Small bowtie systems: an enumeration

M.J. Grannell, T.S. Griggs
Department of Mathematics and Statistics
The Open University
Walton Hall
Milton Keynes MK7 6AA
UNITED KINGDOM
m.j.grannell@open.ac.uk
t.s.griggs@open.ac.uk

G. LoFaro, A. Tripodi
Dipartimento di Matematica
Universita di Messina
Contrado Papardo
Salita Sperone 31
98166 Sant' Agata
Messina
ITALIA
giovanni.lofaro@unime.it
tripodi@dipmat.unime.it

Abstract

There exist 3 near bowtie systems of order 7, 12 bowtie systems of order 9, and 1,411,422 balanced bowtie systems of order 13.

AMS classification: 05B07

Keywords: Bowtie system, Steiner triple system

1 Introduction

Let $X = (V, E)$ be the graph with vertex set $V = \{x, a, b, c, d\}$ and edge set $E = \{xa, xb, xc, xd, ab, cd\}$. Such a graph is called a *bowtie* and will be represented throughout this paper by the notation $ab - x - cd$. The vertex x is called the *centre* of the bowtie. A decomposition of the complete graph K_n into subgraphs isomorphic to X is called a *bowtie system of order n* and denoted by BTS(n). An elementary counting argument gives that a necessary condition for the existence of a BTS(n) is $n \equiv 1$ or 9 (mod 12). In a BTS(n), if every vertex of the complete graph K_n occurs the same number of times as the centre of a bowtie, then the bowtie is said to be *balanced*. A necessary condition for the existence of a balanced BTS(n) is $n \equiv 1$ (mod 12).

It is easy to see that, given a BTS(n), by regarding each of the two triangles of every bowtie as separate entities, we have a Steiner triple system STS(n). We call this the *associated* Steiner triple system of the bowtie system. Conversely, if $n \equiv 1$ or 9 (mod 12), it is also true that the triangles of every STS(n) can be amalgamated to form bowties. This is a consequence of the fact that the block intersection graph of every Steiner triple system is Hamiltonian, see for example [1]. If $n \equiv 1$ (mod 12), there exists a cyclic STS(n), see also [1], and this system will have an even number of full orbits. It is then immediate that we can amalgamate triangles from pairs of orbits to form a balanced BTS(n). Hence the necessary conditions for both BTS(n) and balanced BTS(n) given above are also sufficient.

If $n \equiv 3$ or 7 (mod 12), orders for which an STS(n) exists but the number of triangles is odd, it is still a consequence of the Hamiltonicity of the block intersection graph that all triangles except one can be amalgamated in pairs to form bowties. In this case we have a *near bowtie system of order n*; denoted by NBTS(n).

In spite of the very close relationship between bowtie and near bowtie systems on the one hand and Steiner triple systems on the other hand, their properties can be very different. We explore this further in another paper dealing with configurations in balanced bowtie systems, [2]. The present paper is mainly concerned with enumeration results, specifically for non-isomorphic NBTS(7)s, BTS(9)s, and balanced BTS(13)s. The respective numbers are 3, 12 and 1,411,422 which contrast sharply with the number of non-isomorphic STS(7)s, STS(9)s, and STS(13)s, i.e. 1, 1, and 2, see for example [3]. The enumeration of NBTS(7)s and BTS(9)s is done by hand but the number of non-isomorphic balanced BTS(13)s requires the use of a computer. We also discuss the automorphism group of every system. In order to do this we use the elementary but very useful observation that the automorphism group of a bowtie or near bowtie system is a subgroup of the automorphism group of the associated Steiner triple system. Information about the automorphism groups of the unique STS(7) and STS(9) and the two non-isomorphic STS(13)s is taken from [3].

2 Near bowtie systems of order 7

In this section we prove the following theorem.

Theorem 1 *There exist precisely 3 pairwise non-isomorphic NBTS(7)s with automorphism groups S_4 of order 24, K_4 of order 4, and C_3 of order 3, respectively.*

Proof. Represent the associated STS(7) on the base set Z_7 as the set of triangles generated by $\{0, 1, 3\}$ under the action of the mapping $i \mapsto i + 1$ (mod 7). Without loss of generality, choose the triangle 013 (here and in what follows we will omit set brackets and commas) to be the single triangle in an NBTS(7). Any automorphism of an NBTS(7) must stabilize this triangle.

The automorphism group of the STS(7) has order 168 and the orbit length of triangles is 7. Hence the number of elements of the automorphism group stabilizing a triangle is $168/7 = 24$. Those automorphisms which stabilize the triangle 013 are of the form $p_i q_k$ where p_i is a permutation on the set $\{0, 1, 3\}$ and q_k is a permutation on the set $\{2, 4, 5, 6\}$. There are precisely 24 such permutations q_k. Now if $p_i q_k$ and $p_j q_k$ are both permutations which stabilize the triangle 013 then so does the permutation $p_i q_k (p_j q_k)^{-1} = p_i p_j^{-1}$. But this is a permutation on the set $\{0, 1, 3\}$ which also collectively fixes all the other triangles of the STS(7). This is not possible unless $p_i p_j^{-1}$ is the identity permutation, i.e. $p_i = p_j$. So the 24 elements of the automorphism group that stabilize the triangle 013 all have distinct permutations q_k, i.e. the group is the symmetric group S_4. For each permutation q_k, there exists a unique permutation p_i such that $p_i q_k$ is an element of the automorphism group. In fact we find that the subgroup, say $G = \langle \alpha, \beta \rangle$, which stabilizes 013 is generated by the permutations $\alpha = (0\ 1\ 3)(2\ 4\ 6)$ and $\beta = (1\ 3)(2\ 5\ 6\ 4)$.

The other 6 triangles of the STS(7) can be amalgamated into 3 bowties in 15 different ways, which are listed below. Those systems denoted by letters A, B, and C are non-isomorphic as consideration of the centres of the bowties shows. Further, under the action of the group G, these systems can be taken to be the base systems of the 3 orbits, respectively of lengths 1, 6, and 8, into which the 15 near bowtie systems are partitioned. Permutations from the group G which can be applied to obtain the other systems are given.

1.	$14-2-35$	$36-4-05$	$15-6-02$	C
2.	$14-2-35$	$34-6-15$	$45-0-62$	B
3.	$14-2-35$	$34-6-02$	$04-5-16$	apply (1 3)(4 5) to C
4.	$12-4-36$	$23-5-04$	$15-6-02$	apply (0 1)(2 5) to C
5.	$12-4-36$	$23-5-16$	$45-0-26$	apply (2 4)(5 6) to B
6.	$12-4-36$	$35-2-06$	$04-5-16$	apply (2 4)(5 6) to C
7.	$12-4-05$	$25-3-46$	$15-6-02$	apply (0 3)(2 4) to B
8.	$12-4-05$	$23-5-16$	$34-6-02$	apply (2 5)(4 6) to C
9.	$12-4-05$	$35-2-06$	$34-6-15$	apply (0 1)(4 6) to C
10.	$24-1-56$	$25-3-46$	$45-0-26$	A
11.	$24-1-56$	$23-5-04$	$34-6-02$	apply (0 1)(2 5) to B
12.	$24-1-56$	$35-2-06$	$36-4-05$	apply (0 1)(4 6) to B
13.	$14-2-06$	$25-3-46$	$04-5-16$	apply (0 3)(5 6) to B
14.	$14-2-06$	$23-5-04$	$34-6-15$	apply (2 6)(4 5) to C
15.	$14-2-06$	$23-5-16$	$36-4-05$	apply (0 3)(5 6) to C

Thus there are 3 pairwise non-isomorphic NBTS(7)s having automorphism groups of order $24/1 = 24$, $24/6 = 4$, and $24/8 = 3$ respectively. The actual permutations which form the automorphism groups are easily determined and details of the systems are given below.

System A.
Bowties $24-1-56$, $25-3-46$, $45-0-26$. Triangle 013.
Automorphism group of order $24 \simeq S_4$ generated by the permutations $\alpha = (0\ 1\ 3)(2\ 4\ 6)$ and $\beta = (1\ 3)(2\ 5\ 6\ 4)$.
System B.
Bowties $14-2-35$, $34-6-15$, $45-0-62$. Triangle 013.
Automorphism group of order $4 \simeq K_4$ consisting of the identity and the permutations (2 6)(4 5), (1 3)(4 5), and (1 3)(2 6).
System C.
Bowties $14-2-35$, $36-4-05$, $15-6-02$. Triangle 013.
Automorphism group of order $3 \simeq C_3$ generated by the permutation $\alpha = (0\ 1\ 3)(2\ 4\ 6)$. □

3 Bowtie systems of order 9

In this section we prove the following theorem.

Theorem 2 *There exist precisely 12 pairwise non-isomorphic BTS(9)s of which 3 have automorphism group C_4 of order 4, 3 have automorphism group C_3 of order 3, and 6 have only the trivial automorphism group.*

Proof. Represent the associated STS(9) as the set of triangles

$\{0,1,2\}$,	$\{3,4,5\}$,	$\{6,7,8\}$,	parallel class A;
$\{0,3,6\}$,	$\{1,4,7\}$,	$\{2,5,8\}$,	parallel class B;
$\{0,4,8\}$,	$\{1,5,6\}$,	$\{2,3,7\}$,	parallel class C;
$\{0,5,7\}$,	$\{1,3,8\}$,	$\{2,4,6\}$,	parallel class D.

The automorphism group of the STS(9) has order 432 and acts doubly transitively on the points of the design.

There are two possibilities for the distribution of the centres of the 6 bowties in a BTS(9):

(a) one point occurs twice and four points occur once, or

(b) six points occur once.

Each possibility is considered in turn. Clearly two BTS(9)s having different distributions of the centres are non-isomorphic.

(a) There are $9 \times 3 = 27$ realizations of a pair of bowties having the same centre. These form a single orbit under the automorphism group of the STS(9). To see this, note that the automorphism group is transitive on the points and observe that, for example, the three pairs of bowties having the point 0 as their centres, namely

(i)	$12-0-36$	$48-0-57$
(ii)	$12-0-48$	$57-0-36$
(iii)	$12-0-57$	$36-0-48$

are mapped cyclically by the permutation (3 4 5)(6 8 7).

Therefore, without loss of generality, let the pair of bowties having the same centre be $12-0-36$ and $48-0-57$. These are stabilized by a subgroup H of order $432/27 = 16$. The reader can easily check that $H = \langle\sigma, \tau\rangle$ where $\sigma = (1\ 4\ 6\ 7\ 2\ 8\ 3\ 5)$, $\tau = (3\ 6)(4\ 7)(8\ 5)$ and $\tau\sigma = \sigma^3\tau$. Further, since the automorphism group of any BTS(9), say B, which contains the bowties $12-0-36$ and $48-0-57$ must stabilize these, then $\mathrm{Aut}(B)$ is a subgroup of H.

We now consider completions of the above two bowties to a BTS(9). We adopt the notation that a bowtie is of type XY where X, Y $\in$ {A, B, C, D}, X $\neq$ Y if the two triangles that form the basic bowtie come from parallel classes X and Y respectively. Noting that the pair of bowties having the same centre are of types AB and CD and are stabilized by the group H, the six choices for the types of the remaining four bowties partition into four cases, each of which is also stabilized by H. We consider each of these cases in turn and give the results of the calculations to determine the number of bowtie systems and their automorphism groups. Details of the calculations are omitted; although they are tedious they are straightforward and involve applying the elements of the group H to the appropriate bowtie systems.

Case I. Bowtie types AB, AB, CD, CD.
There are four possibilities which form a single orbit under the action of the group H. One of these possibilities is the set of bowties

$$35-4-17 \quad 67-8-25 \quad 56-1-38 \quad 37-2-46$$

which is stabilized by the group C_4 generated by the permutation (1 4 2 8) (3 7 6 5) $= \sigma\tau$.

Case II. Bowtie types AC, AC, BD, BD and AD, AD, BC, BC.
There are eight possibilities, four of each type, which form two orbits under the action of the group H. A representative of one of the orbits is the set of bowties

$$34-5-16 \quad 68-7-23 \quad 47-1-38 \quad 58-2-46$$

which is stabilized by the group C_4 generated by the permutation (1 5 2 7) (3 4 6 8) $= \sigma^7\tau$.
A representative of the other orbit is the set of bowties

$$34-5-16 \quad 68-7-23 \quad 17-4-26 \quad 25-8-13$$

which is stabilized by the group C_4 generated by the permutation (1 3 2 6) (4 5 8 7) $= \sigma^6$.

Case III. Bowtie types AB, AC, BD, CD and AB, AD, BC, CD.
There are 32 possibilities, 16 of each type, which form two orbits under the action of the group H. A representative of one of the orbits is the set of bowties

$$35-4-17 \quad 78-6-15 \quad 25-8-13 \quad 37-2-46$$

and a representative of the other orbit is the set of bowties

$$35-4-17 \quad 78-6-15 \quad 58-2-46 \quad 27-3-18.$$

Both systems have only the trivial automorphism group.

Case IV. Bowtie types AC, AD, BC, BD.
There are 16 possibilities which form a single orbit under the action of the group H. One of these is the set of bowties

$$34-5-16 \quad 67-8-13 \quad 14-7-23 \quad 58-2-46.$$

Again the system has only the trivial automorphism group.

Thus there are six pairwise non-isomorphic BTS(9)s in which one point occurs twice and four points occur once as the centres of the bowties. We now consider possibility (b).

(b) Six points occur once as the centres of the bowties. Thus three points do not occur as the centres of the bowties, and trivially these three points do not form a triangle of the STS(9). Using the representation of the STS(9) given at the beginning of the proof, let these three points be 0, 1, and 3. This can be done without loss of generality because the automorphism group acts transitively on triples which do not form triangles.

Now consider the following 3×3 array of bowties containing the triangles 012, 036, and 138.

$01-2-58$	$03-6-78$	$13-8-67$	Row P
$01-2-37$	$03-6-15$	$13-8-25$	Row Q
$01-2-46$	$03-6-24$	$13-8-04$	Row R
Column 1	Column 2	Column 3	

Any bowtie system must contain precisely one bowtie from each column. But bowties P2 and P3, P1 and Q3, R1 and R2 contain a common triangle which leaves 18 possibilities. The automorphism group of a BTS(9) which contains any of the partial bowtie systems must also stabilize the partial system and hence also the sets $\{0,1,3\}$, $\{2,6,8\}$, $\{4,5,7\}$. It is then easily shown to be a subgroup of the group $K = \langle \lambda, \mu \rangle$ where $\lambda = (0\ 1\ 3)(2\ 8\ 6)(4\ 5\ 7)$ and $\mu = (1\ 3)(2\ 6)(5\ 7)$, which is isomorphic to the symmetric group S_3 of order 6. The 18 partial bowtie systems are listed below. Under the action of the group K, these partition into 5 orbits consisting respectively of 1, 2, 3, 6, and 6 systems of 3 bowties. Base systems in each orbit are denoted by I, II, III, IV, and V. Permutations from the group K which can be applied to these base systems in order to obtain the other systems are also given.

1.	$01-2-58$	$03-6-78$	$13-8-04$	I
2.	$01-2-58$	$03-6-15$	$13-8-67$	II
3.	$01-2-58$	$03-6-15$	$13-8-04$	III
4.	$01-2-58$	$03-6-24$	$13-8-67$	IV
5.	$01-2-58$	$03-6-24$	$13-8-04$	(031)(268)(475) to II
6.	$01-2-37$	$03-6-78$	$13-8-25$	(1 3)(2 6)(5 7) to II
7.	$01-2-37$	$03-6-78$	$13-8-04$	(1 3)(2 6)(5 7) to III
8.	$01-2-37$	$03-6-15$	$13-8-67$	(013)(286)(457) to III
9.	$01-2-37$	$03-6-15$	$13-8-25$	(0 3)(2 8)(4 7) to III
10.	$01-2-37$	$03-6-15$	$13-8-04$	V
11.	$01-2-37$	$03-6-24$	$13-8-67$	(013)(286)(457) to II
12.	$01-2-37$	$03-6-24$	$13-8-25$	(031)(268)(475) to I
13.	$01-2-37$	$03-6-24$	$13-8-04$	(031)(268)(475) to III
14.	$01-2-46$	$03-6-78$	$13-8-25$	(1 3)(2 6)(5 7) to IV
15.	$01-2-46$	$03-6-78$	$13-8-04$	(0 1)(6 8)(4 5) to II
16.	$01-2-46$	$03-6-15$	$13-8-67$	(013)(286)(457) to I
17.	$01-2-46$	$03-6-15$	$13-8-25$	(0 3)(2 8)(4 7) to II
18.	$01-2-46$	$03-6-15$	$13-8-04$	(0 1)(6 8)(4 5) to III

We now consider completions of each of the five pairwise non-isomorphic partial bowtie systems to form a BTS(9).

Case I. The partial bowtie system $01-2-58$, $03-6-78$, $13-8-04$ is stabilized only by the permutation μ and the identity. There are two completions, namely $26-4-35$, $16-5-07$, $23-7-14$ and $26-4-17$, $16-5-34$, $23-7-05$ which are mapped to one another by μ. Hence Case

I gives precisely one BTS(9) having the trivial automorphism group.

Case II. The partial system $01-2-58$, $03-6-15$, $13-8-67$ has the unique completion $08-4-26$, $34-5-07$, $14-7-23$ forming a BTS(9) having the trivial automorphism group.

Case III. The partial system $01-2-58$, $03-6-15$, $13-8-04$ also has a unique completion $23-7-68$, $17-4-26$, $07-5-34$, again forming a BTS(9) having the trivial automorphism group.

Case IV. The partial system $01-2-58$, $03-6-24$, $13-8-67$ is stabilized by the cyclic group of order 3 generated by the permutation λ. There are two completions, namely $08-4-35$, $16-5-07$, $23-7-14$ and $08-4-17$, $16-5-34$, $23-7-05$, both of which are also stabilized by the same group. Hence Case IV gives precisely two BTS(9)s each having automorphism group C_3 of order 3 generated by the permutation λ.

Case V. The partial system $01-2-37$, $03-6-15$, $13-8-04$ is stabilized by the group K. There are two completions, namely $68-7-14$, $28-5-07$, $26-4-35$ and $68-7-05$, $28-5-34$, $26-4-17$ which are mapped to one another by μ. Hence Case V gives precisely one BTS(9) having automorphism group C_3 of order 3 generated by the permutation λ.

Thus there are six pairwise non-isomorphic BTS(9)s in which six distinct vertices occur once as the centres of the bowties.

For convenience the 12 pairwise non-isomorphic BTS(9)s are collected together below.

System (a)(I). Bowties $12-0-36$, $48-0-57$, $35-4-17$, $67-8-25$, $56-1-38$, $37-2-46$. Automorphism group of order $4 \simeq C_4$ generated by the permutation (1 4 2 8)(3 7 6 5).

System (a)(II)(i). Bowties $12-0-36$, $48-0-57$, $34-5-16$, $68-7-23$, $47-1-38$, $58-2-46$. Automorphism group of order $4 \simeq C_4$ generated by the permutation (1 5 2 7)(3 4 6 8).

System (a)(II)(ii). Bowties $12-0-36$, $48-0-57$, $34-5-16$, $68-7-23$, $17-4-26$, $25-8-13$. Automorphism group of order $4 \simeq C_4$ generated by the permutation (1 3 2 6)(4 5 8 7).

System (a)(III)(i). Bowties $12-0-36$, $48-0-57$, $35-4-17$, $78-6-15$, $25-8-13$, $37-2-46$. Trivial automorphism group.

System (a)(III)(ii). Bowties $12-0-36$, $48-0-57$, $35-4-17$, $78-6-15$, $58-2-46$, $27-3-18$. Trivial automorphism group.

System (a)(IV). Bowties $12-0-36$, $48-0-57$, $34-5-16$, $67-8-13$, $14-7-23$, $58-2-46$. Trivial automorphism group.

System (b)(I). Bowties $01-2-58$, $03-6-78$, $13-8-04$, $26-4-35$, $16-5-07$, $23-7-14$. Trivial automorphism group.

System (b)(II). Bowties $01-2-58$, $03-6-15$, $13-8-67$, $08-4-26$, $34-5-07$, $14-7-23$. Trivial automorphism group.

System (b)(III). Bowties $01-2-58$, $03-6-15$, $13-8-04$, $23-7-68$, $17-4-26$, $07-5-34$. Trivial automorphism group.

System (b)(IV)(i). Bowties $01-2-58$, $03-6-24$, $13-8-67$, $08-4-35$, $16-5-07$, $23-7-14$. Automorphism group of order $3 \simeq C_3$ generated by the permutation (0 1 3)(2 8 6)(4 5 7).

System (b)(IV)(ii). Bowties $01-2-58$, $03-6-24$, $13-8-67$, $08-4-17$, $16-5-34$, $23-7-05$. Automorphism group of order $3 \simeq C_3$ generated by the permutation (0 1 3)(2 8 6)(4 5 7).

System (b)(V). Bowties $01-2-37$, $03-6-15$, $13-8-04$, $68-7-14$, $28-5-07$, $26-4-35$. Automorphism group of order $3 \simeq C_3$ generated by the permutation (0 1 3)(2 8 6)(4 5 7). □

As an endnote, we remark that a computer search gave 3,348 BTS(9)s associated with the system STS(9). Since $3,348 = 432(1/4+1/4+1/4+1+1+1+1+1+1+1/3+1/3+1/3)$, this is precisely the number predicted by the theorem.

4 Balanced bowtie systems associated with the cyclic STS(13)

The automorphism group of the cyclic STS(13) is of order 39 and has the following automorphism types:

(a) a 13-cycle,

(b) a fixed point and four 3-cycles.

No automorphism ϕ of type (b) can stabilize a bowtie system associated with the cyclic STS(13). Consider a bowtie $ab-x-cd$ whose centre, x, is the unique fixed point of ϕ. Then either

(i) $\phi : \{a,b\} \mapsto \{a,b\}$ and $\{c,d\} \mapsto \{c,d\}$, or

(ii) $\phi : \{a,b\} \mapsto \{c,d\}$ and $\{c,d\} \mapsto \{a,b\}$.

In case (i) either $\phi(a) = a$, which is not possible, or $\phi(a) = b$ and $\phi(b) = a$ which implies that $\phi^2(a) = a$, which is also not possible. In case (ii) assume, without loss of generality, that $\phi(a) = c$ and $\phi(b) = d$. Then either $\phi(c) = a$ and $\phi(d) = b$ which implies that $\phi^2(a) = a$, not possible, or $\phi(c) = b$ and $\phi(d) = a$ which implies that $\phi^3(a) = d$ and hence that $a = d$.

Represent the cyclic STS(13) on the base set Z_{13} as the set of triangles generated by $\{0, 1, 4\}$ and $\{0, 2, 7\}$ under the action of the mapping $i \mapsto i + 1 \pmod{13}$. If a balanced bowtie system associated with the cyclic STS(13) has an automorphism of type (a), (i.e. is invariant under the mapping $i \mapsto i+1 \pmod{13}$), then the two triangles that form the bowtie whose centre is 0 come from different orbits. Thus there are precisely nine such systems generated respectively from the following bowties, where 10, 11, and 12 are represented by T, E, and W respectively.

1.	14 – 0 – 27	2.	3W – 0 – 68	3.	9T – 0 – 5E
4.	14 – 0 – 5E	5.	3W – 0 – 27	6.	9T – 0 – 68
7.	14 – 0 – 68	8.	3W – 0 – 5E	9.	9T – 0 – 27

All other bowtie systems associated with the cyclic STS(13) will be automorphism-free.

A computer search gives 7,339,770 balanced bowtie systems associated with the cyclic STS(13), of which 7,339,761 will be automorphism-free. Each isomorphism class of these will arise 39 times in the search, corresponding to the mappings $i \mapsto ai + b \pmod{13}$ with $a \in \{1, 3, 9\}$ and $b \in \{0, 1, \dots, 12\}$ of the automorphism group of the cyclic STS(13). Thus there are 7,339,761/39 = 188,199 isomorphism classes of automorphism-free balanced bowtie systems associated with the cyclic STS(13). The remaining nine systems, numbered 1 to 9 above in this section, fall into three isomorphism classes. Systems #1, #2, #3 (respectively #4, #5, #6 and #7, #8, #9) are isomorphic under the mapping $i \mapsto 3i \pmod{13}$. The results of this section are summarized in the following theorem.

Theorem 3 *There exist precisely 188,202 pairwise non-isomorphic balanced BTS(13)s associated with the cyclic STS(13), of which 3 have automorphism group C_{13} of order 13 and the rest have only the trivial automorphism group.*

5 Balanced bowtie systems associated with the non-cyclic STS(13)

The automorphism group of the non-cyclic STS(13) is of order 6 and has the following automorphism types:

(a) three fixed points and five 2-cycles,

(b) a fixed point and four 3-cycles.

No automorphism ϕ of type (a) can stabilize a bowtie system associated with the non-cyclic STS(13). Let the three fixed points be x, y, and z. Then $\{x, y, z\}$ is a triangle of the non-cyclic STS(13) and must appear in one bowtie. Without loss of generality it may be assumed that x is the centre. Now each point appears in six triangles of the non-cyclic STS(13). So consider the triangles containing the point y. One of these appears in the bowtie above whose centre is x. Two more appear in the bowtie whose centre is y. All other bowties with a triangle containing the point y appear in pairs $by - a - cd$ and $\phi(b)y - \phi(a) - \phi(c)\phi(d)$. Thus triangles containing the point y appear an odd number of times; a contradiction. Further, by the same argument as used in the previous section, no automorphism ϕ of type (b) can stabilize a bowtie system associated with the non-cyclic STS(13) and hence all such bowtie systems are automorphism-free.

A computer search gives 7,339,320 balanced bowtie systems associated with the non-cyclic STS(13). Since these are all automorphism-free, each isomorphism class will arise 6 times in the search and hence after dividing 7,339,320 by 6, the following theorem can be stated.

Theorem 4 *There exist precisely 1,223,220 pairwise non-isomorphic balanced BTS(13)s associated with the non-cyclic STS(13), all of which have only the trivial automorphism group.*

References

[1] C.J. Colbourn and A. Rosa, *Triple Systems*, Clarendon Press, New York, 1999.

[2] M.J. Grannell, T.S. Griggs, G. LoFaro and A. Tripodi, Configurations in balanced bowtie systems, in preparation.

[3] R.A. Mathon, K.T. Phelps and A. Rosa, Small Steiner triple systems and their properties, *Ars Combin.* **15** (1983), 3-110.