Y can now be assigned to the incomplete blocks B27*, B38* and B46* in any order, say B279, B38X and B46Y whence the completion of the design is unique as shown in Table 14.

Table 14

ABCDE	B15	C26	D37	E48
A1234	B279	C35Y	D459	E25X
A5678	B38X	C47X	D28Y	E369
A9XY	B46Y	C189	D16X	E17Y

7 The cases $v = 18$ through 25

Both the projective plane of order 4, PG(2, 4), and the affine plane of order 5, AG(2, 5), are 2-transitive and unique up to isomorphism. Hence for each one, the designs produced by removing two points are pairwise isomorphic. The same is obviously true if only one point is removed. So by Theorem 3 of [1] the designs for $v = 19$, 20, 21, 23, 24 and 25 are unique up to isomorphism. It remains to deal with $v = 18$ and $v = 22$. The results which follow are based on the full automorphism groups of PG(2, 4) and AG(2, 5) obtained from **designtheory.org** [3, 2].

Consider the case $v = 18$. By [1, Theorem 3], the design is obtained from PG(2, 4) by removing three points. We construct PG(2, 4) by developing $\{0, 1, 4, 14, 16\}$ modulo 21. We then delete points 19 and 20 to obtain the 19-point, 21-block design D;

{0,1,4,14,16}, {0,2,7,8,11}, {1,2,5,15,17},
{1,3,8,9,12}, {2,3,6,16,18}, {2,4,9,10,13},
{3,5,10,11,14}, {4,6,11,12,15}, {5,7,12,13,16},
{6,8,13,14,17}, {7,9,14,15,18}, {0,10,12,17,18},
{0,3,13,15}, {3,4,7,17}, {0,5,6,9},
{4,5,8,18}, {1,6,7,10}, {8,10,15,16},
{1,11,13,18}, {9,11,16,17}, {2,12,14}.

Let σ be the subgroup of the full automorphism group of PG(2, 4) which stabilizes the points 19 and 20. Then σ is generated by the permutations

(0 5)(7 1)(8 15)(11 17)(4 13)(18 3)(14 12),
(2 12)(7 17)(8 18)(11 10)(1 16)(15 13)(9 6),
(0 5 9)(7 1 10)(8 17 13)(11 15 4)(16 3 18),
(0 15)(7 1)(8 5)(11 17)(4 18)(10 6)(9 16)(13 3),
(7 11)(1 17)(4 18)(10 16)(9 6)(13 3)(14 12),
(0 9 6)(7 13 16)(8 4 18)(11 10 3)(1 15 17),
(0 5)(7 17)(8 15)(11 1)(4 3)(10 16)(9 6)(13 18).

and is an automorphism group of D. Furthermore, under the action of σ the points of D fall into two orbits $T = \{2, 12, 14\}$ and $U = \{0, 1, 3, 4, \ldots, 11, 13, 15, 16, 17, 18\}$. Therefore, denoting by D_i the design obtained from D by removing point i, we see that the D_i fall into two isomorphism classes, $i \in T$ and $i \in U$.

Finally, in the case $v = 22$, the design is obtained from AG(2, 5) by removing three points [1, Theorem 3]. We construct AG(2, 5) as

$$\{\{(0,0),(0,1),(1,2),(1,4),(3,3)\} + (i,j) : 0 \le i \le 4,\ 0 \le j \le 4\} \quad \cup$$

$$\{\{(0,0),(1,0),(2,0),(3,0),(4,0)\} + (0,j) : 0 \le j \le 4\}$$

with arithmetic in $Z_5 \times Z_5$. Then points $(4,3)$ and $(4,4)$ are deleted to obtain the 23-point, 30-block design E, where a point (a,b) of AG(2, 5) is denoted by the integer $5a + b$;

{0,1,7,9,18},	{1,2,5,8,19},	{0,3,14,21,22},
{2,3,6,9,15},	{0,4,6,8,17},	{2,4,13,20,21},
{3,4,5,7,16},	{0,5,10,15,20},	{1,6,11,16,21},
{2,7,12,17,22},	{7,8,11,14,20},	{5,9,11,13,22},
{8,9,10,12,21},	{2,10,14,16,18},	{3,10,11,17,19},
{4,11,12,15,18},	{0,12,13,16,19},	{1,13,14,15,17},
{6,18,19,20,22},	{1,3,12,20},	{1,4,10,22},
{5,6,12,14},	{6,7,10,13},	{3,8,13,18},
{4,9,14,19},	{7,15,19,21},	{8,15,16,22},
{9,16,17,20},	{5,17,18,21},	{0,2,11}.

Let τ be the subgroup of the full automorphism group of AG(2, 5) which stabilizes the points $(4,3) = 23$ and $(4,4) = 24$. Then τ is generated by the permutations

(1 8)(19 5)(16 20)(18 4)(14 21)(10 13)(6 7)(9 17)(15 12)(3 22),
(1 5 8 19)(16 9 20 17)(18 15 4 12)(14 3 21 22)(10 6 13 7),
(1 18 9 7)(8 16 15 22)(19 10 3 17)(5 14 6 12)(21 4 13 20),
(1 19 8 5)(16 17 20 9)(18 12 4 15)(14 22 21 3)(10 7 13 6).

and is an automorphism group of E. Let E_i denote the design obtained from E by removing point i. Then for i in the triple of E, we have $E_0 \simeq E_2 \simeq E_{11}$ via isomorphisms generated by the permutations

$$(2\ 0)(4\ 3)(7\ 5)(9\ 8)(12\ 10)(14\ 13)(17\ 15)(19\ 18)(22\ 20): \quad E_0 \mapsto E_2,$$

$$(11\ 2)(4\ 16\ 5\ 3)(19\ 20\ 22\ 6)(12\ 10\ 21\ 8)(15\ 14\ 17\ 13): \quad E_0 \mapsto E_{11}.$$

On the other hand, the points of E which are not in the triple form a single orbit under the action of τ and therefore the corresponding E_i form a single isomorphism class.

8 Summary

We conclude with a table of the results. The values of $n_1^{(5)}(v)$ are the numbers of non-isomorphic designs which achieve the bound $g_1^{(5)}(v)$.

v	$g_1^{(5)}(v)$	$n_1^{(5)}(v)$	v	$g_1^{(5)}(v)$	$n_1^{(5)}(v)$
5	1	1	15	20	2
6	6	1	16	20	1
7	10	1	17	20	1
8	13	1	18	21	2
9	15	3	19	21	1
10	16	1	20	21	1
11	16	1	21	21	1
12	18	1	22	30	2
13	19	3	23	30	1
14	19	1	24	30	1
			25	30	1

References

[1] J.L. Allston and R.G. Stanton, A note on pair coverings with maximal block length five, *Utilitas Math.* **28** (1985), 211–217.

[2] R.A. Bailey, P.J. Cameron, P. Dobcsányi, J.P. Morgan, and L.H. Soicher, Designs on the Web, *Discrete Math.* **306** (2006), 3014–3027.

[3] P. Dobcsányi and L.H. Soicher, An on-line collection of t-designs, **http://designtheory.org/database/t-designs/** (2005).

[4] M.J. Grannell, T.S. Griggs, and R.G. Stanton, Minimal perfect bi-coverings of K_v with block sizes two, three, and four, *Ars Combin.* **71** (2004), 125–138.

[5] M.J. Grannell, T.S. Griggs, and R.G. Stanton, On λ-fold coverings with maximum block size four for $\lambda = 3$, 4, and 5, *J. Combin. Math. Combin. Comput.* **51** (2004), 137–158.

[6] M.J. Grannell, T.S. Griggs, and R.G. Stanton, On λ-fold coverings with maximum block size four for $\lambda \geq 6$, *Utilitas Math.* **66** (2004), 221–230.

[7] M.J. Grannell, T.S. Griggs, R.G. Stanton, and C.A. Whitehead, On the covering number $g_1^{(4)}(18)$, *Utilitas Math.* **68** (2005), 131–143.

[8] M. Grüttmüller, I.T. Roberts, and R.G. Stanton, An improved lower bound for $g_1^{(4)}(18)$, *J. Combin. Math. Combin. Comput.* **48** (2004), 25–31.

[9] M. Grüttmüller, I.T. Roberts, S. D'Arcy, and J. Egan, The minimum number of blocks in pairwise balanced designs with maximum block size 4: the final cases, *Australas. J. Combin.* **36** (2006), 303-313.

[10] I.T. Roberts, S. D'Arcy, J. Egan, and M. Grüttmüller, An improved bound on the cardinality of the minimal pairwise balanced design on 18 points with maximum block size 4, *J. Combin. Math. Combin. Comput.* **56** (2006), 203–221.

[11] E. Seah and D.R. Stinson, personal communication.

[12] R.G. Stanton, The exact covering of pairs on nineteen points with block sizes two, three, and four, *J. Combin. Math. Combin. Comput.* **4** (1988), 69–78.

[13] R.G. Stanton, An improved upper bound on $g^{(4)}(18)$, *Congr. Numer.* **142** (2000), 29–32.

[14] R.G. Stanton, A lower bound for $g^{(4)}(18)$, *Congr. Numer.* **146** (2000), 153–156.

[15] R.G. Stanton, An improved lower bound on $g^{(4)}(17)$, *Congr. Numer.* **171** (2004), 41–50.

[16] R.G. Stanton and D.R. Stinson, Perfect pair-coverings with block sizes two, three, and four, *J. Combin. Inform. System Sci.* **8** (1983), 21–25.

[17] R.G. Stanton and A.P. Street, Some achievable defect graphs for pair-packings on 17 points, *J. Combin. Math. Combin. Comput.* **1** (1987), 207–215.