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FRAMEWORKS FOR LEARNING, TEACHING AND RESEARCH: THEORY AND PRACTICE

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Abstract

I have chosen to reverse the order of the key words in the conference title, and to interpolate a third term in order to fit with my view of the role and functioning of frameworks. I begin by introducing a framework for learning in which systematic variation can be used to provoke learners into becoming aware of mathematical structure. Structural Variation Grids have evolved over several years and I indicate some of the history of their development. I then use some frameworks for teaching based on Ference Marton's notion of variation, some based on George Polya's descriptions of mathematical thinking, some based on Jerome Bruner's three modes of re-presentation, and one based on my own work on the structure of attention, in order to provide theory-based justifications for pedagogical and didactic choices that the Structural Variation Grids afford. These frameworks can be used to enhance and enrich the learning potential of particular instances of grids, but also any other mathematical task in any mathematical topic. Like most frameworks for teaching, the ones I will use can be transformed into frameworks for learning through the process of scaffolding and fading (Brown *et al* 1989), itself a framework for teaching. In the final section I suggest why and how these frameworks work, and this includes a description of the methods used to justify the claims in this paper.

My aim is to exemplify what I think is at the heart of learning and of being taught, at the heart of professional development, and indeed at the heart of research as well, namely, the emergence and elucidation of informative frameworks as collections of related distinctions. I try to demonstrate and describe conditions which can make a framework become active for individuals, and I elaborate on what makes them effective for people for whom they are active. I end with some advice on how to cope with new frameworks when they are encountered.

Structural Variation Grids

Historical Context

Many years ago I experienced a lesson given by Laurinda Brown based on the function game (Banwell *et al* 1972, see also Rubenstein 2002). It was conducted entirely in silence to great effect. Participants were invited to conjecture the result of applying an unknown function to different inputs, based on examples provided by her at the beginning. Everything was done in silence, with sad or happy faces drawn according to whether the *keeper of the rule* agreed or disagreed with the conjecture. The one rule was that no-one was allowed to say what they thought the rule was. Those who thought they knew 'the rule' were encouraged to offer examples which would help others come to the same conjecture, and also to try to test and challenge their conjecture. Apart from the silence, the format has strong resonances with the game *Eleusis* described by Martin Gardner (1977; 2001 p504–512). Gardner observes that the rules provide an analogy with science, because nature never tells you whether your conjectured rule is correct.

I was stimulated to look for the first opportunity to try working in silence and it came in a lecture to 300 Open University students, in which I presented the first few terms of a sequence:

$2 + 2 = 2 \text{ x } 2 \qquad 3+1\frac{1}{2}=3\times1\frac{1}{2} \qquad 4+1\frac{1}{3}=4\times1\frac{1}{3} \qquad 5+1\frac{1}{4}=5\times1\frac{1}{4}.$

I paused at each equal sign, and at the end of each equation, in order to show that I was doing the calculations myself. I have since done this with thousands of people over many years. Each time, no matter who the people are, everyone seems to know what the next term will be even if they struggle with the arithmetic to check the validity of their conjecture. I have used many sequences like this, getting participants to re-present the first term in the format of the others, to go backwards into the negatives (starting with 0, then -1, -2, …), to use not just whole numbers but rationals (starting with ½ or ¾), irrationals (starting with $\sqrt{2}$ or $\sqrt[3]{7}$) and beyond, according to the sophistication of the audience. The main thrust is towards expressing the general equation, and then justifying it using algebra. Sequences like this can be used to provoke learners into wanting a way to manipulate generalities (letters), as well as a providing a source for appropriate rules for that manipulation: the rules of algebra as generalizations of the rules of arithmetic. This contrasts with algebra presented simply as rules for 'alphabet arithmetic'.

In 1998 I was asked by some teachers in Tunja Colombia to suggest how to work with learners on factoring when they did not have facility with or even belief that (-1) x (-1) = 1. My response was what I then called *Tunja Sequences* (Mason 1999, 2001) which used the same principle of a developing sequence of specific instances of a factored quadratic such as

$1^2 - 1^2 = (1 - 1)(1 + 1) \qquad 2^2 - 1 = (2 - 1)(2 + 1) \qquad 3^2 - 1 = (3 - 1)(3 + 1) \ldots$

Here learners could be expected to detect the pattern and to express it in general, verbally, and even algebraically. By being exposed to a number of such sequences derived from factored quadratics, learners could be expected to become adept at expressing and justifying generality (the heart, root and purpose of algebra). Having generalised, they can work out the rules for expanding brackets, and for factoring quadratics, simply by using their natural powers to detect what is changing and what is invariant.

Recently, while writing a book on the teaching of algebra (Mason *et al* 2005) I wanted to extend these Tunja sequences to allow a second parameter to vary, and thus was born Structural Variation Grids. Tom Button kindly provided me with a basic Flash template which I then modified to produce different Grids, some of which are described in the next section.

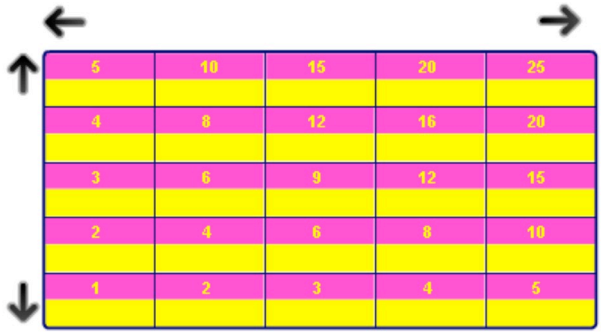
Using these grids briefly with teachers has already generated considerable excitement, and this is what has encouraged me to present them in this forum. I am confident that many of you will have done or used something similar at various times. The reason for presenting them here is to exhibit them as an exemplar of a pedagogic framework for learning.

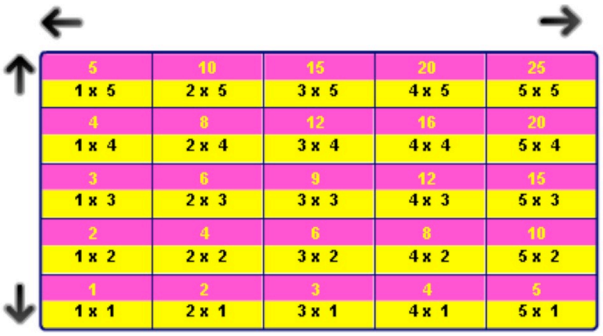
Sample Grids

Multiplication Table and (-1) x (-1)

Task 1: Number Grid

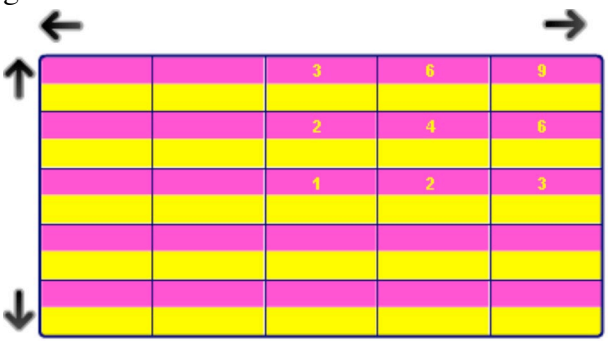
Say (to someone else or to yourself) what you see in the left-hand grid below.
Explain any patterns you see in the following grid. Conjecture and justify the entries in the square which is 20 cells to the right of the bottom left hand corner, and 13 cells up. Generalise.

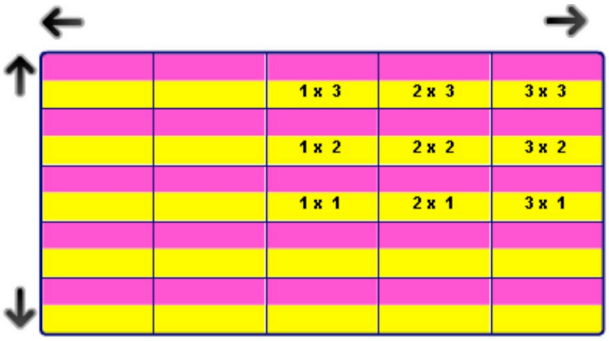




People rapidly come to the conclusion that the grid is actually both a mathematical use of a table format, and an integration of the 'multiplication tables', as shown in the grid on the right. In the Flash version of the grid you can click on any cell to reveal or obscure its contents. Thus you can start with the grid empty and reveal just a few cells. Note that it is also possible to start by revealing multiplications and then revealing the answer.

The arrows shift the window in the direction indicated on an infinite grid, so you can extend the patterns in any direction, but particularly down and to the left. You can then check for consistency between extending down then left, and left and then down starting from the shifted grids below.

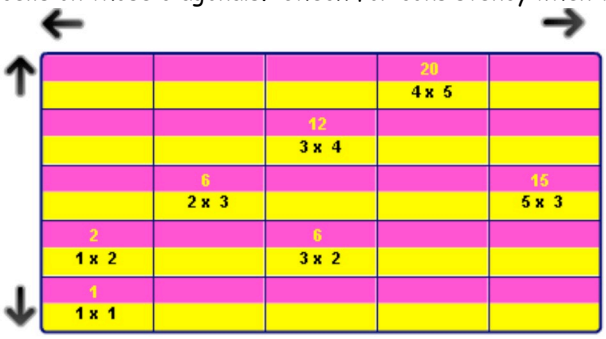


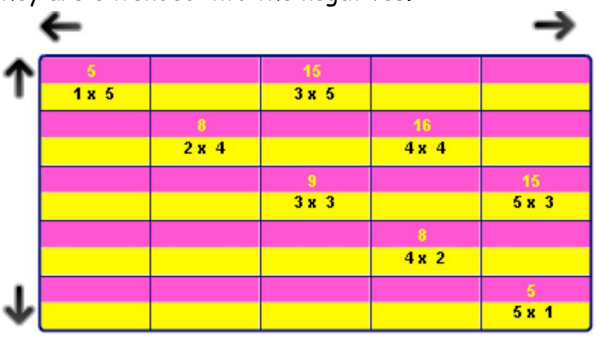


Task 2: Number grid (cont'd)

Get some people to extend the patterns downwards and then to the left; get others to extend it to the left and then downwards. Do you predict the same cell entries?

Describe the entries in a diagonal such as the ones shown below so that you can predict the entries in other cells on those diagonals. Check for consistency when they are extended into the negatives.





Note that again you can choose whether to display both rows in a cell, and the order and speed at which to reveal them. Note also the difference between predicting and checking a new entry, and colouring in patterns on an already extant grid. The empty cells invite learners to anticipate, to imagine and to conjecture, and to reason on the basis of properties they perceive.

Other Number Grids

Numerous variations on the same theme are of course possible. For younger children the entries in a cell could use other operations. In the sample cells here I have used cell (3, 2) as a generic example extracted from different grids.

3 + 2
5

3 – 2
1

3 ÷ 2
1.5

5(3 + 2)
25

5(3 – 2)
5

By predicting (anticipating) the entries in various cells, learner quickly get a sense of the structure of arithmetic by exercising their own powers of observation and generalisation. By locating all the cells with a specified entry they encounter mathematical structure.

Working with Zero

Another variant uses a similar grid structure with fractions in order to expose the reasons for outlawing certain operations using zero. Flash versions of these are in preparation.

Task 3: Zero in Fractions

What values would you expect where the question marks appear in the grid to the right? The … indicates an opportunity to extend and generalise. By following different patterns to predict values for the empty cells, explain why they must remain empty.

…	…	…	…	…
?	1 / 4	2 / 4	3 / 4	4 / 4
?	1 / 3	2 / 3	3 / 3	4 / 3
?	1 / 2	2 / 2	3 / 2	4 / 2
?	1 / 1	2 / 1	3 / 1	4 / 1

Extend the grid to the left and downwards.

Do the same for the following grid. It may be helpful to look for patterns in the format as presented, and then to perform the calculations and look for patterns in the answers. (Much more of the grid is shown in order to be clear about what is possible. Of course in a live presentation this would not be necessary.)

-1 / 1	-1/2 / 1	-1/3 / 1	…	?	…	1/3 / 1	1/2 / 1	1 / 1
-1 / 1/2	-1/2 / 1/2	-1/3 / 1/2	…	?	…	1/3 / 1/2	1/2 / 1/2	1 / 1/2
-1 / 1/3	-1/2 / 1/3	-1/3 / 1/3	…	?	…	1/3 / 1/3	1/2 / 1/3	1 / 1/3
…	…	…	…	?	…	…	…	…
…	…	…	…	?	…	…	…	…
-1 / -1/3	-1/2 / -1/3	-1/3 / -1/3	…	?	…	1/3 / -1/3	1/2 / -1/3	1 / -1/3
-1 / -1/2	-1/2 / -1/2	-1/3 / -1/2	…	?	…	1/3 / -1/2	1/2 / -1/2	1 / -1/2
-1 / -1	-1/2 / -1	-1/3 / - 1	…	?	…	1/3 / - 1	1/2 / -1	1 / -1

Notice the possibilities afforded for working on the division of fractions as well, by rehearsing a column, and then generalising it, then rehearsing a row and generalising that. You can even rehearse diagonals with different slopes, and generalize them. Learners are in this way exposed to the structure of division of fractions. Work on such grids would be presumed to take place in the context of other images such as the division of a rectangle into cells.

Task 4: Zero and Exponents

Extend the left-hand grid outwards. In how many different ways can you use sequences to justify values for the question marks? Explain why one entry must remain empty. Extend it to the left and downwards.

…	…	…	…	…	…

?	…	$1^{1/3}$	$1^{1/2}$	1^{1}

?	3^1	3^2	3^3	3^4	…
?	2^1	2^2	2^3	2^4	…
?	1^1	1^2	1^3	1^4	…
	?	?	?	?	…

?	…	$(1/2)^{1/3}$	$(1/2)^{1/2}$	$(1/2)^1$
?	…	$(1/3)^{1/3}$	$(1/3)^{1/2}$	$(1/3)^1$
…	…	…	…	…
	…	?	?	?

What is the same, and what is different about the left and right-hand grids? Extend the right-hand grid to the left and downwards to justify values for the question marks. Why are there difficulties extending the upper-right quadrant below the row of question marks but not in extending it to the left of the column of question marks?

Factoring Quadratics

A factored quadratics grid with the upper and the lower cell entries filled in are displayed below.

(x - 1)(x + 3)	x(x + 3)	(x + 1)(x + 3)	(x + 2)(x + 3)	(x + 3)(x + 3)
(x - 1)(x + 2)	x(x + 2)	(x + 1)(x + 2)	(x + 2)(x + 2)	(x + 3)(x + 2)
(x - 1)(x + 1)	x(x + 1)	(x + 1)(x + 1)	(x + 2)(x + 1)	(x + 3)(x + 1)
(x - 1)x	xx	(x + 1)x	(x + 2)x	(x + 3)x
(x - 1)(x - 1)	x(x - 1)	(x + 1)(x - 1)	(x + 2)(x - 1)	(x + 3)(x - 1)

x^2+1x-6	x^2+2x-3	x^2+3x	x^2+4x+3	x^2+5x+6
x^2+0x-4	x^2+1x-2	x^2+2x	x^2+3x+2	x^2+4x+4
x^2-1x-2	x^2+0x-1	x^2+1x	x^2+2x+1	x^2+3x+2
x^2-2x	x^2-1x	x^2+0x	x^2+1x	x^2+2x
x^2-3x+2	x^2-2x+1	x^2-1x	x^2+0x-1	x^2+1x-2

Using pedagogic devices like those suggested for the number grids, learners can work out for themselves from just a few cell entries, what other cells are likely to contain. Negatives can be pursued by shifting the window, and various forms of data can be provided. For example, a diagonal from top left to bottom right somewhere in the extended grid can be used to predict the other two corner entries. Again the aim is to get learners to generalise, not just for predicting what will appear in a specified cell, but how the upper and lower entries relate to each other. Thus they encounter both factoring and expansion of brackets. By looking for patterns they can work out how to factor an expression as well as how to multiply out brackets.

Other Possibilities

Structural Variation Grids can be used for laying out any two-parameter families of operations. A variant form using a spreadsheet to display grids using numbers from arithmetic sequences, and looking at properties of entries forming specified geometric relationships appears in Hewitt *et al* (2005). Other possibilities include pairs of simultaneous linear equations, and pairs of quadratics differing only in the sign of the constant term. In every case the aim is to provoke learners into using their natural powers rather than telling them things they can work out for themselves. The Zero grids show how particular topics require didactic as well as pedagogic decisions.

Structured Variation Grids as a Framework for Learning

There are many formats or layouts which are used with learners and which can develop into a framework for learning. Examples include number-lines (empty or not), Cuisenaire rods, Dienes or Multi-Base blocks, bundles of ten sticks, the balance metaphor for equations, graphical presentation of functions, and grid multiplication which is also known as Gelosian multiplication and is related to Arabic diagrams for the expansion of $(a + b)(c + d)$. There are advantages to

using the same structure in different contexts because learners bring to each successive use the way of working used previously. This reduces the overheads in getting to grips with the rules and affordances of a new format or tool. Thus with Structural Variation Grids, once learners have become used to extending and expressing patterns and justifying their expressions of generality, they are likely to behave in the same way again. The grid displays structure and stresses consistency as a driving force in extending mathematical concepts (Mazur 2003 p73). Learners are likely to become imbued with mathematics as sense-making because the grids provide visual access to the structural patterns which justify calling negatives, fractions and the like, *numbers*. Consistency and continuity become second nature rather than new concepts. The grid also provides a format for learners to use for exploring two parameter patterns for themselves.

Frameworks for learning can become tedious, and downright boring if their use is too formulaic and if the pedagogical practices fail to call upon learners' natural and developing powers. Formats and frameworks also need to exhibit sufficient variation so that learners do not make assumptions about the role or importance of invariants which the teacher does not intend learners to include. Exposure to a restricted range of examples can lead to the formation of inappropriate concepts analogous to the *figural concepts* identified by Fischbein (1987, 1993).

Informing Pedagogical & Didactic Practices

As devices these grids have already proved to be attractive to a range of teachers. They are formats which afford the possibility for anticipating, conjecturing, and generalising, but their form also invites pedagogical and didactical choices. *Pedagogical* is used here to refer to general teaching strategies such as beginning in silence, asking learners to say what they see or what is the same-and-different about objects presented to them. *Didactical* is used here in the European sense of specific to the mathematical objects used in the grid and to the mathematical aims of the lesson. Pedagogical and didactical choices are informed by frameworks which arise from research of various kinds, but which act to remind practitioners about choices of actions, and which could inform research into those choices.

In the following subsections I make use of a number of different frameworks in order to highlight features of the grids which seem to make them potentially fruitful as one framework for learning what could be used throughout the school curriculum. Most of the frameworks I shall mention were developed in and for a 200 hour distance-taught course entitled *Developing Mathematical Thinking* which first appeared in 1981. Some 400 mathematics teachers studied it as a form of professional development in the first year. It ran for 6 years (with numbers declining to 180 per year) and profoundly influenced a generation of teachers and teacher educators in the U.K.. It was said at the time that every mathematics teacher-education group in the country had someone who had been either a student or a tutor (or both) on the course. Evidence for the success of the course is that we would (and still do) come across people in other contexts not just using the language introduced in the course, but making use of the language to inform and justify their current practice. The 'language' being referred to was a collection of frameworks: mostly triples of words which could be used to trigger sensitivities to notice opportunities for actions, as well as access to the actions themselves. I begin with a framework that has emerged relatively recently.

Dimensions of Possible Variation

One of the essential features of Structural Variation Grids is that they permit rapid exposure to several examples which display variation in one or two different aspects. These examples can be chosen according to pedagogic and didactical purposes according to the perceived needs of

the learners. Human beings have brains which are well suited to detecting variation, especially systematic or structured variation, and the grids enable a quick succession of examples to be presented.

Ference Marton has for some years been developing the observation that what people discern is variation (Marton & Booth 1997, Marton & Trigwell 2000, Marton & Tsui 2004; see also Runesson 2005). This has led him to propose that learning consists of discerning freshly, that is, of becoming aware of new *dimensions of variation*. A *dimension of variation* is an aspect which can vary in an example and still it remains an example. In other words, a concept or technique is understood to the extent that the person is aware of what can be varied and what must nevertheless remain invariant. Often it is relationships which are invariant rather than aspects of objects themselves. Anne Watson and I (Watson & Mason 2002, 2005) extended this idea to *dimensions of possible variation* to indicate that at any time teacher and learner may be aware of different aspects or dimensions which could be varied, even if they are not varied in the current situation. Furthermore, we noted that often in mathematics learners have a restricted notion of the *range of permissible change* in any specific dimension. For example, when generalising sequences and grids, learners typically think of whole numbers while the teacher may be aware of rationals and reals as possibilities.

The vital aspect of variation as a description of learning is experiencing sufficient variation in sufficiently quick succession to be aware of it as variation of some feature, and hence as a dimension of possible variation.

We have found Marton's idea of great interest and use for two basic reasons: first, it fits with our own view that invariance in the midst of change is a central theme of mathematics, and second, it proves to be fruitful for analysing learning and the potential for learning afforded by tasks, including sets of exercises (Watson & Mason in press).

With and Across The Grain

The layout of Structural Grids is designed to afford plenty of opportunity for learners to experience structure through sequential variation. Whereas Tunja sequences are sequential, the two dimensionality of the grid permits many variations in how learners are exposed to enough data to be able to predict the entries in as-yet-unrevealed cells. This in itself can be attractive and motivating to learners who are becoming used to being invited to detect pattern and structure. But the pedagogic significance emerges when learners are asked not only to predict cell entries but to justify why the upper and lower entries in a cell must always be equal. For the Zero Grids, this means making sense of potential values in a particular cell by using the structure of sequence of cells which include that cell. Trying to make sense of the over-all patterns in a grid calls upon mathematical sense-making. It is convenient to describe these two aspects of using grids using a well known metaphor associated with wood: *going with and across the grain*.

Anne Watson (2000) came across some learners asked to copy and complete a table based on the following structure

7 x 1 = 7	1 x 7 = 7	7 ÷ 1 = 7	7 ÷ 7 = 1
7 x 2 = 14	2 x 7 = 14	14 ÷ 2 = 7	14 ÷ 7 = 2
…	…	…	…

She noted that when learners follow a simple number pattern to anticipate the next and future terms, they are acting in a manner which is similar to going with the grain of a piece of wood: fresh wood splits relatively easily along the grain. This matches my experience of offering people sequences of terms in which everyone (mathematicians and non-mathematicians

alike) quickly work out the pattern and can predict the next and future terms. Going with the grain on sequences and grids means following simple patterns such as writing all the 7s in the first column, then all the multiplication signs, then the numbers 1, 2, 3, … and so on. Cutting across the grain reveals the structure of wood, so *going across the grain* can be used to refer to the act of making mathematical sense of relationships, here, between the different entries in a row of the table, which is presumably what the authors intended learners to encounter.

Copy-and-Complete has become a classic form of task in UK textbooks, referring to a partially filled in table in the text which learners are expected to copy into their books and then fill out according to some underlying pattern. Copying is a clerical activity. It calls upon some hand-eye coordination but does not draw on mathematical thinking, especially when it is carried out by inserting all the invariant elements first, and then inserting the things that vary. Even if learners end up with a completed table, they may not have encountered the target technique or actually done much thinking. Mathematical thinking only begins when attention is directed to what a specific term is saying, or to what each row or column or other subsequence in a grid is saying, not in particular, but as an instance of a generality, that is, by going across the grain.

The phrase *with and across the grain* can shift from description to action when it reminds teachers to prompt learners to make mathematical sense and so turn copy-and-complete from a clerical exercise into a significant and relevant mathematical experience. Another way of saying this is that in order for doing a task to influence learning, it is necessary to prompt learners to see the general through (each of) the particulars, and then to see each of the particulars in (as instances of) the general. This two way process was summarised by Alfred Whitehead (1932):

> To see what is general in what is particular and what is permanent in what is transitory is the aim of scientific thought. (p4)

I prefer to rephrase it more expansively: 'to see the general through the particular and the particular in the general' and 'to be aware of what is invariant in the midst of change' is how human beings cope with the sense-impressions which form their experience, often implicitly. The aim of scientific thought is to do this explicitly.

With and Across the Grain, when internalised as a description of actions which a teacher can take to direct learner attention, has become a teaching framework which enhances or structures learning. When taken up by learners, it acts as a framework for learning. As with other teaching-learning frameworks, it serves to bring to mind actions which might enrich learning, but which might otherwise have slipped by unnoticed.

Specialising & Generalising; Conjecturing & Convincing

Structural Variation Grids can be used to prompt learners to make use of their natural powers to detect and express generality, to make conjectures, to test them by specialising or particularising, and to try to justify those conjectures to others. If these powers are seen by the teacher as essential to learners making sense of mathematics and making mathematical sense, then they will seek out opportunities to provoke learners to use those powers. By drawing learner attention to the spontaneous use of those powers, and offering a label by which to refer to them in the future, teachers can promote the development and refinement of those powers. George Polya (1962) promoted them as components of mathematical behaviour, and Mason *et al* (1982) promoted them as processes which contribute to mathematical thinking. As the focus of educators has changed over the years, the collective noun to describe these processes, practices, or powers has to be changed to match current concerns. Speaking of learners' natural powers (Mason 2002) certainly attracts teacher attention and finds resonance with their experience.

Enactive–Iconic–Symbolic and related frameworks for teaching

Some of the patterns in Structural Variation Grids are so elementary that most people find themselves enacting them without even really being aware. Usually this is because the sequence uses something very familiar such as 1, 2, 3, … . There is a sense in which their body responds to implicitly perceived structure rather than passing thorough the intellect, even though it manifests itself through the intellect in words and symbols. Sometimes when a pattern is not immediately detected, people nevertheless have a sense of pattern, even though they cannot immediately articulate it. You might say they have an image or overall shape or sense. Once a pattern starts to emerge, it can be developed and expressed in words, pictures, icons such as clouds to stand for 'the number I'm thinking of but am not going to tell you', and even using letters as symbols for as-yet-unknown or unspecified numbers, as in traditional algebra. This description builds on distinctions proposed by Jerome Bruner (1966) who identified three modes of representation: enactive, iconic and symbolic. In designing the Open University course mentioned earlier, we found that these resonated with our experience, but that we wanted to elaborate on some of the ramifications. The result was a metaphorical interpretation of his distinctions, and three closely related frameworks. The basic framework remains the same as Bruner's but with elaboration:

> *Enactive* mode: manipulating familiar and confidence-inspiring entities, whether they are physical (blocks, sticks, counters, rods, …), as Bruner suggested, or meta-physical (numerals as numbers, letters as variables or as generalities, familiar diagrams, screen manipulable objects, etc.);
>
> *Iconic* mode: images, pictures and drawings which depict what they are (as in a cloud for a number I am thinking of or don't yet know) as Bruner suggested, but including also a pre-articulated as-yet-inchoate 'sense of'.
>
> *Symbolic* mode: symbols whose use is a convention and so by their nature have to be explained, as Bruner suggested, but they are abstractly symbolic only so long as they remain unfamiliar.

For example, Helen Drury (personal communication) working with a year 10 top set used the Factor Grid for the first time as a computer display in whole-class mode and then invited learners to fill in a blank grid for themselves. They had a choice either to try to fill in the expanded expressions above the factored versions immediately, or else to begin by writing out the factored cells before completing the expanded expressions.

Filling in the entries themselves afforded an opportunity for enactive subconscious awareness of patterns to be manifested and experienced. This is why paying attention to *how* you fill out a table or draw a picture enactively can be so useful when trying to articulate a generality: so useful in fact that a slogan such as *Watch What You Do*, along with *Say What You See* can be useful for reminding learners to do more than simply try to get answers. Learners who tried to do the expanded expressions immediately mostly struggled to find the complex patterns, especially in the constant term. They tended to enter all the x^2s first, which is efficient, and which may direct attention to significant patterns, but it may also divert attention inappropriately, as with copy–and–complete. One learner spotted that opposite corners had opposite signs, and another described similarities with the arithmetical multiplication grid. Pedagogic decisions had to be made about whether to invite them to report their observations or to leave others to make similar discoveries. The important work involved trying to make sense of the upper and lower entries in each cell, leading to a deeper enactive awareness of how factoring quadratics works.

Developments from E-I-S

Once symbols become familiar (for example, numerals for numbers), the symbols become less symbolic in effect and more enactive. To capture this specifically we developed the triple *Manipulating—Getting-a-sense-of—Articulating* (MGA for short) as a spiral of development (using Bruner's notion of spiral learning). It was used by teachers, and subsequently by their learners, to remind learners to backtrack to something more familiar and manipulable when they get stuck or when something seems to be beyond immediate grasp.

The use of MGA as an acronym illustrates the framework perfectly. Unless you are already familiar with MGA, you are likely to need to expand it in your mind, to read out the full form of words and then think about the meaning. Over time and with use you may find MGA becoming a useful shorthand for triggering actions and awareness in yourself, for making sense of experiences, and for communicating with others. The acronym can actually help you to articulate some observation or to describe some phenomenon.

In relation to Structural Variation Grids, filling out a grid for themselves from the contents of a sequence of cells in one row and another in one column, or even from the content of a few sporadically placed cells, enables learners to work with familiar entities (expressions in cells whose content is known) in order to get a sense of the overall structure of a particular grid. At a more meta-level, familiarity with one grid enables them to recall what they did previously when they tackle a new one, so that over time they get a sense of structure indicated by a two-way grid, and two-way grids as a format for arithmetic structure.

Hand in hand with MGA is the triple *Do–Talk–Record* (in the sense of writing-up not exploratory writing-down). Writing our course in the early 80s we were well aware of the importance of learner-learner talk and collaboration. This framework allowed us to remind teachers that pushing learners to make written records too quickly can be at best unproductive and frustrating for all concerned, and at worst, actually harmful. It is valuable if not essential to allow learners time to talk about what they have been doing, and indeed to get them doing things (enactively manipulating the familiar) so that there is something mathematical to talk about. Talking to others, trying to justify your ideas and conjectures is an excellent way to externalise your thinking, get it outside of yourself so that you can look at it critically. That makes it easier not to be indentified with your idea but to treat it as a conjecture to be modified.

Alongside these three frameworks we also found it useful to include something to remind teachers that learners do not usually master ideas on first exposure. So we suggested a triple of *See—Experience—Master* as a reminder that first encounters are a bit like seeing a fast vehicle go by. It takes repeated encounters to begin to discern details and to recognise relationships amongst those details. Only then does it make sense to try to achieve mastery, to develop facility and fluency and to minimise the amount of attention needed to carry out techniques and procedures.

Finally, we embedded these frameworks in the notion of a *classroom rubric* or *ways of working* which correspond to what is now described as socio-cultural practices of a *community of practice* (Lave & Wenger 1991) and as *sociomathematical norms* (Yackel & Cobb 1996). Fundamental to the effective functioning of a classroom ethos is a *mathematical* or *conjecturing atmosphere* (Mason, Stacey & Burton 1982, see also Mason & Johnston-Wilder 2004a). This is a way of working in which everything said is treated as a conjecture, uttered in order to think about it more clearly and to modify it as appropriate. Those who are very confident take the opportunity to listen and to suggest illustrative examples and counter-examples, and those who

are not so confident take opportunities to try to express their thinking in order to help them clarify that thinking, just as in the *Eleusis* game.

Structure of Attention

Inviting learners to say 'what is the same and what is different' about several entries in a Structural Variation Grid, or simply to 'say what you see' initiates a movement of their attention. They may gaze at the whole (and be aware that there appear to be missing or hidden entries in a grid) or at the whole of a particular element such as a cell entry; they may discern details such as particular entries, or details within an entry (such as two rows to each cell, or the presence of various mathematical signs); they may recognise relationships within a cell (such as an equation) and between cells (such as all having two factors or the upper part being a calculation and the lower part an answer or vice versa); they may perhaps perceive some relationships as properties which apply across all visible cells and so might apply to all cells; they may even be able to reason on the basis of those properties in order to justify their prediction of what will appear in different cells, or where a particular entry is to be found.

As a teacher with a class, the problem is that different learners may be attending in different ways. If as teacher you are attending in one way, say talking about properties of cells, when learners are busy discerning details or recognising relationships between particular entries, there may be a mismatch and consequent breakdown in communication. By being aware of what you are attending to, and how, you can either direct learner attention appropriately, or put your own focus of attention to one side and try to enter the experience of some of the learners.

What is Attention?

For William James, philosopher and psycholgoist, attention is what makes it possible to perceive, conceive, distinguish and remember. It is the basis of all our psychological functioning (James 1890 p 424). As might be expected, he deals with a number of important issues concerning attention in general. For example, he argues on the basis of experiments that attention is not simply what the eyes are looking at, or indeed any other particular source of sense impressions (p 438). He links attention to anticipative imagination (p 439-411) as a prerequisite for discerning anything at all. James develops this theme of discernment, or discrimination, to make use of what he calls Helmholtz's law, that

> we leave all impressions unnoticed which are valueless to us as signs by which to discriminate things (p 456).

In other words, we notice what we are attuned to discern. James goes on to discuss pedagogic implications such as that it is useful for teachers to work with learners to strengthen and attract their attention in order to improve motivation, since people engage with what catches their attention (James 1890, p 446). To do this requires being aware of what in learners' previous experience can be used as a basis of previous attention-experience, what John Dewey referred to as 'psychologising the subject matter' (Dewey 1902, p 12).

James sees attention as a form of 'free energy', since when you make an effort to attend to something you can sustain it for only very short periods before attention wanders (p 420) requiring a further expenditure of effort, but when attention is engaged it requires no energy expenditure at all for it to remain focused for long periods of time.

I agree with James that *'my experience is what I agree to attend to'* (his emphasis), although his wording might be taken to imply voluntary agreement, which is certainly not always the case. At each moment, as my attention shifts, I am the totality of that attention; the totality of my experience *is* my attention. Attention is not just as what puts me in touch with the world of my

experience, but what creates and maintains that world. This is meant to include things of which I am subliminally or covertly aware, sometimes through body awareness, sometimes through social awareness, sometimes through emotional resonance, and sometimes through cognitive awareness. None of these need be conscious. This makes attention difficult to study directly, because it is no good asking people 'what are you attending to?' since the very question alters the focus and locus of that attention.

Where I differ with James is in his metaphor of attention or consciousness as a flowing stream, for it seems to me that his own descriptions (e.g. James 1890 p456 quoting Müller), as well as my observations, lead to the conclusion that attention is briefly sharp and alert, and then slowly declines into absence of awareness until some fresh stimulus wakes it up again. The sense that we have of experience flowing by is actually much more episodic and fragmentary (Mason 1988), as attempts to reconstruct recent and distant experiences demonstrates all too clearly.

Task 5: Focus, Locus, and Multiplicity

Can you gaze into the distance while asking yourself what you think attention is?
Can you be aware while you are reading that this paper is just one contribution to a whole collection of papers, and can you then shift so that you are focusing intently on the wording of the next part of the task, oblivious to the other papers?
How many different things can you attend to at once? For example, in a lecture, can you attend to the speaker's voice, use of display, clothes, and content of what they are saying? Can you at the same time as reading this imagine yourself going and getting something to drink, and being aware of some background music or other sounds?

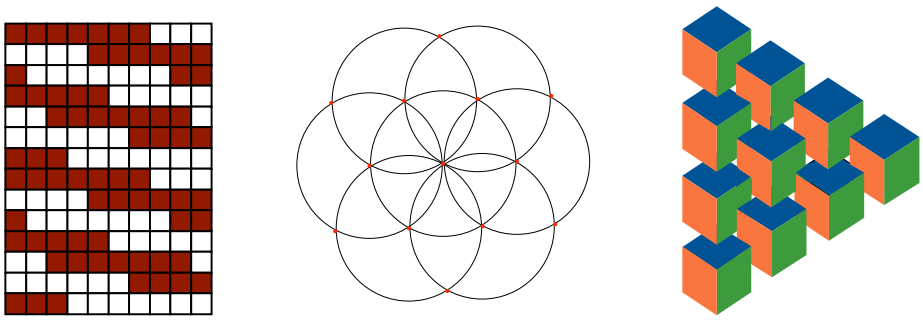
You can attend to things physically present and also to things not physically present (locus); you can 'gaze' while pondering, and you can concentrate very specifically on some small detail (focus). You can be aware of one single detail, and you can be multiply aware cognitively, multiply aware enactively, and multiply aware affectively (multiplicity). Once focused, attention can be diverted by rapid movement within your field of vision, especially if it is peripheral, and changes in other sense impressions can also attract your overt attention.

There are deep physiological questions about whether you actually attend to several things at once, or whether you rapidly cycle through a variety of foci, the way computers now do. There is also an issue about whether consciousness directs behaviour or is subject to a 'user illusion' of being in charge, as Tor Norretranders (1998) proposes. Whatever may be the case, personal experience is sufficient to highlight important aspects of attention which can be used to improve both teaching and learning.

Interrogation of experiences suggests that attention can be focused or diffuse, localised or global, single or multiple. But even focused localised attention has different forms.

Task 6: Say What You See

Say (to yourself, to a colleague) what you see in the three pictures, and try to pay attention to how your attention alters.



Which diagram seems the most complex?
How many different rectangles can you find in the central diagram of the circles, where the vertices have to be on the points of intersection of the circles?

It is quite likely that the middle diagram seemed quite simple compared to the others, at least until you started to work within the diagram looking for distinct quadrilaterals. The counting question serves to focus attention, calling then upon not just discerning particular vertices, but relating their positions so as to form rectangles. To be a rectangle is to impose a property on the specific points so that they satisfy a relationship. This task is just part of a complex of tasks using this same diagram and developed by Geoff Faux (private communication).

The picture on the right makes perfect local sense, but when you gaze at the whole there are some inconsistencies with the way material space is structured: one part of the picture seems to pull against other parts.

By contrast, in the picture on the left, you may have found yourself beginning to count the dark or the white squares in each row, then discovering that there are the same number in each block if you carry over to the next line. You may have gazed at the whole without seeing very much until you noticed a repeating pattern, although this may not have emerged until you did some counting. Did you think to look for patterns in the columns? Recognising possible relationships between alternating columns leads to perceiving a potential property, which can then be justified on the basis of an assumption about how the rows are generated.

Whether attention is the subjective experience of physiological functioning, as Théodule Ribot (1890) would have it, or the engine for physiological response to environment, as William James (1890) proposes, there seem to be quite distinctive if subtly different forms of attention:

Holding Wholes (gazing)

Discerning Details (features & attributes)

Recognising Relationships (part-part, part-whole)

Perceiving Properties (leading to generalisation)

Deducing from Definitions (reasoning on the basis of explicitly stated properties stated independently of particular objects)

Shifts between these are rapid, often subtle, but vital in order to engage in mathematical thinking. While gazing, some sudden movement, perhaps even apparent motion produced from circadian eye movement can suddenly switch attention to awareness of details amongst a mass of other, undiscerned detail. As details are detected and discriminated, the mind automatically looks for relationships: differences and samenesses. To do this requires something being relatively invariant as a background against which to detect change. Recognising relationships tends to focus on particulars, whereas perceiving properties is a move to the more general, to the particular as exemplary or paradigmatic. Formalising in mathematics is the overt action which

accompanies a shift from perceiving properties to taking certain properties as definitive and so as the basis for further reasoning. Discerning these subtle shifts in the structure of attention develops Marton's notion of learning as discerning variation, because it provides a more detailed structure of what might attended to, and how.

Here is an opportunity to 'see' whether you recognise some of the subtle moves being suggested by this framework.

Task 7: Some Sums

What do you make of the assertion that $\sum_{k=\sum_{1}^{n-1} j}^{\sum_{1}^{n-1}(j+1)} (2k+1) = n^3$?

Is your first reaction panic? Did you find your attention drawn to or away from the little summation signs? Did you try some particular values for n? Did you sit back and gaze at the whole, then pick out details when you gained a little confidence, perhaps? Did you recognise a sum of consecutive odd numbers? Were you able to focus on the upper and lower limits of the main sum? Did you detect a relationship between them? That could tell you how many terms there are in the sum. Did you recognise the lower sum so that you could write it in terms of n? That would then enable you to write the upper sum in terms of n as well, making it much easier to try some examples, but you might recognise that the sum of consecutive odd numbers is a difference of the squares of the term preceding the first, and of the last term. So you could actually check it out in general without having to try some particular cases!

Notice that reasoning on the basis of properties of sums of consecutive natural numbers and of consecutive odd numbers is only possible if you can attend to the structure without having to keep in mind a particular example. This requires that you see through particularities of specific symbols (summation signs, bracketed terms), and appreciate what compound symbols are saying as a whole. Recognising a relationship between the shape of the upper and lower summation signs shortcuts the algebra of evaluating the upper sum independently of the lower.

Gazing is an under-rated form of attention. It includes mulling things over as you wait for a bus, take a shower, or wash dishes. Discerning different forms or structures of attention is only useful if it serves to inform future practice, whether by refining research probes, or by suggesting ways of acting with learners so that there is a better match between what learners and teachers are attending to, and how they are attending.

Why might the structure of attention matter?

At a classroom level, if learner attention and teacher attention are significantly differently structured then confusion is a most likely outcome. More particularly,

> if some learners are attending gazing when the teacher is discerning specific details;
>
> if some learners are discerning details when the teacher is talking about relationships amongst details;
>
> if some learners are recognising relationships amongst discerned elements when the teacher is talking about properties of objects in general;
>
> or if some learners are thinking about properties when the teacher is reasoning on the basis of, or deducing from those properties;

then here is likely to be a mismatch, a failure of communication. If some learners are focusing on what they are supposed to do, and others on what the object itself is, then again a mismatch is

likely. One of the abiding problems in mathematics education is how to promote mathematical reasoning (proof). Colette Laborde (2003) observed in the context of dynamic geometry that

> It is a matter of getting pupils to understand that in geometry they have to rely on what they see to get ideas on how to solve a problem, but they do not have the right to use that when it is a matter of reasoning 'rigorously'; they must then restrict themselves to the theoretical plane. (p$$$)

This is the difficult move, the move which is uncommon outside of mathematics, the move which marks out the natural mathematician from others who have to struggle to make it.

Similar mismatches are likely to occur in an in-service, continuing professional development context. If task-exercises are offered, as in this paper, as an opportunity to engage in mathematical thinking, attention may be fully caught up in the mathematics, leaving little attention free for meta-concerns and the use or influence of some chosen framework. If behaviour such as styles of questioning or prompting is modelled, teachers may attend to specific details rather than recognising relevant relationships, so that they are left with behaviours but no criteria for when to use it, and no sensitivity to features of a situation which might usefully trigger awareness of that behaviour. Where examples of pedagogical decisions are offered, attention may be absorbed by recognising relationships within the particular rather than drawing back to perceive properties which could apply in many different situations. Hence it is less likely that a new situation will trigger relevant awareness. Even where an explicit framework is presented, as here, it may be that most attention is taken up with relationships so that there is insufficient attention available for perceiving those specific relationships as properties, so the label does not become richly imbued with personal meaning.

van Hiele Levels

Anyone familiar with van Hiele levels (van Hiele-Geldof 1957, Burger & Shaunessy 1986) will be aware of close similarities with the structures of attention being proposed. Here is one version (based on Burger & Schaunessy *op cit*), to which I have appended a version cast in terms of what reasoning might look like:

Level 1: Visualization (reasoning based on direct perception)

Level 2: Analysis (reasoning based on relating component parts and attributes)

Level 3: Abstraction (reasoning based on necessary conditions as known facts)

Level 4: Informal Deduction (reasoning based on relating properties)

Level 5: Formal Deduction (reasoning from axioms systematically)

Pierre van Hiele generalised these beyond geometry (van Hiele 1986) but in the process made them even more abstract and, for me, harder to connect to moment-by-moment experience, which is where I think it is important to focus in order to influence and improve learners' experiences of being taught mathematics. I came to the structures of attention through an entirely different route based on Eastern sources (Bennett 1956-1966).

The difference between the structures identified here and the van Hiele levels lies precisely in the notion of levels. Rather than seeing these structures as levels or even as hierarchical qualities in the way researchers have developed the van Hiele ideas to date, I am proposing the radical stance that these so-called levels are actually descriptions of the way that people attend all the time, often with rapid shifts from one to another.

There are of similarities also with the onion model of understanding developed by Pirie & Kieren (1994) but there is not sufficient space here to elaborate on this connection.

Reflection

Notice the steps taken to try to engage the reader with each framework and especially this one concerning the structure of attention: presenting task-exercises designed to highlight aspects of the phenomenon of interest in (different ways in which attention is structured); commenting on that experience, with the hope that there will be resonances with or challenges from the reader's past experience; justifying the potential value of the framework in terms of addressing perplexing questions in teaching and learning mathematics; and linking to the literature.

Theory of Frameworks

The examples of frameworks provided in this paper illustrate my view of a framework as a collection of labels which serve to bring to mind useful distinctions, and relevant relationships as instances or manifestations of properties as well as ways of working with learners. The label acts as a reminder, as a re-sensitisation not only to notice but also to act. Thus *enactive–iconic–symbolic* distinguishes three forms of representation, and *do–talk–record* distinguishes three learner activities. They also trigger possibilities such as re-presenting in a different mode or inviting learners to do this in order to develop flexibility in moving between modes, and such as constructing tasks which call upon learners to try to describe what they are thinking about what they have been doing, before being urged to make written records. But making distinctions is only the beginning, as the *structure of attention* framework indicates. What also matters are relationships between distinguished elements, and the abstraction rendered by perceiving specific relationships as more general properties.

Encountering Frameworks

The exposition given here of a number of frameworks has been necessarily brief and truncated. Working with teachers over many years, and writing materials to try to support their professional development has highlighted again and again the necessity of several experiences before a framework begins to be active. First, it is vital to engage people in immediate mathematical or classroom experience in which they are likely to experience some aspect of a framework to be proposed. In text, this means offering mathematical tasks and then offering comments based on the sorts of things people tend to notice when engaging with those tasks. Labelling salient experiences with what might become a framework then permits later reference back to such experiences. Where classroom video is available, incidents can be shown and then used to resonate personal experiences with what seem to participants to be similar elements. Individuals can be stimulated to report incidents that seem to have something in common, and negotiation can take place as to what similarities and differences people are aware of. It really helps if people have the opportunity to discuss such examples in order to come to some agreement as to what the framework labels refer. The aim is to build up a rich network of associations with past experiences and appropriate actions, triggered by the label. This means that generic labels using words that might even be used in a relevant situation are more useful than labels such as the names of people involved in particular past incidents.

Thus frameworks as labels can act to sensitise people to notice situations that might have gone unnoticed previously. In order for frameworks to become active and informative, whether for teaching or research, it is vital that they also become associated with actions which can be initiated as a result of noticing. Techniques for enhancing and enriching the use of frameworks in this way have been described in detail in Mason (2002) as the *discipline of noticing*.

How Frameworks Arise

Whenever a situation or incident is recognised as not just particular, but as an instance of some general structure, it becomes a representative of a phenomenon. Thus does a phenomenon come into existence in the mind of the observer. If, as is often the case, someone wishes they could have acted differently in the situation, or else wants to remember to act in a particular way in future, then there is an opportunity for a framework to develop. Some word or words which capture an essence of the phenomenon, and which can be strongly associated with a desirable action, produces a framework.

Many frameworks arise out of the literature, as was illustrated by some of those described earlier. Others arise as a result of probing and considering what actions you would like to take in certain situations. As with diagrams and other forms of notation, the ones you create for yourself are the ones which are most vivid, along with frameworks which crystallize awareness you have had but hadn't yet articulated. At the core of any useful framework are distinctions which enable you to discern details and to recognise relationships in the particular situation. These then need to bring to mind actions you would like to initiate. By setting yourself to notice phenomena in the future and to use that noticing to act in fresh ways, a framework of labelled distinctions and actions can inform future practice.

Validity

Frameworks, however publicly negotiated and developed, are essentially personal. Despite widespread currency of the labels, without careful and ongoing negotiation of meaning and interpretation, frameworks can divide as much as unite. Examples abound, such as the many uses of labels such as *ZPD*, *activity theory*, *constructivism*, and *problem solving*.

It is the sensitivity to notice and to act, triggered by resonance between a framework and a particular situation, which renders the framework useful. Frameworks are not then either valid or invalid, but rather informative or not informative for a person at a time in a situation. There is no claim of universality. The deeper issue about validity is whether someone really is sensitised to notice more and to act more effectively, or whether a framework supports the individual in misapprehended solipsism and prejudice. The only way to guard against self-delusion is to engage in ongoing practices of offering incidents and tasks to new colleagues to see if they recognise what is being pointed to, and whether they too find the distinctions informative in their future practice. This is the basis for the discipline in the *discipline of noticing*. Thus validity is at once personal yet in need of frequent testing against an ever widening community.

A Note About Method

My method is and always has been to reflect deeply on my own experience of doing mathematics and of being mathematical with others. I am constantly seeking situations analogous to those of learners so that I can get a taste of what they are experiencing. I use that to inform my pedagogic and didactic choices. I try to stimulate others to notice what I think I am noticing so as to guard against solipsism, and I try out actions which seem to be an improvement, in a ongoing development of noticing, sensitisation and action. Stimuli are constantly being refined and honed to meet new situations. The data I am offering you as reader is the collection of memories and awarenesses which come to mind as you engaged with the stimuli offered in the paper, in the form of tasks and commentary. If you didn't engage in the tasks, you are unlikely to have gained access to much data. Validity for you resides in the extent to which you find your past experience resonated or challenged, and your future actions informed. I do not seek any absolute validity, because it seems to me impossible: we are dealing with human interactions.

Human beings are a remarkable combination of predictable mechanicality and creative agency. I see all 'truths' in mathematics education as people, time, place, and situation dependent.

Distinctions are not 'natural cleavages' of the world, but rather lie in the eye of the beholder. They are psychological in the sense that they involve a re-structuring of attention in the mind of the individual. They are social in the sense that they are often encountered in the practices of others, and if they resonate with or challenge previous experience, can be adopted and adapted into a personal practice.

How Frameworks Can Inform Research

I have concentrated here on frameworks for learning and for teaching, confident that any framework which is informative for these purposes will be informative for research purposes as well. However it is important not to allow theoretical frameworks and distinctions to displace careful observations. To be useful as data, observations and transcripts depend on being accounts-of incidents, so that readers feel they could recognise such a situation had they been present, and even that they recognise the type of situation in their own experience. If justifications and explanations which partially account-for the situation are intermingled with accounts-of incidents, then the reader is unable to question or disagree with what is said, and may not be able to recognise similarities and differences with instances from their own experience. Similarly, if analysis is intermingled with accounts, such as substituting framework labels for specific observations, then again it is difficult if not impossible for a reader to question or disagree with the analysis. Distinctions triggered by frameworks need to be justified from observations of behaviour rather than baldly asserted as part of the data.

Attempts to classify people according to distinctions offered by a framework are especially unhelpful, because it is behaviour that is being observed and classified, not the person. But even classifying behaviour can be misleading, because what is observed is only a fragment in time. The person may be experiencing a much richer flow of awarenesses not displayed in behaviour. The best research leads people to reveal their awareness and their dispositions as well.

Frameworks such as those described in this paper can also inform the researcher in the design of their study, such as when seeking tasks to reveal dimensions of variation of which subjects are aware or can access, to get them doing and talking as well as making records, to provoke them into displaying mathematical thinking and to stimulate them to expose the subtle shifts in the structure of their attention.

Dangers

Frameworks are rather like geometric diagrams: they have implicit structure which you need to know about in order to make good use of it. A framework such as *enactive–iconic–symbolic* or the van Hiele geometry levels may be used in a community as if they were agreed and well defined, when actually what is resonated within different people by those terms is markedly different. One has only to look at the use of the *zone of proximal development* as a term in mathematics education papers over the last twenty years to be reminded of this! This is the danger of frameworks. It is a danger inherent in any signifier which refers to abstractions from specifics. To be confined to a world of specifics is to lose the power of generalisation, while to be restricted to a world of abstractions without referents to specifics is to live in an ivory tower. The rich use of frameworks as described here straddles the two worlds of particulars and generalities, of concrete experiences and abstract notions.

Conclusions

What can be learned from these observations? Every time someone offers you a framework, model, or even just a list, ask yourself questions such as the following:

> What is being stressed? What is being distinguished? What am I sensitised to notice now that I did not discern before?
>
> What examples come to mind from immediate or recent experience, and from past experience, and how are these experiences informed by the framework?
>
> What is being ignored?
>
> What associated actions would I like to use in such a situation, and why?
>
> What other possibilities are afforded as a result of being sensitised by this framework?

When you are using technical terms with a colleague, offer an account of an incident which you think exemplifies some aspect, and seek agreement as to its appropriateness. In this way mathematics education can develop from a ragbag of distinctions into a theory-based discipline.

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