# STUDIES IN ALGEBRAIC THINKING NO 1

# UP & DOWN SUMS
# OR
# WHY ARE WE DOING THIS, MISS?
# (INNER & OUTER ASPECTS OF TASKS)

John Mason Draft August 2006

Many learners act as if they believe that their job at school is to obtain the answers to many of the mathematical tasks they are set. But if the purpose of school were simply for learners to complete the tasks set for them by teachers, the whole process of education would be much simpler than it proves to be. Getting the answers to mathematical problems cannot be what is important, if only because the answers are already known, in most cases. What *is* of importance is the process of obtaining an answer. To learn is not to finish tasks, but to have one's sensitivities to notice and to choose pertinent actions, enhanced; to become aware not just of relationships between particular entities, but to develop a disposition to seek out such relationships; not just to perceive properties, but to make use of properties to draw conclusions; and in the case of mathematics, to reason on the basis of explicitly announced properties. So there must be more to tasks than the overt or outer aspect: what learners are asked to do. Dick Tahta (1981) drew attention to the distinction between inner and outer aspects of tasks, and in this article I want to illustrate developments of his distinction, while at the same time drawing on other aspects of learning and teaching mathematics.

## Outer & Inner

My aim is to use a few related tasks to indicate how teachers can make pedagogic and didactic (that is, topic specific) choices informed by psychological and socio-cultural constructs. I proceed by inviting the reader to engage in some tasks and to reflect upon what is noticed as a result.

### Up & Down Sums (1)

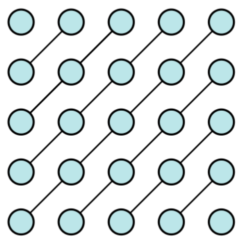
In a live session based on this task, participants would be exposed to each line in turn, with pauses after the first entity, after the equals sign, and at the end of each equation:

Generalise and justify:

$1 = 1^2$ $\quad\quad$ $1 + 2 + 1 = 2^2$ $\quad\quad$ $1 + 2 + 3 + 2 + 1 = 3^2$

*Comments*

This pattern is very familiar. It can be seen as an application of Gauss's method for adding an arithmetic progression (to be discussed later) and it can be read from a diagram such as the one shown here corresponding to the fifth equation in the sequence.



In order to read the diagram appropriately it is necessary to discern relevant details (diagonals), seek out relationships (diagonals as sequence, overall forming a square), and to perceive these relationships as properties applying more generally to any term in the sequence.

The outer task is to quite explicit: generalise and justify. To succeed you have to have some idea of what these words might mean, gleaned from previous experience of their use in relation to other tasks. More particularly, it means locating some pattern, some structure, and 'continuing it', and then finding a way to justify the conjecture, not simply by pointing to how you came to see the pattern, but why the equation must always hold.

It may be worth mentioning here that the origins of the word *theorem* lie in 'seeing', so that a proof could be thought of as 'arranging that others see what you see', where 'see' is used metaphorically and comprehensively to mean 'perceive', 'become aware of', 'appreciate relationships'.

## With and Across the Grain

To promote generalisation and comprehension, a tactic such as 'Say What You See', in which participants are invited to point to any details they notice in the equations (or any other stimulus) can often generate a group-wide awareness which is more complex than that observed initially by any individual. If used frequently enough in situations where it does indeed lead to enhanced and enriched awareness, it is possible that learners will begin to internalise the tactic as something they can usefully do for themselves.

To detect and continue a pattern, the individual has to discern aspects which are invariant, and aspects which are changing in some principled manner. Watson (2000) uses the image of cutting *with and across the grain* as a reminder that detecting and continuing a sequential pattern is at best pattern-spotting. The pedagogic issue is what you do with and about the pattern you detect.

Did you check that it accounts for (fits) the data accumulated so far?

Did you check that any assertions (here equations) are actually correct?

Did you try running the sequence backwards?

Using the image of splitting and cross-cutting wood, going *with* the grain is easy: it splits along the fibres, along the invariances and the systematic variation. For example here, each line starts with 1, and ends with the line-number squared; just before the equals sign there is also a 1; the numbers go up in sequence and then down again. And so on. But what matters most is going *across* the grain: seeing what the equation actually says mathematically. This is how you make contact with mathematical structure. Sometimes it helps to Watch What You Do to continue the pattern, because your body may be aware of relationships which are not yet articulated by the intellect.

## Invariance & Specialising

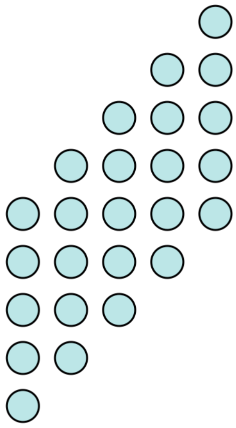
The 'inner task' includes making use of your power to detect invariance in the midst of change, to imagine and to express what you imagine (here in the form of the next, or even a general equation which follows the same pattern), to try other instances yourself, and to use this experience to try to articulate permissible variation and relative invariance. Paulo Boero (2001) drew attention to the vital role of anticipation in mathematics, particularly in algebra. You do not embark on random calculations; rather, you anticipate something and then check it out. This often involves specialising: trying particular cases to try to get a sense of what might be happening in general, as George Polya (1962) recommended.

The purpose of specialising, as Polya saw it, is to gain experience of the underlying structure. Thus doing a few examples is not simply to get some answers, nor even to get some answers so that a pattern can be found in those answers, what Dave Hewitt (1992) called *train spotting*. Rather, the purpose of specialising is to pay attention to *how* you go about checking, in case it reveals something. Thisn is where Watch What You Do can be so helpful. For example, you could try out the instance

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 6 + 5 + 4 + 3 + 2 + 1$$

by adding up from the left. You will of course get the answer. But will it be informative? If instead you gaze for a while at the terms, you may notice something. For example, arranging them one way makes it easier to add up:

$1 + 2 + 3 + 4 + 5 + 6 + 7$
$6 + 5 + 4 + 3 + 2 + 1$

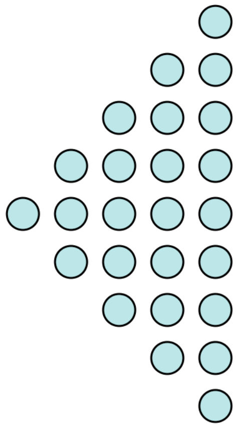


which reveals that each column adds up to 7, so there are seven 7s.

This makes sense of the $7^2$ by revealing different roles for the two 7s: one as a 'lots of' and the other as a 'sum'.

It also suggests a way of 'seeing' other instances of the sequence of equations, and even a way to generalise. The diagram can be interpreted as displaying the fifth equation in the sequence, with the rows giving the terms, and the columns an efficient way of calculating the sum. By looking 'through the particular to the general', paying attention to (stressing) relationships rather than specific numbers of elements (in this case), those relationships can be perceived as properties which hold for any equation in the sequence, and any corresponding diagram.

Another way to organise the summation is as



$1 + 2 + 3 + 4 + 5 + 6 + 7$
$\quad 1 + 2 + 3 + 4 + 5 + 6$

$= \quad 1 + 3 + 5 + 7 + 9 + 11 + 13$

Now the columns add up to 1, 3, 5, 7, 9, 11, 13, respectively, which means the total is the sum of the first seven odd numbers.

If you are on familiar territory, you may recognise this as $7^2$. If you have not previously encountered this fact, then you would have another generalisation concerning the sum of consecutive odd numbers to express as a conjecture, and justify. The diagram depicts the fifth equation, with the rows giving the original sum, and the columns the sum of consecutive odd numbers. This diagram can be achieved by transforming the previous diagram, and vice versa, and so can be seen together as displaying the fact that the sum is a square of the equation number.

## Alternative Presentations

The initial task could be presented in the form of diagrams rather than arithmetic, with an invitation again to generalise and to express in arithmetical form. The diagrams also offer a way of seeing how transforming the layout can make the counting much easier. To rehearse the experience of going with and across the grain, it would be useful at this point to draw a sequence of diagrams corresponding to each of the two displays above, and then to go across the grain by describing to yourself how the transformation of the diagram works in general.

Another inner aspect of the task is to experience multiple ways of perceiving and thinking about something. Within mathematics it can often make things easier and clearer to pause and look for an alternative way of perceiving or re-presenting something. More broadly, in a world in which recognising and valuing diversity of viewpoint is vital to social harmony, mathematics lessons can illustrate this theme without anyone being very explicit about it. an

Another inner aspect is the personal propensities which may come to the fore, such as a tendency to dive in as soon as an action comes to mind, without pausing to gaze in order to let the whole suggest several possible actions. Thus, through working on mathematics, learners can be covertly learning about becoming aware of choices that they can make not only in mathematics lessons, but in life more generally. Helen Drury (private communication) calls this *affective* generalisation, which goes hand in hand with *enactive* generalisations which are body-led and which lead to *cognitive* generalisations through the tactics previously mentioned: Say What You See and Watch What You Do.

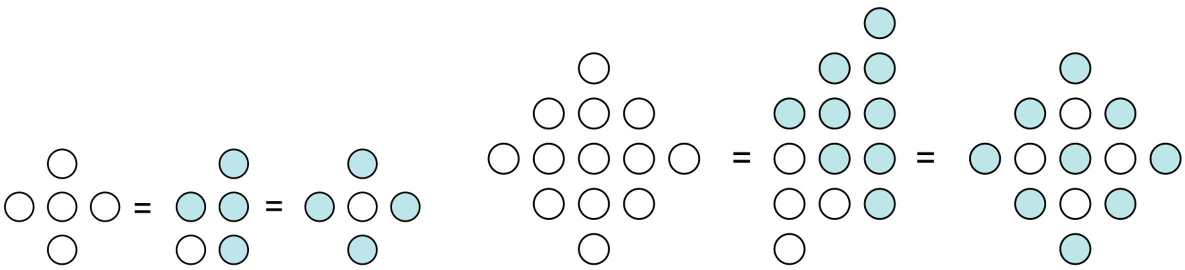
All of this analysis has emerged from one single task. But 'seeing' ideas go by is rarely enough for them to be integrated into functioning or to inform future behaviour. It is useful to think in terms of *seeing* ideas go by, and only after gaining more and more *experience* expecting to gain *mastery* of the idea. This trio of *See–Experience–Master* provides a useful framework for informing pedagogic choices: giving learners long enough with multiple experiences before expecting fluency and facility.

## More Experience

Here is an opportunity to notice more specifically some of the things you do in order to generalise.

### Up and Down Sequence (2)

Two instances of arithmetical facts are shown here (the second and the fifth of a sequence). Do they generalise?



$1 + 3 + 1 = 2\text{x}2 + 1 = 1^2 + 2^2$ $1 + 3 + 5 + 3 + 1 = 3\text{x}4 + 1 = 2^2 + 3^2$

*Comment*

Notice that the format of this task is different, which suggests that there are many different ways of posing similar problems. Here it is necessary to imagine other similar expressions, and to look for relationships between the fact that these are the fourth and seventh terms, and the role played by those numbers in the corresponding expressions.

Notice also the effect of the double equals sign: you can work on the left hand side in two ways to reveal both of the compact expressions, or you can relate one compact expression to the other in your generalisation. Did you start with drawing more pictures, or did you go straight to another arithmetical example?

An excellent aid to seeing a generalisation is to Watch What You Do as you make a copy for yourself of the given data, and as you work out what you think might be the next or some subsequent instance. Very often your body is or becomes aware of pattern before your intellect has woken up to it. It does not take much to notice the pair of consecutive squares and the rising and falling (up and down) pattern.

Again, the outer task is to locate some structure which you can generalise, while the inner task includes enriching your awareness of tactics that you can employ, such as how your body participates in the detection and expression of pattern, and how that pattern is connected with articulating what is changing (and how it is changing) and what is staying invariant. Invariance is not just about being fixed, for relationships which hold for more than one instance can be seen as properties which 'hold' in general, that is, are invariant. One relationship here is between the pair of consecutive squares and their relation to the position number of the particular instance in the sequence. Another is the length of the sequence of terms on the left hand side of any one equation and the instance number. Notice that there is more to be seen in a more complicated example, such as the fifth or the seventh, than in early terms in the sequence.

Another inner aspect of the task is the opportunity to become aware of your propensity to work with symbols or with diagrams. People who are at home with symbols could extend and develop their powers by spending some time to looking for a diagrammatic version of the equations analogous to the diagram used with the first task; people more at home with diagrams could extend and develop their powers by working on developing symbolic expressions. Indeed, it might even be useful to pose the task in terms of 'reading a diagram' as an arithmetic statement, and then generalising, especially for learners who are still gaining fluency in the expression of generality in symbols.

### Up and Down Sequence (3)

The following are the fourth and seventh terms of a sequence of arithmetical statements. What might others be?

$1 + 4 + 7 + 10 + 7 + 4 + 1 = 4 \times 8 + 2 = 4^2 + 2 \times 3^2$

$1 + 4 + 7 + 10 + 13 + 16 + 19 + 16 + 13 + 10 + 7 + 4 + 1 = 7 \times 17 + 2 = 7^2 + 2 \times 6^2$

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*Comment*

Some people approach a task like this by 'filling out' intermediate objects in the sequence so as to get a clearer sense of the sequence and its pattern. In looking for the role of the instance number in the expressions it is perfectly possible to gaze at the two instances, and 'see through' the particular to a general statement. It is however vital not to allow attention to be diverted by the presence of all of the 4s and the 7s in relation to the fourth and seventh equation in a sequence. Two of the 4s and two of the 7s are irrelevant to the overall pattern, but two of the 4s and two of the 7s are highly pertinent, contributing to a structural relationship which, when re-viewed as a potential property, aids the expression of generality.

## Learning from experience

One thing we don't often learn from experience
is that we don't often learn from experience alone.

In order to promote learning from experience, it is very helpful to 'go across the grain' by drawing back and trying to make sense of what has happened, what structure has been revealed, what inner aspects have been noticed while working on the task(s). It can also be very helpful to involve other people so that there is an impetus to try to articulate clearly and succinctly, rather than simply pausing momentarily and allowing a few thoughts to drift by on your mental screen. If no-one is available, then it can be helpful to try to describe what you have noticed to a pet, even, in extremis, to a virtual pet goldfish!

Conjecturing is a core mathematical activity. But as Polya pointed out, it really helps to externalise them, to express them as if to someone who will try to make sense of them, and this applies both to mathematical and to pedagogic or didactic conjectures. Indeed, this article is an example of this very action, in which I as author am trying to bring to articulation and to coordinate a variety of different awarenesses of a pedagogic, didactic, and mathematical nature.

## Reflection

The sequence of three tasks (Up & Down Sequence 1, 2, 3) can also be seen as *going with the grain*, following a developing pattern. Stopping and juxtaposing corresponding terms from each of the three provides an opportunity to *go across the grain*, to try to reveal structure which might not have been evident in any one of the tasks. Here then is a reflective, 'cutting across the grain' task.

### Up & Down Sequences

Here are the fourth terms of the sequences from the previous three tasks:

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1 + 2 + 3 + 4 + 3 + 2 + 1 = $4^2$

1 + 3 + 5 + 7 + 5 + 3 + 1 = 4x6 + 1 = $4^2 + 3^2$

1 + 4 + 7 + 10 + 7 + 4 + 1 = 4x8 + 2 = $4^2 + 2 \times 3^2$

What is the same and what is different about these three equations? What is invariant and what is changing? Generalise!

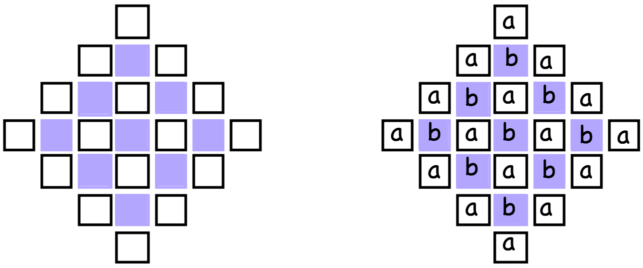
*Comment*

You may have felt it useful in the first line to recast $4^2$ as 4x4 + 0 in order to reveal a clear 'with the grain' structure in the right hand sides.

## Linking and Re-Presenting

The first Up & Down sequence arose from a pattern of dots in a square, and the second could be presented as two embedded squares. Can these be generalised so as to present not only the third Up & Down sequence, but generalisations?

The following presentation occurred to me when preparing for a workshop:




On the left is the array of two interwoven squares. Counting the squares in each row (or in each column) gives Up & Down sequence 2; replacing the contents of each square with constants $a$ and $b$ as shown on the right and then adding the contents for each row (or each column) suggests the general Up & Down sequence

$a$, $2a + b$, $3a + 2b$, … $(n - 1)a + (n - 2)b$ $na + (n - 1)b$ $(n - 1)a + (n - 2)b$, … $a$

Based on the arithmetic progression with first term $a$ and constant difference $b + a$. The first Up & Down sequence is generated by $a = 1$ and $b = 0$. The reason for the arithmetic sequence is seen by noting that each of the rows adds and extra cell of each type (until the middle, then decreases by the same).

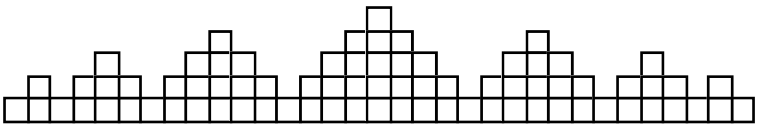
## Further Extensions

### Up & Down Up & Down

What about a sequence which uses up and down in an up and down type fashion:

1, 2, 1, 2, 3, 2, 1, 2, 3, 4, 3, 2, 1, 2, 3, 4, 5, 4, 3, 2, 1, 2, 3, 4, ,3 ,2 1, 2, 3, 2, 1 ,2, 1

which can perhaps be appreciated more easily visually:



Now the techniques developed so far may need to be extended, and perhaps there are connections with sums of squares.

## DofPV and RofPCh

One of the features which varies between the three tasks is the difference between the terms, which in each case is constant. The fact that it has been systematically varied might suggest a generalisation in which the difference is expressed as a parameter such as $d$. Changing the common difference changes the details of the right hand side, but can still be presented as a product plus a constant or as a combination of consecutive squares. The three presentations (one symbolic and two diagrammatic) could be exploited to inform if not provide reasoning which justifies why the corresponding conjectures remain valid.

The constant difference is an example of a *dimension of possible variation*: something that can be changed without affecting the essence of the particular. With each DofPV there is a corresponding range of permissible change(RofPCh): what values can this parameter take? For example, could the common difference be a fraction, a decimal, or negative? Often when this question comes to mind, you find that you have had a very limited sense of the possibilities. The very fact of discerning the difference *as* constant, and articulating that, brings to light the possibility that it can be different constants, and might even suggest that perhaps the difference could change from being constant to some other systematically changing difference, or even ratio.

Thinking in terms of DofPV is a stimulus to going across the grain in the search for structure, for appreciation and understanding. Asking yourself what features or aspects could be changed while leaving the same idea intact can lead to unexpected generalisations, and the effect is to produce a sense of satisfaction, for something particular has been extended to something more encompassing.

These notions apply not just to parameters but to methods: one way to approach the first sequence was to reorganise the terms so that there was a constant sum. This too can be seen as a DofPV. The method works in the other two sequences as well, so perhaps it is something which could be used in other situations. Indeed this is the basic idea attributed to the young Gauss when asked to add up the numbers from 1 to 100 (Hayes 2006). A related DofPV arises by becoming aware of how, in the second arrangement, the sum of the odd numbers in the first sequence was revealed., Perhaps it could be used in the second and third sequences to get an analogue version, even though it may be more awkward to express.

For example, rearranging the second sequence

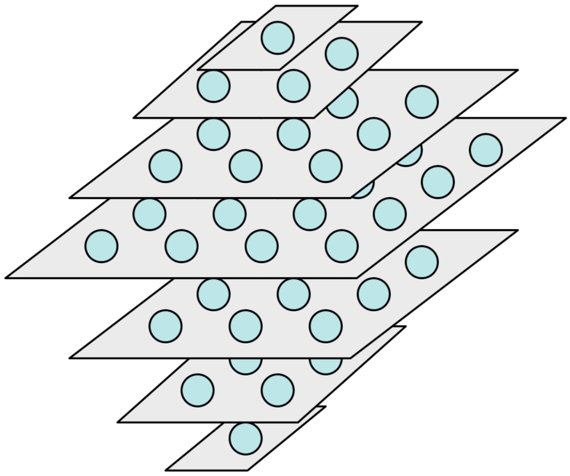
$$\begin{array}{l} 1 + 3 + 5 + 7 \\ \quad + 1 + 3 + 5 \end{array}$$

gives multiples of 4 in each column except the first, so there is an arithmetical fact that one more than the sum of consecutive multiples of four (starting from 4x1) is the sum of two consecutive squares. Rearranging the third sequence in a similar fashion yields the arithmetical fact that one more than the sum of consecutive numbers of the form $6n - 1$ (starting from 5) is the sum of double a square plus the square below. Note how what is stated here is a generalisation achieved by seeing through one particular instance to a generality whose status remains a conjecture until it has actually been verified algebraically or diagrammatically.

There is another dimension of possible variation which has not as yet been exploited. The first term in every case has been 1. Could it be changed to something else and still a general formula achieved? The search is on for a way to display this more general situation as a constant plus a product, or as a combination of two squares.

What about the number of terms in an up & down sequence? The generalisations being worked on have treated it as a DofPV, but what is its RofPCh? Could it be a half? (It is not at all clear what this might mean). Could it be negative? (Not really in this instance). What about the first term: what sort of a number could it be (any number at all: integer, fraction, real, complex).

Thinking diagrammatically, is there some way to extend these up & down sequences into three dimensions, for example, by looking at an up and down sum of consecutive squares in layers:



### Characterising

Whenever you have a result, such as that $1 + 2 + \ldots + n + (n-1) + \ldots + 1 = n^2$, you can ask yourself two related questions: if someone gives me a number, could it be the answer to a similar question. Here it is plain that the answer is always a perfect square, and given any positive integer perfect square you can produce an up & down sequence (using a difference of 1) whose sum is that square. So the possible answers to up & down sequences with difference 1 have been characterised. The same can be done for other constant differences, for allowing the first term to be different, and for the whole class of such sequences. Thus, given a number $N$, in how many ways can it be represented as an up & down constant difference sequence?

Characterising is a special form of the mathematical theme of *Doing & Undoing*: whenever you find you can 'do' something, ask yourself how you might undo it. For example, if you can find the sum of members of a class of sequences, can you, given a number, find a member of that class which would give that sum? If you can represent a sequence of arithmetical facts diagrammatically, can you start with a diagram and decode it as an instance of a collection of arithmetical (algebraic) facts? If you can solve a problem to get an answer, what other similar questions would give rise to that same answer, and what other numbers could be achieved as answers to similar questions.

### Didactic Objects

Pat Thompson (2002) coined the expression *didactic object* to refer to something such as physical apparatus, and animation or diagram, a collection of symbols, or indeed anything which is used as a focus of attention so as to promote interaction between learner, mathematics and (usually) teacher. Of course this interaction occurs within a milieu which has been generated by previous interactions and includes both explicit and implicit *ways of working* on and with mathematical objects. The ethos and atmosphere which grows up, the classroom rubric (Floyd *et al* 1981), the local practices established in the community (Winbourne & Watson 1998) all provide the ecological background for any specific interaction. One feature of the term *didactic object* is the ambiguity of *object* as a thing and as a goal. So the teacher, in presenting a didactic object such as the equations in one of more of these tasks, also has a didactic objective or aim, namely to engage learners in activity through which they are likely to

encounter one or more mathematical themes;

experience the use of and perhaps develop some of their mathematical powers;

become aware of one or more of their natural propensities and dispositions, with the chance to work against or at least reduce the influence of ones which are unhelpful, while accentuating and supporting ones that are helpful mathematically;

become aware of tactics which are used pedagogically by the teacher, but which they could use for themselves;

confirm and augment their sense of agency with respect to mathematics, and their sense of themselves as mathematical thinkers.

### Explicitness and Implicitness

Tahta (1981*)* pointed out that to try to make an inner aspect of a task explicit is to banish it to the exterior, the outer task. It removes the possibility for learners to come across that aspect for themselves, to 'make the connection'. It is perfectly reasonable to make the pedagogic choice to draw learner attention to some inner aspect of the task *after* there has been opportunity to gain the relevant experience, but if the experience is labelled and signalled in advance, then learners are likely to be on the lookout for it, and so fail to generate it themselves. This means they may not yet be in a position to generate it for themselves in the future. Remarks made during reflection about what learners might have noticed either fit with and so enrich their experience and the possibility of having that insight come to mind in the future, or else signal something to be looked out for in the future.

### So why <u>are</u> we doing this, Miss?

Knowing details of why you are doing something in an educational context can interfere with the educational aims, hence the importance and value of distinguishing inner aspects from the outer task. In order to appreciate what the tasks they are given might be for, learners may need to be reassured that they are encountering important mathematical themes, and to become aware of how they are developing their natural mathematical powers. The question may not necessarily signal an interest in applications or 'relevance to the world outside school'. It is usual for people to ask such questions not when they are being successful, but when they feel they are not coping. 'Why am I doing this?' does not usually come to mind in the midst of a flow. It comes to mind when there is considerable disruption and loss of confidence. What may matter most might be sensitivity to learners' desire for some story which goes beyond 'passing exams' or 'succeeding in life'. Offering them coherent articulations every so often which draw upon and make explicit their recognition of their use of their own powers, and the reoccurrence of persistent mathematical themes can provide the necessary reassurance.

### Follow up

More details of the mathematical powers, themes and pedagogical constructs introduced can be found in Mason & Johnston-Wilder (2006).

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