STUDIES IN ALGEBRAIC THINKING NO 3

MARBLE SHARING

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This one of a number of case studies in how a single task can be used to access a domain of similar tasks, through using the notion of *dimensions of possible variation* (DofPV)

DofPV will be used to show how a task can be generalised and extended to produce a domain of tasks, and also applied to how the task is presented to learners. Once learners learn to use DofPV for themselves, they can learn more effectively from doing exercises, and become mathematical explorers rather than just mathematical clerks doing routing calculations with familiar techniques.

Task 1 Marbles

If Anne gives John one of her marbles, she will then have one more than twice as many marbles as John then has. However, if instead, John gives Anne one of his marbles, he will have one more than a third as many marbles as Anne then has. How many marbles have they each currently?

The first thing to note is that the language is quite complicated. One aspect of opening a task up to exploration, of using a task to become aware of a domain of associated tasks, is that tasks can be simplified as well as made more complicated. It is all part of elaborating a task domain.

How might Task 1 be simplified? Instead of two conditions, the task could give one condition and then ask how many marbles either John or Anne had to finish with if one of them has a specified starting number. For example

Task 2 Simpler Marbles

If Anne gives John one of her marbles, she will then have one more than twice as many marbles as John then has. If John started with 12 marbles, how many did Anne start with?

Notice that this simplifies in two ways: it reduces the number of constraints, and it also converts the task from indirect to direct: when the quantity at the end of a process is revealed and the starting values are sought is *indirect*; when calculations are to be done on specified numbers is *direct*. Arithmetic tasks are direct, in that a sequence of calculations are performed on starting numbers to reach an answer; problems susceptible to algebra tend to be indirect, in that it is necessary to start with the as-yet-unknown (as Mary Boole referred to it: see Tahta 1972) and set up equations and inequalities which can then be solved, indirectly. Algebraic thinking is more likely to be useful in indirect calculations than in direct calculation, even though many if not most problems can be done arithmetically, that is directly, with sufficient insight.

This is an important point. Experience in arithmetic conditions learners to proceed from the known to find the unknown, calculating as you go; algebra is powerful because it enables you to "acknowledge your ignorance" as Mary Boole declared, to denote that ignorance by a symbol (in the role of 'as-yet-unknown'), and then to express the conditions and constraints of the situation using those symbols. For example, if you can check whether a proposed answer is correct, then it is highly likely that you can use a symbol for that answer, and express the checking process as a sequence of equalities and inequalities. These then constitute the algebraic conditions to be solved.

If the language of the task is overly challenging, then that too can be simplified, perhaps by simplifying the constraint. Diagrams or picture sequences may prove helpful to capture the imposed condition.

Dimensions of Possible Variation & Range of Permissible Change

What features of either Task 1 or Task 2 can be varied while still retaining the basic structure?

Certainly the numbers can be changed: but changed to what? For example, the number 'one' appears in four places in Task 1 and in two places in Task 2. Must they stay the same? A moment's thought suggests they could vary independently, though it might be interesting to preserve a certain symmetry by keeping them all the same. So altogether there are six different numbers to change independently, or six dimensions of possible variation.

But what are their ranges of permissible change? Will any numbers do? Since marbles are being counted, non-negative integers are required, so the range of permissible change of the numbers appears to be non-negative integers. But any such integers? John and Anne have to start and finish with a whole number of marbles, so there may be some implicit constraints. Task 3 pursues this direction.

What about the multiples? One is already fractional, but both could have fractions as their RofPCh, and not just unit fractions.

There are other features which could change. For example, perhaps instead of 'one more', there could be 'one less', which is the same as allowing the amount more to be negative. So the RofPCh those two numbers is the integers not just the non-negatives.

The context could be changed, so that other people could be exchanging coins or pencils, or even quantities of liquid. Changing the context could change the RofPCh of some or all of the numbers. The RofPCh of the context is any situation in which each person can consider giving some measured part of their whole to the other, who can then measure the total.

The dimensions located so far (the explicit numbers, the context) are surface details: by no means trivial, but far from being all. Before pursuing more subtle dimensions, it is worth noting that when learners see tasks as individual and distinct, unrelated to others, unconnected in a web of similar tasks, they are in a weak position to recognise the type of a question on a test or examination. By working explicitly on extending their own appreciation of the DofPV and the associated RofPCh of each dimension, learners equip themselves to solve large, usually infinite classes of 'similar tasks'. Furthermore, by participating in the construction, elaboration and complexification of tasks, learners appreciate how pedagogic tasks arise, and are therefore more likely to treat them as a challenge than as a burden, an opportunity to pit their wits rather than a imposition to be escaped as quickly as possible. It makes much more sense to 'resolve other people's problems' (as set in books, on work-cards, in tests and on examinations) when you have already constructed similar ones yourself.

Probing More deeply

Story-problems often have implicit numbers which can also be varied. For example, the number of people involved. The more people there are, the more conditions that need to be stated in order to force a single solution, a lesson which is important for learners to encounter. Of course tasks do not have to have unique solution. Sometimes finding a relationship is sufficient. (See Task 3 and 4.)

Motivational Comments

Many authors have argued that these 'word-problems' or 'story –problems' are irrelevant and uninteresting. They ask, very reasonably, in what sorts of situations would someone know the data of the task but not know the answer to the question? Why not just count them? Who cares anyway? A possible response is that relevance and interest are not properties of the task itself, but rather are descriptive of a state a person or people find themselves in connection with the task. Put another way, things are neither relevant nor irrelevant, because it depends on the person and the situation. Problems and situations are neither inherently interesting nor inherently uninteresting. It is people who become interested. One of the outcomes of working with dimensions of possible variation is precisely that what looks initially as particular to the point of being peculiar, can be come interesting if the learners' natural powers of mathematical curiosity and sense making are brought into play. If instead of 'suffering' story-problem tasks to be inflicted upon them, learners are invited to work with dimensions of possible variation to complexify simple tasks for themselves, the puzzle aspect which challenges you to sort out someone else's attempt to be clever and devious can be taken as a challenge. Puzzles have been collected and distributed for over 500 years, presumably because human beings like to be puzzled!

The task has now become generalised:

Task 3 Implicit Constraints in the General

If Anne gives John a of her marbles, she will then have b more than m times as many marbles as John then has. However, if instead, John gives Anne c of his marbles, he will have d more than n times as many marbles as Anne then has. How many marbles have they each currently?

What implicit constraints must apply if the answers are to be integers?

An arithmetic approach making use of trying particular cases would experiment with different sets of numbers varied systematically. By paying attention to how the various calculations work out, it may be possible to get a sense of what conditions must hold amongst the six quantities a, b, c, m, and n. For example

Time as a DofPV

The tasks so far clearly state that if one person does something, the result is a number related to what the other person then has. But it could have been related to the starting values:

Task 4

If Anne gives John a of her marbles, she will then have b more than m times as many marbles as John had. However, if instead, John gives Anne c of his marbles, he will have d more than n times as many marbles as Anne had.

What effect does this have on the equations?

Almost certainly it will be easier to start by using particular numbers for the various parameters, preferably different so that you can track their role in the final answer, and locate any constraints

Note the similarities with age problems, which have their own slightly different DofPV (see for example … ref).

In the tasks so far, two conditions are provided as parallel options. But the giving and taking could be in sequence rather than in parallel.

Task 5 Give and Take In Sequence

If Anne gives John a of her marbles, she will then have b more than m times as many marbles as John then has.

Multiple presentations

Task 1 was stated in words. It could have been performed on two bags of marbles (contents unknown), or using surrogates for the marbles. The words could be used in parallel with the actions, or the actions could be performed in silence. The question could be included, or learners could be asked to think up questions about the situation involving the numbers of marbles. The task could be stated in a particular case, as in Task 1, but it could also be stated in various degrees of generality. It could start with a really simple version and then be gradually complexified, or it could start with various degrees of complexity and learners urged to simplify for themselves before re-complexifying or re-generalising. If learners always encounter the same format, whether simple to complex or particular to general, they will not be encouraged to make use of their own powers both to specialise or particularise, and to generalise, to simplify and to complexify.

Historical Versions

Two persons, *A*, and *B* were talking of their Ages: Says *A* to *B*, seven years ago I was thrice as old as you at that Time; and seven Years hence I shall be just twice as old as you will be: I demand their present Ages? [Mole 1788 problem V p129]

Two purses together contain 300 sovereigns. If we take 30 out of the first and put them in the second, then there is the same sum in each. How much does each contain? [Wright 1825 p203]

A person has two snuff boxes. If he put 8*l*. into the 1st, then it is half as valuable as the other. But if he take these 8*l*. out of the first and put them into the 2nd, then the latter is worth three times as much as the former. What is the value of each? [Wright 1825 p204]

Person, A, having taken any number he pleases out of a heap of counters, another person, B, is told to take p times as many. The person who conducts the game specifies p but does not know how many counters A took. A is now told to hand to B a certain specified number, q, of the counters which he holds, and B is told to give in exchange to A p times as many counters as A has left. Show that B will have at the end $[p + 1]q$ counters. Give a numerical illustration. [Nunn 1919 problem 3 p150]

A son asking his father how old he was, received this answer : Your age is now one fourth of mine; but 5 years ago, your age was only one fifth of mine. What then are their two ages ? [Hutton 1833 problem 1 p80]

Seven years ago A's age was three times B's, and seven years hence A's age will be double B's. Find their ages. [Thomson 1898 problem 12 p207]

Six years ago A was one-third more than three times as old as B. Three years hence A's age will be $2\frac{1}{6}$ times B's. Find their present ages. [Thomson 1898 problem 14 p208]

Four years ago A was 1/5 more than 5 times as old as B, and four years hence A will be 1/3 more than 3 times as old as B. What is the age of each? [Thomson 1898 problem 15 p208]

A man is now twice as old as his son; 15 years ago he was three times as old. Find the age of each. [Wentworth 1898 problem 7p25]

A man is now twice as old as his son; 20 years ago he was four times as old as his son. Find the age of each. [Wentworth 1898 problem 50 p30]

A's age now is two fifths of B's. Eight years ago A's age was two ninths of B's. find their ages. [Wentworth 1898 problem 91 p167]

A boy, being asked his age and that of his sister, replied, "If I were 3 years older, I would be 3 times as old as my sister; but, if she were 2 years older, she would be one-half as old as I am." How old was each ? [Waterloo text ≈1900 p236]

In 5 years A will be twice as old as B. Five years ago A was three times as old as B. Find the age of each now. [Hawkes, Luby, Tauton 1910 problem 13 p212]

A's age is 5/2 B's age. In 10 years A's age will be twice B's age. Find their ages now. [Hawkes, Luby, Tauton 1910 problem 14 p159]

The age of A is 2/3 that of B. Fourteen years ago A's age was 1/2 B's age. Find their ages now. [Hawkes, Luby, Tauton 1910 problem 15 p159]

SOURCES

Ballard, P. 1928, *Teaching The Essentials of Arithmetic*, University of London Press, London.

Bézout, 1792, *Cours de Mathematiques, a l'usage du corps de l'artillerie*, Paris.

Bland, M. 1832, *Algebraical Problems, producing Simple and Quadratic Equations with Their Solutions*, J. Smith, Cambridge.

Bonnycastle, J. 1803, An Introduction to Algebra, with Notes and Observations, 11th Ed, J. Johnson, London.

Bourdon, M. 1840 *Éléments D'Algèbre*, Brussels.

Bouthilier, J. 1809, Traite *D'Arithmetique pour usage des Élèves*, Quebec.

Diophantos, [Trans. Heath], 1964 *Diophantus of Alexandria : a study in the history of Greek algebra*, Dover, New York.

Fine, H. 1901, *A College Algebra*, Ginn & Co., Boston.

Hawkes, H. Luby, W. & Touton, F., 1909, *Complete School Algebra,* Ginn and Company, Boston.

Hutton, C. 1833, *A Course of Mathematics composed for the use of the Royal Military Academy*, New Edition by W. Ramsey, Tegg, London.

Kilpatrick, J. 1967, Analyzing the Solution of Word Problems in Mathematics: an exploratory study, Unpublished PhD Dissertation, Stanford; *Dissertation Abstracts International* 28 4380A; *University Microfilms* 68-5, 442.

Maclaurin, C. 1779, *A Treatise of Algebra in 3 Parts*, J. Nourse, London.

Marton, F. & Booth, S. 1997, *Learning and Awareness*, Lawrence Erlbaum, Mahwah.

Mole, J. 1788, *Elements of Algebra; to which is prefixed, a choice collection of arithmetical questions, with their solutions, including some new improvements worthy the attention of arithmeticians.*

Nunn, T. P. 1919, *Exercises in Algebra [Including Trigonometry], Part I*, Longmans, Green and Co. London.

Saunderson, N. 1740, *The Elements of Algebra in Ten Books*, London.

Tahta, D. 1972, *A Boolean Anthology: selected writings of Mary Boole on mathematics education*, Association of Teachers of Mathematics, Derby.

Thomson, J. 1874, *New Practical Arithmetic for Grammar Departments,* 15th edition, Clark & Maynard, New York.

Thomson, W. 1898 *Chambers's Algebra For Schools*, Chambers, London.

Torbert, I. 1834, [3rd edition] *The Farmer and Mechanic's Manual: or the present currency of the United States, concisely explained, by rules entirely original, and applied to practical purposes, in a manner not before published, together with mensuration and an appendix containing rules that are seldom used, but sometimes necessary*, Torbert, Philadelphia.

Wells, W. & Hart, W. 1923, *First year Algebra,* Heath & Company, Boston.

Wentworth, G. A. 1898, *New School Algebra*, Ginn & Co. Boston.

Wright, J. 1825, *Self Examinations in Algebra*, Black, Young, & Young, London.