Diagonally switchable 4-cycle systems

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Abstract

A diagonally switchable 4-cycle system of order n, briefly DS4CS(n), is a 4-cycle system in which by replacing each 4-cycle (a, b, c, d) covering pairs ab, bc, cd, da by either of the 4-cycles (a, c, b, d) or (a, b, d, c) another 4-cycle system is obtained. We prove that a DS4CS(n) exists if and only if $n \equiv 1 \pmod 8$, $n \geq 17$ with the possible exception of $n = 17$.
AMS classification: 05B30
Keywords: 4-cycle system, configurations, avoidance

1 Introduction

A 4-*cycle system of order* n, briefly 4CS(n), is a decomposition of the complete graph K_n into 4-cycles. Such a decomposition exists if and only if $n \equiv 1 \pmod 8$, [5]. In this paper we consider a class of 4-cycle systems having a particular property. In order to define this property first note that a 4-cycle (a, b, c, d) covers the pairs ab, bc, cd, da but <u>not</u> the diagonals ac, bd. Using the four points a, b, c, d two further 4-cycles, (a, c, b, d) and (a, b, d, c) may be constructed by replacing, respectively, each pair of non-adjacent edges of the original 4-cycle by the diagonals. We refer to such transformations as *diagonal switches.* If a 4-cycle is written (a, b, c, d) and we wish to replace it by the 4-cycle (a, c, b, d) we will refer to this as a type 1 transformation; if we wish to replace it by the 4-cycle (a, b, d, c) we will refer to this as a type 2 transformation. Of course, if we write the 4-cycle (a, b, c, d) as (a, d, c, b) and apply a type 1 transformation then we get $(a, c, d, b) = (a, b, d, c)$, so that the classification into types 1 and 2 is a purely lexicographical convenience. A set of 4-cycles, not necessarily a 4CS(n), is said to be *diagonally switchable* if each 4-cycle can be transformed by diagonal switches so that the set of transformed 4-cycles covers the same pairs as the original 4-cycles. For example, the set of 4-cycles $\{(A_0, A_1, A_2, B_3), (B_0, B_1, B_2, A_3), (A_0, A_2, A_3, B_1), (B_0, B_2, B_3, A_1), (A_0, A_3, A_1, B_2), (B_0, B_3, B_1, A_2)\}$ is diagonally switchable, by applying type 1 transformations to all six 4-cycles, to the set of 4-cycles $\{(A_0, A_2, A_1, B_3), (B_0, B_2, B_1, A_3), (A_0, A_3, A_2, B_1), (B_0, B_3, B_2, A_1), (A_0, A_1, A_3, B_2), (B_0, B_1, B_3, A_2)\}$. We determine, with one value left in doubt, the existence spectrum of 4-cycle systems of order n having the diagonally switchable property, and denote such systems by DS4CS(n). In particular we prove the following theorem.

Theorem 1.1 *There exists a* 4*-cycle system of order* n *having the diagonally switchable property if and only if* $n \equiv 1 \pmod 8$*,* $n \geq 17$*, with the possible exception of* $n = 17$.

Such systems are related to a problem considered in [3]. In that paper, three of the present authors studied configurations in 4-cycle systems. There are precisely four configurations of pairs of 4-cycles which can occur in a 4CS(n). These are as follows, with each letter representing a distinct point.

(1) (a, b, c, d) (w, x, y, z) (disjoint),

(2) (a, b, c, d) (a, x, y, z) (intersecting at a point),

(3) (a, b, c, d) (a, x, b, z) (intersecting in two points, with no common diagonal),

(4) (a, b, c, d) (a, x, c, y) (intersecting in two points, with a common diagonal).

All of the configurations (1), (2) and (3) occur in every 4CS(n) but configuration (4), the *double-diamond*, may be avoided. The following theorem was proved in [3].

Theorem 1.2 *A double-diamond-avoiding* 4*-cycle system of order* n *exists if and only if* $n \equiv 1 \pmod 8$*,* $n \geq 17$.

Since, in a 4CS(n) having the diagonally switchable property, all diagonals of the 4-cycles of the system appear as edges of 4-cycles of the transformed system, every DS4CS(n) is necessarily double-diamond avoiding. Hence, except for the value $n = 17$, this paper provides an alternative proof of Theorem 1.2, which provides some motivation for studying the systems. However the systems constructed in the proof of Theorem 1.1 have a much richer structure than those given in the earlier result.

It is also worth remarking that the pair of 4CS(n)s arising from a DS4CS(n) contain no common 4-cycles, but that if we consider the 4-cycles in each 4CS(n) as blocks of points, then the two systems contain the same blocks. We note that it is not possible to find three such 4CS(n)s (that is, a triple of 4CS(n)s in which no 4-cycle is repeated, but which contain the same blocks if we consider each 4-cycle as a block). To see this, consider a pair ab which does not occur as a diagonal in a 4CS(n). Since there are only two distinct 4-cycles on the points a, b, c, d that contain the edge ab, it follows that no triple of such systems exists.

2 Some lemmas

In order to prove Theorem 1.1 we first need some preliminary lemmas. The design theoretic terms and ideas used in this and later sections such as group divisible design, transversal design and equivalence with mutually orthogonal Latin squares, and Steiner system are standard and can be found in most books on Design Theory and in particular in [1].

Lemma 2.1 *The complete* 4*-partite graph* $K_{2,2,2,2}$ *can be decomposed into a set of* 4*-cycles that is diagonally switchable.*

Proof. Denote the four partitions of $K_{2,2,2,2}$ by $\{A_i, B_i\}$, $i = 0, 1, 2, 3$. The decomposition is then that given in the Introduction. □

Lemma 2.2 *The complete* 5*-partite graph* $K_{2,2,2,2,2}$ *can be decomposed into a set of* 4*-cycles that is diagonally switchable.*

Proof. Denote the five partitions of $K_{2,2,2,2,2}$ by $\{A_i, B_i\}$, $i = 0, 1, 2, 3, 4$. The decomposition into 4-cycles is $\{(A_i, A_{i+1}, B_{i-1}, B_{i+2}),$ $(A_i, A_{i+2}, B_{i-2}, B_{i-1}) : i = 0, 1, 2, 3, 4\}$ where all subscript arithmetic is modulo 5. The transformed set of 4-cycles, all by type 2 transformations, is $\{(A_i, A_{i+1}, B_{i+2}, B_{i-1}), (A_i, A_{i+2}, B_{i-1}, B_{i-2}) : i = 0, 1, 2, 3, 4\}$. □

Lemma 2.3 *The complete* 6*-partite graph* $K_{2,2,2,2,2,2}$ *can be decomposed into a set of* 4*-cycles that is diagonally switchable.*
Proof. The 15 4-cycles are:
(A_1, B_4, B_3, A_5), (A_1, B_2, B_4, A_3), (A_0, A_1, A_4, B_2),
(A_0, B_1, A_2, B_4), (A_0, A_2, A_1, B_5), (B_0, A_2, A_5, B_4),
(B_0, B_3, A_2, A_4), (B_1, B_2, B_3, A_4), (A_3, A_5, A_4, A_0),
(B_3, B_1, A_5, A_0), (A_1, B_3, B_5, B_0), (B_1, B_5, B_2, B_0),
(A_3, B_2, A_5, B_0), (A_3, B_5, B_4, B_1), (A_3, A_4, B_5, A_2).
The transformed set of 4-cycles is obtained by applying type 1 transformations to all of the above. □

Lemma 2.4 *The complete* 4*-partite graph* $K_{8,8,8,8}$ *can be decomposed into a set of* 4*-cycles that is diagonally switchable.*
Proof. Take a transversal design TD(4, 4), that is, a pair of mutually orthogonal Latin squares of order 4. Inflate each point by factor 2. Each block then becomes a copy of $K_{2,2,2,2}$ which can be decomposed as in Lemma 2.1. □

Lemma 2.5 *The complete* 5*-partite graph* $K_{8,8,8,8,8}$ *can be decomposed into a set of* 4*-cycles that is diagonally switchable.*
Proof. Take a transversal design TD(5, 4), that is, a set of three mutually orthogonal Latin squares of order 4. Inflate each point by factor 2. Each block then becomes a copy of $K_{2,2,2,2,2}$ which can be decomposed as in Lemma 2.2. □

Lemma 2.6 *The complete* 6*-partite graph* $K_{8,8,8,8,8,8}$ *can be decomposed into a set of* 4*-cycles that is diagonally switchable.*
Proof. Take a Steiner system S(2, 5, 25) and remove all blocks through a specified point to obtain a 5-GDD (group divisible design) of type 4^6. Inflate each point by factor 2. Each block then becomes a copy of $K_{2,2,2,2,2}$ which can be decomposed as in Lemma 2.2. □

3 Some systems constructed by computer

For technical reasons, in particular as outlined in the remarks following Theorem 4.2, we will construct DS4CS($8s + 1$), for all s : $3 \le s \le 46$. Those for $s = 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 17$ were constructed by computer and are given below.

s = 3, n = 25
Let the vertices of the complete graph K_{25} be $\{(x, y) : x = 0, 1, 2, 3, 4;\ y = 0, 1, 2, 3, 4\}$. The DS4CS(25) is then constructed under the group action $(x, y) \mapsto (x + 1, y) \pmod 5$, on the following set of 15 starter 4-cycles.

(1) $((0,0),(0,2),(3,1),(2,2))$, $((0,0),(3,2),(4,2),(3,3))$,
$((0,0),(4,2),(4,4),(2,4))$, $((0,1),(2,1),((3,3),(2,3))$,
$((0,1),(0,2),(2,4),(1,4))$, $((0,1),(0,3),(4,2),(4,3))$,
$((0,1),(3,3),(0,2),(3,4))$, $((0,1),(2,4),(3,2),(4,4))$.

(2) $((0,0),(1,0),(1,1),(1,4))$, $((0,0),(2,0),(0,1),(1,1))$,
$((0,0),(2,1),(0,2),(4,1))$, $((0,0),(1,2),(3,2),(0,3))$,
$((0,0),(1,3),(1,4),(4,3))$, $((0,0),(2,3),(4,3),(0,4))$,
$((0,0),(3,4),(0,3),(4,4))$.

To obtain the transformed 4-cycle system, apply a type 1 (respectively type 2) transformation to those 4-cycles obtained from a starter listed under 1 (respectively 2).

This convention will be used in describing the other 4-cycle systems in this section. All of these are defined on the set Z_n and are cyclic. They are obtained under the group action $i \mapsto i+1 \pmod n$ on the sets of starters listed.

s = 4, n = 33
(1) $(0,1,3,6)$, $(0,5,9,17)$.
(2) $(0,10,1,21)$, $(0,14,7,22)$.

s = 5, n = 41
(1) $(0,1,3,6)$, $(0,4,11,16)$, $(0,9,17,27)$.
(2) $(0,13,1,23)$, $(0,15,4,24)$.

s = 6, n = 49
(1) $(0,1,3,6)$, $(0,4,11,23)$, $(0,5,36,17)$.
(2) $(0,8,24,9)$, $(0,10,35,14)$, $(0,20,9,36)$.

s = 7, n = 57
(1) $(0,1,3,6)$, $(0,4,11,19)$, $(0,5,33,18)$, $(0,9,41,17)$.
(2) $(0,10,31,11)$, $(0,12,43,16)$, $(0,22,9,43)$.

s = 8, n = 65
(1) $(0,2,5,13)$, $(0,4,16,6)$, $(0,5,22,44)$, $(0,28,37,38)$.
(2) $(0,7,46,15)$, $(0,20,1,24)$, $(0,25,14,49)$, $(0,29,15,47)$.

s = 9, n = 73
(1) $(0,3,41,21)$, $(0,1,3,7)$, $(0,5,13,19)$, $(0,9,20,34)$,
$(0,10,57,15)$.
(2) $(0,17,4,48)$, $(0,22,10,46)$, $(0,23,64,24)$, $(0,28,12,55)$.

s = 10, n = 81

(1)	$(0,5,32,33)$,	$(0,8,24,13)$,	$(0,34,56,35)$,	$(0,30,4,41)$,
	$(0,3,34,9)$.			
(2)	$(0,2,44,32)$,	$(0,7,3,67)$,	$(0,10,63,18)$,	$(0,15,38,57)$,
	$(0,20,14,43)$.			

s = 11, n = 89

(1)	$(0,15,24,40)$,	$(0,1,3,6)$,	$(0,4,11,19)$,	$(0,5,17,27)$,
	$(0,11,29,43)$,	$(0,13,54,23)$.		
(2)	$(0,21,4,63)$,	$(0,28,63,29)$,	$(0,33,8,47)$,	$(0,36,16,67)$,
	$(0,37,13,57)$.			

s = 13, n = 105

(1)	$(0,6,34,57)$,	$(0,9,23,16)$,	$(0,8,5,17)$,	$(0,47,15,49)$,
	$(0,1,11,13)$,	$(0,4,22,46)$,	$(0,11,52,31)$.	
(2)	$(0,19,70,20)$,	$(0,25,30,70)$,	$(0,26,68,30)$,	$(0,27,72,33)$,
	$(0,29,81,37)$,	$(0,36,21,83)$.		

s = 14, n = 113

(1)	$(0,23,21,24)$,	$(0,6,55,28)$,	$(0,1,5,10)$,	$(0,7,15,26)$,
	$(0,9,23,36)$,	$(0,12,29,44)$,	$(0,16,76,46)$.	
(2)	$(0,18,40,88)$,	$(0,20,41,74)$,	$(0,31,101,38)$,	$(0,34,88,37)$,
	$(0,35,16,73)$,	$(0,42,13,81)$,	$(0,47,6,58)$.	

s = 17, n = 137

(1)	$(0,5,91,102)$,	$(0,3,20,28)$,	$(0,82,15,98)$,	$(0,2,24,31)$,
	$(0,1,5,14)$,	$(0,6,18,33)$,	$(0,16,42,63)$,	$(0,18,59,84)$,
	$(0,20,82,58)$,	$(0,23,73,44)$,	$(0,27,76,30)$,	$(0,32,80,43)$.
(2)	$(0,10,23,42)$,	$(0,34,100,40)$,	$(0,36,83,38)$,	$(0,52,128,59)$,
	$(0,56,129,57)$.			

4 Main constructions

In this section we give three constructions which collectively enable us to prove Theorem 1.1. The first of these is direct and allows us to deal with the case where $n \equiv 1 \pmod{24}$. In fact we only need it for six values of n but it is still a simple construction in its own right. We present all three constructions as theorems.

Theorem 4.1 *There exists a DS4CS(*$24t+1$*),* $t \geq 4$*.*
Proof. Take a 4-GDD of type 12^t. These exist for all $t \geq 4$, [2]. Now inflate each point by factor 2. Each block then becomes a copy of $K_{2,2,2,2}$ which can be decomposed as in Lemma 2.1. Further adjoin an extra point, say ∞, and on each inflated group of points of the 4-GDD together with the point ∞, place a copy of the DS4CS(25) given in Section 3. □

The next two constructions are recursive: the first of which is the main construction used to establish the result.

Theorem 4.2 *If there exist a set of three mutually orthogonal Latin squares of order n, a DS4CS($8n+1$) and a DS4CS($8m+1$) where $0 \leq m \leq n$, then there exists a DS4CS($8(4n+m)+1$).*

Proof. The set of three mutually orthogonal Latin squares of order n is equivalent to a transversal design TD(5, n). Remove $k = n - m$ points from one of the groups of the transversal design. Now inflate each remaining point by factor 8. Each block then becomes either a copy of $K_{8,8,8,8}$ (if it previously contained one of the removed points) or a copy of $K_{8,8,8,8,8}$, and these can be decomposed as in Lemmas 2.4 and 2.5. Further adjoin an extra point, say ∞, and on each inflated group of points of the transversal design together with the point ∞, place a copy of a DS4CS($8n$+1) or, in the case of the group which has had k points removed, a DS4CS($8m$+1). □

We should perhaps remark at this point that, since there is no DS4CS(9), the theorem is vacuous for $m = 1$. In addition, for $m = 2$ we have been unable to determine the existence or otherwise of a DS4CS(17). However we will never apply the theorem in either of these cases. The values of n for which the theorem is known to hold are $n \geq 4,\ n \neq 6, 10$. The exclusion of the number 10 from the range of values is because it is not known whether there exist a set of three mutually orthogonal Latin squares of order 10. The resolution of this problem or the existence of a DS4CS(17) in the affirmative would result in a huge simplification of the proof of Theorem 1.1.

The third construction is a modification of the previous construction.

Theorem 4.3 *If there exist a set of four mutually orthogonal Latin squares of order n, a DS4CS($8n+1$) and a DS4CS($8m+1$) where $0 \leq m \leq n$, then there exists a DS4CS($8(5n+m)+1$).*

Proof. The proof is analogous to the proof of the previous theorem where the blocks of the transversal design become either a copy of $K_{8,8,8,8,8}$ or $K_{8,8,8,8,8,8}$ and can be decomposed as in Lemmas 2.5 and 2.6. □

5 Remaining "small" systems

We are now in a position to complete the proof that there exists a DS4CS($8s$+1) for all $s:\ 3 \leq s \leq 46$.

1. For $s = 12, 15, 18, 21, 27, 30$, use Theorem 4.1.

2. For $s = 16, 19, 20$, use Theorem 4.2 with $n = 4,\ m = 0, 3, 4$ and the DS4CS(33) and DS4CS(25) given in Section 3.

3. For $s = 23, 24, 25$, use Theorem 4.2 with $n = 5,\ m = 3, 4, 5$ and the DS4CS(41), DS4CS(33) and DS4CS(25) given in Section 3.

4. For $s = 28$, use Theorem 4.2 with $n = 7,\ m = 0$ and the DS4CS(57) given in Section 3.

5. For $31 \leq s \leq 42$, use Theorem 4.2 with $7 \leq n \leq 9,\ 3 \leq m \leq 6$ and the appropriate diagonally switchable 4-cycle systems given in Section 3.

6. For $s = 43, 44, 45$, use Theorem 4.2 with $n = 9,\ m = 7, 8, 9$ and the DS4CS(73), DS4CS(65) and DS4CS(57) given in Section 3.

7. For $s = 29$, use Theorem 4.3 with $n = 5,\ m = 4$ and the DS4CS(41) and DS4CS(33) given in Section 3.

8. For $s = 46$, use Theorem 4.3 with $n = 8,\ m = 6$ and the DS4CS(65) and DS4CS(49) given in Section 3.

We now have DS4CS($8s$+1) for all $s:\ 3 \leq s \leq 46,\ s \neq 22, 26$. We give proofs of the existence of diagonally switchable 4-cycle systems of these two orders in the next two theorems.

Theorem 5.1 *There exists a DS4CS(*177*).*
Proof. Take a transversal design TD(6, 16), that is, a set of four mutually orthogonal Latin squares of order 16. Delete four points from each of two rows of the design to obtain a decomposition of the 6-partite graph $K_{16,16,16,16,12,12}$ into blocks of size 4, 5 or 6. Now inflate each point by factor 2. Each block then becomes a $K_{2,2,2,2}$ or $K_{2,2,2,2,2}$ or $K_{2,2,2,2,2,2}$ which can be decomposed as in Lemmas 2.1, 2.2 or 2.3. Further adjoin an extra point, say ∞, and on each inflated group of points of the decomposition together with the point ∞, place a copy of the DS4CS(33) or DS4CS(25) given in Section 3. The result is a DS4CS($2 \times 4 \times 16 + 2 \times 2 \times 12 + 1$), that is, DS4CS($8 \times 22 + 1$). □

Theorem 5.2 *There exists a DS4CS(*209*).*
Proof. Take a PBD(27,$\{4, 5, 6\}$), [4] and remove all blocks through a specified point to obtain a $\{4, 5, 6\}$-GDD with groups of size 3, 4 or 5. Now inflate each point by factor 8. Each block then becomes a copy of $K_{8,8,8,8}$ or $K_{8,8,8,8,8}$ or $K_{8,8,8,8,8,8}$ which can be decomposed as in Lemmas 2.4, 2.5 or 2.6. Further adjoin an extra point, say ∞, and on each inflated group

of points of the group divisible design together with the point ∞, place a copy of the DS4CS(25) or DS4CS(33) or DS4CS(41) given in Section 3.

□

6 Proof of the main result

Finally, we are able to prove Theorem 1.1.

Proof. We have constructed DS4CS($8s$+1) for all s : $3 \leq s \leq 46$. So suppose that $s \geq 47$. Express s in the form $s = 4n + m$ where $n \geq 11$ and $m \in \{3, 4, 5, 6\}$. Then there exist a set of three mutually orthogonal Latin squares of order n and a DS4CS($8m$+1). So by Theorem 4.2, the existence of a DS4CS($8n$+1) implies that of a DS4CS($8s$+1). If $n \leq 46$ the result follows. Otherwise replace s by n and apply the above argument recursively. □

References

[1] I. Anderson, *Combinatorial Designs: construction methods*, Ellis Horwood, 1990.

[2] A.E. Brouwer, A. Schrijver and H. Hanani, Group divisible designs with block size 4, *Discrete Math.* **20** (1977), 1-10.

[3] D.E. Bryant, M.J. Grannell, T.S. Griggs and M. Mačaj, Configurations in 4-cycle systems, *Graphs Combin.* to appear.

[4] H. Lenz, Some remarks on pairwise balanced designs, *Mitt. Math. Sem. Giessen* **165** (1984), 49-62.

[5] C.C. Lindner and C.A. Rodger, Decomposition into cycles II: cycle systems in *Contemporary Design Theory: a collection of surveys* (ed. J.H. Dinitz and D.R. Stinson), John Wiley 1992.