STUDIES IN ALGEBRAIC THINKING NO 2

SUMS TO ONE:
A STUDY IN GENERALISATION
AND
IN THE USE OF FUNDAMENTAL CONSTRUCTS FROM MATHEMATICS EDUCATION

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INTRODUCTION

This is a study of how a simple task developed into a number of more challenging explorations. On the surface therefore it illustrates the thesis that every mathematical task can be seen as but a particular example of a domain or space of related tasks. It is also a study of how various fundamental constructs in mathematics education inform both reflection on experience, and pedagogically effective practices.

The task itself began as one of a number of tasks (Mason 1996) designed to reveal to undergraduates that they actually knew spontaneously to specialise (to make use of particular examples in order to get a sense of what might be going on structurally), and then to re-generalise using symbols. Generalisations, extensions and variations of the task developed over a long period of time. In the process, the task itself became useful for other pedagogic aims. Constructs illustrated include dimensions-of-possible-variation which is a development of ideas of Ference Marton (Marton & Booth 1997) which intersect with a major mathematical theme: the study of invariance in the midst of change; the structure of attention (closely allied to the van Hieles' framework); scaffolding-and-fading (Seeley Brown *et al* 1989, Bruner 1996); manipulating–getting-a-sense-of–articulating (Floyd *et al* 1981); and example-spaces (developed by Watson & Mason 2002, 2005) as part of concept-images, among others.

The pedagogic assumptions underlying these notes is that to be effective as learners, students need to be active rather than passive, asserting conjectures rather than merely assenting to what is going on, anticipating what is coming rather than dwelling in what has just happened. Furthermore, the most effective way to prompt learners to use and develop their own powers of mathematical thinking is to get them to use those powers spontaneously, and then to draw attention to those powers. Subsequently, if they do not use those powers spontaneously, their attention can be drawn to the possibility by reference to prior experience.

It has proved difficult to separate mathematical exploration, commentary on mathematical exploration, and pedagogical remarks, but in order to alert the reader to what to expect, for the most part, mathematical commentary is displayed left and pedagogical commentary is displayed right.

Mathematics

Task 1: One Sum

I have written down two numbers which sum to one. Which do you think will be larger, the square of the larger plus the smaller, or the square of the smaller plus the larger?

Following Polya's film (1965), I like to get people to commit themselves to a conjecture, then test that conjecture. It is all too easy for learners to 'sit at the back of the class' and think that they know, just as it is all too easy as reader of these notes to glance at tasks and then carry on reading rather than stopping and actually working on the task yourself. Committing yourself to making a conjecture gives you something to test out, something to aim for. When you discover that your conjecture needs modifying, this disturbance may act as a memory aid to recall the experience, and hence may contribute to learning, whereas 'guessing correctly' leaves little or no impression.

Before reading this far you have probably already tried out one or two cases. This is what Polya (1962) called specialising, and he recommended it whenever you are stuck on a problem (see also Mason, Burton & Stacey 1982). Most people specialise spontaneously in this task, yet when they get stuck in the midst of making sense of some difficult mathematics, or when working on a hard problem, it is curious how specialising does not always come readily to mind.

The pedagogical issue is how to support learners in having richer possibilities come to mind when they are stuck or struggling. Alerting them to their own spontaneous use of their own natural powers to specialise and to generalise is an important step in prompting them to educate their awareness of their own powers. Later, when they seem not to be using those powers spontaneously, they can be prompted to use them, at first directly, then later more indirectly, until they begin to use them spontaneously themselves. This is how scaffolding-and-fading can be used to pedagogical advantage. (Bruner 1996, Seeley Brown *et al* 1989, Love & Mason 1992).

Trying a few cases is all very well, but the abiding mathematical question is whether what you found to be the case for your numbers is necessarily the case for my pair of numbers. In other words, does 'it' always work? The notion of always-sometimes-never true is important in mathematics, because locating the range-of-permissible-change in which some statement is valid is an important part of appreciating the statement. Put another way, many mathematical results can be seen as the statement of an invariant with a specification of what is permitted to change. Invariance only makes sense in the presence of variation, and variation only in the presence of some invariance. Understanding a concept involves appreciating what it is possible to vary in an example and still it remain an example, and over what range the change is permissible. Appreciating a technique or the exemplary nature of a task similarly involves dimensions of possible variation and corresponding ranges of permissible change.

Whenever an indefinite pronoun such as *this*, *that*, or *it* appears in learners' explanations there is not only potential ambiguity, but a real possibility that learners are unaware of slides between different referents for the 'it'. By developing the habit of asking learners "what is the it?", learner attention can be drawn to confusions.

Some people who are algebraically confident immediately reach for symbols, and with a tiny bit of manipulation they verify that their conjecture always works. The algebraic solution is so quick and complete that it is hard not to then drop the problem and go on to something else. However, a good deal more is waiting to be found. The next task demonstrates a way of seeing which requires no algebra.

Task 2: Depicting

How might the situation in Task 1 be depicted?

For example, 'squaring the larger' might suggest looking at squares; two numbers summing to one might therefore suggest a unit square with some sort of a division to show two numbers summing to one. I deliberately do not wish to display a diagram at this point, because it is more fruitful to come to it, perhaps with some guidance, for yourself.

If you have already drawn your diagram, then when you read what follows you will have to suspend what you have done to try to make sense of my description (or ignore mine altogether); if not then you have to make sense of my description without the assistance of your own. Neither actions are trivial; both can be problematic.

Task 2a: Depicting

Imagine a square, with a chord drawn at right-angles to one edge, going across the square. Imagine another chord at right-angles to the first, making two smaller squares in opposite corners inside the original square.

As a teacher it is often necessary to suspend your own images in order to try to appreciate what learners are describing and displaying about their own thinking. Learning to hold various versions arising from the same stimulus is therefore of advantage when working with learners.

Whether you had already drawn your own, or whether you were starting from scratch with my description, you experienced a situation in which there is potential interference and potential support from at least two of the three different modes of representation which Bruner (1966) described as enactive, iconic, and symbolic. I see these as three different worlds which we occupy: material, mental and symbolic including verbal. In this case, the task affords the possibility to experience movement between the mental the material, and between the verbal and the material, and as will emerge shortly, back again.

On Presentation

Tasks are not single items, but rather particular manifestations of a whole domain of associated tasks, as this study will indicate. The notion of dimensions-of-possible-variation is useful for thinking about task presentation, as well as for the substance and focus of tasks. For example, Task 2a is presented as an invitation to imagine. A sensible way of coping is to build up a diagram following the instructions. It is also possible to try to work mentally for as long as possible, in order to strengthen the natural power that everyone has to imagine. Note however, that *imaging a square* does not necessarily mean to picture a square. Some 40% of the population do not respond strongly to the language of pictures for what they experience in the iconic or mental world. Some people have a sense. Almost a visceral sense of square, some people work with the language, and most people have an element of all of these.

Task 2a could have been presented using a diagram. Different powers would then be called upon to make sense of the diagram. In a workshop it could be presented in text as an instruction, or as a fully developed worksheet, or it could be described orally. It could even be presented in silence! A diagram could be presented fully completed, or could be built up step by step, perhaps even electronically. These are just some of the dimensions-of-possible-variation in task presentation. The choice of the form of presentation will be informed by the purposes of the task and the experience and expectations of the learners.

Presenting a task, perhaps in silence, building up a diagram or sequence of expressions affords learners the opportunity to anticipate what is coming next. For example, drawing one chord across the square and then pausing offers an implicit opportunity to anticipate not only that a perpendicular chord will be drawn next, but also where it will be drawn.

It may be useful to draw learners' attention to the possibility of anticipation by asking whether they are aware of anticipating what comes next. If learners are not anticipating but merely waiting for what comes next, then they are not taking full initiative and responsibility as learners. They are attending in the sense of 'being present physically' but not attending in the sense of paying as full attention as is possible to what is happening. Anticipation has been highlighted by Boero (2001), and is one aspect of a conjecturing atmosphere or classroom ethos.

Conjecturing Atmosphere & Ethos

In a conjecturing atmosphere, everything said is treated as a conjecture to be tested and probably modified. Those who are certain, who 'know that they know' choose to listen and question more, while those who are tentative, take more risks by trying to articulate their current thinking. It is the attempt to articulate which often helps clarify the concept.

Legrand (1993) promotes the use of *scientific debates* in which learners discuss, negotiate meaning, try to offer counter-examples and examples in order to test what is said and to try to prove or refute conjectures as they arise. This is mathematical thinking in action. Once there are competing conjectures, mathematical thinking really gets going as people try to refute or support their conjecture. The most important feature, unlike political debates, is that everyone is eager to modify their conjecture to take account of what others offer. Mathematically justifiable assertions usually take many modifications before they settle down.

Back to the tasks

Once the diagram of a unit square has been augmented by perpendicular chords, the calculations in the task can be depicted as areas. But to do this requires discerning the edges of the square as having been split into two parts. Further, the two parts are related, because the sum of their lengths is the side of the square and hence one unit. While this discernment may seem trivial, there are many situations in which what seems obvious to the author proves to be opaque to learners.

There is of course a vital shift of attention in seeing both numbers (the square and the unsquared) which are to be added together, as areas. It may take a while to realise the significance of the square having unit lengths. This feature that what seems like a length (one of the chosen numbers) can also be seen as one times that number and hence as an area is yet another example of the power of mathematical notation which emerges from ambiguity, from multiple interpretations of the same symbols.

The areas depicted by the square of the larger plus one times the smaller, and the square of the smaller plus one times the larger are the same, looked at as broken into two parts in two different ways. Thus the two calculations both calculate the same area, and so must be the same.

> When people meet something new (a new diagram, a new idea, a new concept) thir attention is often on the whole, undifferentiated. To make sense they need to discern features, aspects, parts, and to perceive relationships between those parts. None of this is automatic, though it makes use of natural powers that all humans possess, and it follows natural patterns in human sense-making. It behoves the teacher to make sure that learners are discerning what the teacher is discerning, fore-grounding and back-grounding appropriately, so that the intended relationships can be perceived. It is a further shift, again quite subtle, to move from being aware of relationships amongst parts in a particular (such as a diagram or expression) and being aware of those relationships as properties which can be held, or not held, by objects. Thus seeing the two inside squares in one diagram does not guarantee that the learner is aware that the chords could be anywhere along the edge making two squares, and not just in the particular place where they are shown.
>
> Paying attention to relationships and then shifting to being aware of those relationships as putative properties which objects may or may bnot have involves a shift in the structure of how learners are attending to the object (the diagram in this case). The van Hieles (1986) studied these shifts in the structure of attention in the form of levels of geometric thinking, but they are more usefully thought of as ways of attending. It is not a matter of 'being able' or 'not being able' to property-make, as Pirie & Kieren (1994) call it, but rather a matter of rapid changes in how we attend moment by moment. If learners and teacher are not attending in the same way, then there is unlikely to be effective communication between them.

Task Interlude

Many people, when told that I have two numbers which sum to one, or when asked to construct two such numbers, restrict themselves to numbers between 0 and 1. A few choose the extreme pair, 0 and 1, some choose 1/2 and 1/2 (but then find that the calculation has no bite to it!). The sorts of numbers which first come to mind display the person's immediate example-space (Watson & Mason 2002, 2005).

> We have found that asking people to construct mathematical objects, even quite simple ones, reveals a good deal about the range of things of which they are, at the time of asking, aware and confident. If learners have a narrow sense of objects being talked about by the teacher or text, they are unlikely to appreciate the full force of what is being said. So we use tasks such as the following to prompt learners to extend their example-space, and to experience making a choice from a range of possibilities rather than jumping at the first examples which comes to mind.

Here is an example of a creative construction task.

Task 3: Another & Another

Write down two numbers that sum to one.

Write down another pair.

Write down another pair.

> Here we are exploiting the well known psychological impact of 'three repetitions': some people are content with three simple examples, but many find that by the third, they feel like challenging themselves, becoming more extreme, more 'interesting'. Creativity is released. By discussing choices that people make, others become aware of choices they hadn't thought of. On another occasion they may then find themselves being more adventurous (Watson & Mason 1998, Bills, Bills, Watson & Mason 2005]

In the case of Task 3, there are of course possibilities to use fractions, decimals, and negative numbers (perhaps even complex numbers, though 'larger' and 'smaller' would have to be replaced by some other means of identifying one number of the pair).

> The type of number or number representation is one dimension-of-possible-variation connected with pairs of numbers which sum to one. Within that dimension there is a perceived range-of-permissible-choice (most people don't think immediately of extremely large or extremely small numbers). By inviting people to construct objects and then to discuss those choices, the extend their awareness of what could be changed, and the extent of permissible change. For Marton (Marton & Booth 1997), this is one way of specifying 'learning': extending awareness of dimensions-of-possible-variation and their associated ranges-of-permissible-change.

Having considered types of numbers, what other features of the original task could be altered and still have a task of the same type?

Task 4: DofPV

What features of the depiction (and hence of the original task) could be varied and still the same flavour remain?

If the diagram is dominant, then you might think of adjusting one of the chords that divides the square into regions. Seeing or imaging the chords moving might easily raise the possibility that the two chords might become independent of each other. Before pursuing that option, other features which might change include changing the numbers (summing to one could be summing to something else), having three or more numbers summing to one, changing the operation from 'sum' to something else. Other, more remote possibilities include changing the outline figure from a square, say to a rectangle or even to some other shape, and even allowing the chords to be at angles other than parallel to the sides of the square (or other shape). The use of numbers might be changed to other mathematical objects which can be combined in some way.

For example, matrices are sufficiently number-like, despite not commuting under multiplication. Perhaps integrals of polynomials over a fixed interval, or the integral of a particular function over different intervals can replace 'number' (see Task Y). Finally, and this will be pursued later, what about moving to higher dimensions (Task Z)?

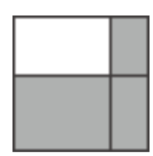
> There are close links with the powerful device of asking 'what if not…?' type questions, and using the tactic promulgated by Brown and Walter (1983) of reading out an assertion and stressing one particular word (for example 'the sum of two numbers is **one …**') which almost automatically invites the question 'why *one*, why not something else?'). The extra power afforded by the notion of dimensions-of-possible-variation is that it suggests some structure amongst the features, and with corresponding ranges-of-permissible-change. Different people may be aware of different ranges. But it does even more. It extends the tactic from a way of opening up tasks to exploration, to a core feature of teacher's informal assessment of learners awareness, providing access to learners' example-spaces. Furthermore, it directs teacher attention to what it means to appreciate-understand a concept, for a concept is appreciated to the extent that the learner has to hand not just examples, but awareness of what it is about those examples which is exemplary. That is, what features of the examples can change, and to what extent, and still the object remains an example of the concept.

Another way of directly focusing attention on a dimension-of-possible-variation is to display an animation (electronic or mental) of the vertical and horizontal line segments moving back and forth, at first coordinated to produce the squares in the corners, then independently.

> Explicitly varying the pair of numbers which sum to one helps to remind learners that the diagram does not just depict a particular case, but rather, makes it possible to see the general through the particular (Whitehead 1932, Mason & Pimm 1984). A reasonable pedagogic conjecture is that unless they are alerted to it, many learners will focus on the particular and at best only dimly be aware that a diagram is intended to 'speak the general'.
>
> Note however that if the teacher is always the one to invoke mental imagery, to invite anticipation and conjecture, to promote consideration of dimensions-of-possible-variation and ranges-of-permissible-change, then learners are liable to become dependent on the teacher for such prompts. Once a pattern of questioning and prompting has been established, scaffolding the focusing of attention in specific ways (Seeley Brown *et al* 1989), it is important to begin fading the intervention through, for example, the use of meta-cognitive questions ("what question am I going to ask you?"; "what question did I ask you earlier/yesterday?") so that learners are encouraged and supported both in becoming aware of the questions and prompts the teacher has been using, and taking the initiative to ask them of themselves for themselves. Scaffolding and subsequent fading can usefully be thought of in terms of a spectrum of intervention densities (Floyd *et al* 1985), from the directive to the prompted to the spontaneously used by learners themselves.

Task 5: Moving Chords

In the diagram, the chords remain parallel to the sides of the square, but they cut the sides differently. Read the shaded area in two different ways to obtain a (more complex) version of the original task.



> Here an anticipated affordance of the task is an opportunity to work on moving from material object (diagram) to mental image (of a generality seen through the particular)

to symbolic expression (verbal or symbolic). Often it takes several attempts to reach an acceptably succinct statement that everyone agrees to in a group working together.

One version might be that the given two pairs of numbers both summing to one, the product of the larger of each pair added to the smaller from one pair is the same as the product of the smaller of each pair added to the larger of the other pair. Those algebraically confident can express this in symbols.

The '=' sign is often taken by learners, especially those growing up in the calculator age, to mean 'do the calculation', or 'get the answer'. Thus whereas they accept the statement that 3 + 4 = 7, they balk at 7 = 3 + 4. Here the term 'is the same as' means 'equal in value', as distinct from 'equal in appearance'.

Task 5a: More Moving Chords

What role is played by the terms 'larger' and 'smaller' in stating an equality in Task 5?

In fact, it seems that the two terms are only being used to identify numbers in from the pairs. The statement could equally well take the form 'the product of one number from each pair added to the other number from one pair is the same as the product of the other numbers from each pair added to the other number from the other pair'. However a good grasp of the way English refers to objects using 'other' is essential!! Again there is an opportunity to experience discerning of objects referred to by the 'abstract words', to experience relating these to correspond to what is written, and to see these relations as properties of pairs of numbers.

One thing among many that this task highlights is that having worked mentally to identify objects, the language emerges quite easily, but encountering the language without the images makes decoding and interpreting quite a challenge!

Structure Informs Meaning

One of the powers of symbols over diagrams of course is that the full range-of-permissible-change is any number (or even number-like object), whereas the diagram suggests an implicit permissible-range-of-change for the pairs of numbers as being between zero and one. It is not difficult to extend the diagram to include negatives, but some additional 'rules' have to be considered and agreed.

Task 6: Depicting Negatives

By allowing the chords to escape the confines of the square, introduce 'signed-distances' so as to be able to draw and read diagrams with negative lengths. What are the rules for calculating areas using signed lengths, so that the calculations are consistent with the arithmetic?

This is a nice example of how the mathematical structure can be used to decide what the rules ought to be, so as to be able to use diagrams with negative lengths (see for example Sawyer 1959). Among other things, it becomes 'necessary' to require that the area formed by two negative segments be considered positive, whereas the area formed by a positive and a negative segment be considered negative. Extending meaning is a fundamental mathematical theme which recurs again and again. Here the interpretation of diagrams, seen as notation, is extended to encompass directed segments I order to admit negative numbers.

Changing the Sums

Task 7: Other Sums

How might the original statement of equality be adjusted to take account of the two numbers summing to S rather than to one? What about the two-pair version? Must the pairs sum to the same number?

An area diagram may be informative!

Changing the Operation

Task 8: Some Difference

Replace the word 'sum' in the original task with 'subtract'.

Is there a way of symbolically connecting the 'subtract' version with the 'sum' version?

What happens if 'subtract' is replaced by '(absolute value of) difference'?

As Polya (1962) suggested, asking yourself if you have seen something similar before which might be informative is a very useful question to ask before diving into calculations.

Task 9: Products

If two numbers have a product of one, is there a corresponding operation to replace squaring to produce an analogy to the original task?

Changing the Objects

Task 10: Integrals

Which do you think is larger,

$\left(\int_{-1}^{0}\left(x^2+x/6\right)dx\right)^2+\int_{0}^{1}\left(x^2+x/6\right)dx$ or $\left(\int_{0}^{1}\left(x^2+x/6\right)dx\right)^2+\int_{-1}^{0}\left(x^2+x/6\right)dx$?

What is it about the choices made that makes this particular example work?

What are some dimensions-of-possible-variation and corresponding ranges-of-permissible-change in those dimensions which preserve the setting as integration? For example, notice that the coefficient of x plays no role whatsoever, and that what really matters, in preserving the structure of the original task, is that the integral from –1 to 1 is equal to 1.

Task 10 variant: Bury the Bone

What is the same, and what is different about, the two statements

$x^2 + 1 - x = (1 - x)^2 + x$

and

$$\left(\int_{-1}^{0}\left(5x^4+4x^3-1/2\right)dx\right)^2+\int_{0}^{1}\left(5x^4+4x^3-1/2\right)dx=\left(\int_{0}^{1}\left(5x^4+4x^3-1/2\right)dx\right)^2+\int_{-1}^{0}\left(5x^4+4x^3-1/2\right)dx$$

Make up even more obscure ways of hiding the basic identity within mathematical calculations (what I like to call 'burying the bone' (Watson & Mason 2005)

Being invited to 'bury the bone' is more than a technical exercise. It invites learners to experience for themselves how tasks can be complexified, thereby alerting them to a range-of-permissible-change, and thereby increasing the likelihood that they will recognise the essential task in some complex version appearing on an examination. It also reveals the extent of their awareness of dimensions-of-possible-variation and corresponding ranges-of-permissible-change as a specification of their sense of the associated task-domain and various concepts involved. For example, some learners might not think to change the limits of integration, or to change the degree of the polynomial (or to reach beyond polynomials).

Task 11: Matrices

What additional constraint is required when using matrices (say two by two matrices, with 'one' replaced by 'the identity matrix') in order to make the statement 'given two pairs of matrices which both sum to the identity matrix, the product of one from each pair added to the other of one pair is equal to the product of the other from each pair added to the other of the other pair' correct?

Because matrix multiplication is not commutative, care must be taken to specify the order in which products are taken. Furthermore, having replaced 'number' by 'matrix' in the original task, area diagrams are no longer sufficient to justify the corresponding assertion, because the matrices do not refer to the area in any way. But the matrices do conform to the axioms to which algebra conforms, except of course for commutativity.

Into Higher Dimensions

Task 10: 3D

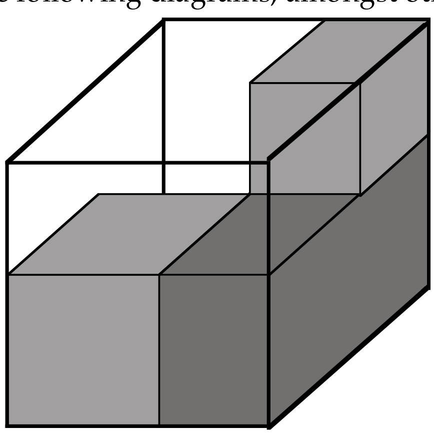
What might a three-dimensional version of the original task look like?

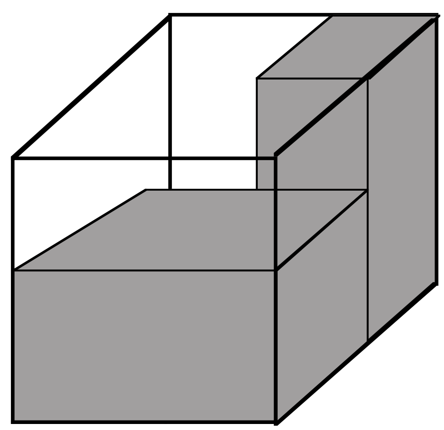
Of course having considered various dimensions-of-possible-variation in the original task, it is possible to generalise many different aspects at the same time, but for the purposes of these notes it seems sensible to work on one dimension-of-possible-variation at a time! The tasks which follow will shift between a single pair and multiple pairs, but always summing to one.

The focus here is to locate a structure within the first task which can be extended into three (and by implication, more) dimensions. What then are salient features of the original?

Stress could be placed on the comparison of two expressions; that expressions were the sum of areas (generalisable to volumes); and on the fact that one expression involved a square (generalisable to a cube) with something added, which in the diagram was a rectangle (generalisable to a cuboid) with one edge the length of the square. Having analysed the original for generalisable features, imagining or drawing a cube partitioned into eight regions, it is necessary to find a way to select certain regions, the sum of whose volumes can be interpreted in two different ways.

Starting from a cube in one corner, and aiming for a cube in the diagonally opposite corner, yields the following diagrams, amongst other possibilities.





Translating into symbols produces $x^3 + x(1-x) + (1-x)^3 = x^2 + (1-x)^2$.

Task 11: Three pairs in 3D

Extend the equality in the previous task to three different pairs each summing to 1.

Extend further to the pairs each having a different sum.

Notice how it is possible to be quite analytical, to allow the symbols to do most of the work in making the generalisations. For example, a cube becomes a product, and hidden 1s become symbols for the corresponding sum.

Task 11: Beyond 3D

What might associated statements look like in higher dimensions?

Thinking in four and more dimensions is not easy, so it is useful to find some symbolic approach which, derived from intuition, can then drive intuition beyond the easily imagined. One way of thinking is to see the unit cube as partitioned into 8 cuboids by the chordal-planes corresponding to one (or three) pairs summing to one along the edges. This yields a cube whose vertices correspond o the eight cuboids. Give these vertices coordinates (0, 0, 0), (1, 0, 0), … (1, 1, 1), and choose a path along the edges of this cube: for example, (0, 0, 0), (0, 0, 1), (0, 1, 1), (1, 1, 1). Note that only one coordinate can change at each step since the path must be along the edges of the cube. Such a path corresponds to choices of cuboids to make up a 'shaded region' in the original unit cube. The two ways of shading volumes of cuboids correspond to partitioning the path into distinct pairs of adjacent vertices. Thus

(0, 0, 0); [(0, 0, 1), (0, 1, 1))]; (1, 1, 1) and [(0, 0, 0), (0, 0, 1)]; [(0, 1, 1), (1, 1, 1)]

code the two choices of shaded volumes in the figures.

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