On Being Stuck On A Mathematical Problem:
what does it mean to have something come-to-mind?

John Mason
ProMath May 2014

## Abstract

Everyone gets stuck sometimes, and it can be frustrating, even debilitating rather than stimulating. However, being stuck is an honourable and useful state because that is when it is possible to learn about mathematics, about mathematical thinking, and about oneself. This applies especially to teachers, because the best way to sensitise yourself to learners’ struggles is to experience parallel struggles yourself.

People are usually eager to get unstuck, to locate and enact some hopefully helpful action, without attending to how they got stuck in the first place, nor how they got going again. I propose to dwell in the states of becoming and being stuck, and to use this as a springboard to examine and amplify the notion of ‘having some possibility come to mind’ as a means to get unstuck. This will include an expansion of the notions of *system 1* and *system 2* (automatic-habitual reaction and considered response) so as to take account of the full human psyche and development of the role of reflection. My method will be as phenomenological as possible, drawing on specific accounts from my own experience, but hoping to resonate with the experience of participants.

## Introduction

Everyone gets stuck some times. As you doubtless know, I have often said that “being stuck is an honourable state” and that if you don’t get stuck, you are unlikely to learn much about fostering and sustaining mathematical thinking in others, much less in yourself (Mason, Burton & Stacey 1982/2010). Put more sharply, “A solved problem is as useful to the mind as a broken sword on the battlefield” (Shah 1970 p119). Learning to persevere, to enjoy the state of hopeful not-yet-knowing but being on the edge of knowing is valuable, while feeling stymied and hopeless, even helpless, is not so useful because once negative emotions kick in, the current state of feeling hopeless may transfer to a sense of oneself as hopeless, whether at doing this problem, at this topic, or even at mathematics altogether. What is a current state of ‘can’t yet’ can all too easily turn into ‘can’t’ and may be transferred into ‘can’t ever’ which is manifested in the future as ‘won’t try’. Dweck (2000) reports research-based techniques for trying to persuade students to convert back to the language of “didn’t” rather than “can’t”.

Usually we are eager to get unstuck. As soon as an action is enacted we are off and running, without attending to how that came about. But as a teacher it is vital to be aware of how learners become stuck, and what sort of prompts will enable them not simply to get unstuck but to learn from the experience rather than amplify the desire to ‘get unstuck at all costs’. Here I want to dwell in and on the state of being stuck, to consider briefly the psychology and sociology of that state in the light of various theories and commonplace clichés, and to elaborate on the role of reflection in developing a repertoire of mathematical actions which can inform future practice.

## Method of Enquiry

My preference always is to work with lived experience, phenomenologically. That means generating immediate experience which may resonate (or dissonate) with your past experience. I am usually less interested in my own musings than in evoking awareness through invoking actions in those with whom I am working. However, in regard to ‘getting stuck’, there simply is not sufficient time to provide tasks that might get readers stuck in different ways, because different people will get stuck on different things in different situations. So my method here is to recall my own experience, and, through brief-but-vivid accounts, to evoke memories of similar parallel experiences in my readers. It is reader’s memories which constitute the ‘data’. This way of working is consistent with what I call the *discipline of noticing* (Mason 2002).

Accounts of instances from my own experience will be followed by what they bring-to-mind in the way of psychological and sociological insights from the literature and from my experience. Thus I will not waste time adumbrating in advance various frameworks of distinctions which constitute my underlying theories, but rather let these emerge in the analytic considerations following reported incidents. Experiences generalise to phenomena, and these bring-to-mind, that is, resonate and trigger discourse from the literature which can be used to make distinctions and connections in and between experiences. It is worth noting that it is the phrase ‘bring-to-mind’ or ‘have come-to-mind’ that I particularly want to probe in relation to so-called Dual Systems theory reflection, and ‘getting unstuck’ mathematically..

## Being Stuck

In what follows I offer a sequence of incidents from my own experience, which may prompt personal reflection in the reader, followed by some analysis connecting observations to the literature.

### Incident 1

When my son was about 8 or 9, he asked where I was going that day, and I told him I was going to Liverpool to give a talk. He asked what it was about so I told him that the title was “What to do when you get stuck”. After a thoughtful silence he announced “Get out and push!”.

This is extremely sage advice from one so young! The whole issue about being stuck is how to ‘get out and push’. Here, ‘get out’ refers to becoming sufficiently aware of being stuck so that you can withdraw sufficiently from the current action(s) so as to change actions. ‘Push’ refers to actions that become available as a result of being released from being stuck. Until you are aware of being stuck you are, indeed, immersed in ‘being stuck’. Even when you are aware of being stuck, you may not have any alternative actions available, and even if you have a possible action in mind, you may not have sufficient will-power to initiate it. Becoming aware of being stuck is by no means inevitable, nor is it sufficient for getting unstuck, but it is a good start!

### Reflection 1A

Have you ever been stuck but only much later realised that you were in fact stuck?

For example, have you ever been looking for something, finding yourself repeatedly looking in the same places, convinced that it must be there somewhere? Or have you ever carried out the same computation several times, often on a fresh piece of paper, vaguely hoping or expecting that the calculation will ‘work out’ (the way you hope or expect) this time? One state of being stuck is to be so immersed that I am blissfully unaware of it. Usually I am ever-hopeful that I am on the verge of making progress. I experience this frequently when programming: ‘the next tweak will sort out the current bug’ is almost a mantra!

There is an eastern teaching story that speaks to this condition, concerning someone who, late at night encounters a person searching for a lost key under a lamp post. After joining the search for some time, the first asks the second where the object was lost. “Ah, over there”, came the response, pointing to the other side of the street where it is dark. “Then why are you looking here?”. “There is more light here”. A more succinct version is the adage that “the thing you are looking for is always in the last place you look”, making use of the idiom that ‘the last place you would look’ is unexpected or considered to be unlikely, as in ‘that is the last place I would look’. For example, a friend lost a contact lens; it was eventually found in the door lock recess, as a result of the joking remark “it couldn’t be in the …, could it?”!

The adage can actually come to attention through some combination of metaphoric resonance between the situation and other experiences you have had, and metonymic triggering through affective memory. It can then assist in the process of becoming aware of being stuck.

### Reflection 1B

Have you ever been stuck, with a vague sense of being stuck but without seeming to be able to do anything about it?

This has happened to me many times. I recall sitting at a desk in my master’s year at college, struggling to complete an assignment by the next day, and repeatedly being vaguely aware of being stuck, of staring unfocused at the ceiling, of not knowing what to do, of not having any suitable action come to the fore, before eventually becoming sufficiently aware that I could in fact act by going and getting a drink, by standing up and walking about, or by stretching. Sometimes letting go enables fresh neural networks to be activated!

In between the state of being sufficiently explicitly aware of being stuck so as to be able to take action to deal with it, and the state of being immersed or even lost in being stuck there seems to be a whole spectrum of states with varying degrees of being vaguely aware of not making progress, of repeating actions already enacted, of not knowing what to do, and of no action taking place at all.

### Analysis 1

There is a complex psychology here, because the hope or thought that progress is imminent can block admitting being stuck and so can stunt progress. Yet taking a positive rather than a negative stance may sometimes actually enable progress. For example, when trying to debug an applet, I have noticed a fear that to break off work will be to lose the thread of the changes being made, while at the same time being aware that progress is not in fact being made, and that I am clutching at straws. Often a break of a few minutes, hours or days brings fresh and sensible possibilities to mind that were blocked by the previous tinkering (Mason 1985).

It seems to me that these states of being stuck can all be accounted-for in terms of whether sufficient energy is available so as to initiate some considered action, trying to go around the obstruction rather than repeatedly ‘crashing on the rocks’ by trying to barge through. We see this when watching a fly buzzing at a window, while a dog, faced with a window, will seek a way around. Put another way, we can investigate and account-for being stuck in terms of whether there is sufficient energy to activate a self other than the self that is dominating but which is absorbed in being stuck.

The human psyche can be thought of as comprised of multiple selves, as for example in Plato (Hamilton & Cairns 1961); Bennett (1964); Hudson (1968); Minsky (1986); deGeest (2006) and others. Each ‘self’ has characteristic triggers (as in Minsky’s ‘default parameters’), characteristic ways of transforming energy, characteristic ways of acting, and characteristic sensitivities. The characteristic energy transformation of a particular self may be negative, positive or ambivalent. The ways of acting can be introverted or extroverted, socially sensitive or socially unaware, and so on. Different selves are seen in the Bhagavad Gita as different combinations of the three Gunas: *rajas* (initiative), *tamas* (receptiveness) and *sattva* (detachment), which together structure *Prakriti*, the realm of the seen (Ravindra 2009 p72). Getting unstuck can be seen as experiencing a shift of dominant self so that fresh action is possible through a different combination of *rajas*, *tamas* and *sattva*, and hence of processing and transforming energy, leading to a change of state. Change of state is not always immediate, as the next incident illustrates.

### Incident 2A

I recall vividly as an undergraduate trying to get to grips with linear functionals on a vector space. These are linear maps from the vector space to the field of scalars. The dual of a vector space is the space of linear functionals on *V,* and is denoted *V*\*. As I recall, the situation involved the linear functionals on *V*\*, so the dual of the dual, *V*\*\*. If *L* is a linear functional from *V* to *K*, then we are concerned with linear functions *f* mapping *V*\* to *K*, among which are the vectors *v* in *V*, because *fv* (*L*) = *L*(*v*) is such a mapping. I recall a state of mystery and wonder at what this notation was saying. I remember repeatedly writing down examples, with the fog, the ‘cloud of unknowing’, gradually dispersing until I could see that it was ‘just’ what is written. I was able to show by means of this ‘example’ that *V* is isomorphic to a subspace of *V*\*\*, and I enjoyed the power of the notation, but I recall being left in doubt as to what other elements might look like, and being both reassured and mystified by the fact that for finite dimensional vector spaces, *V*\*\* and *V* are isomorphic, but not when *V* is of infinite dimensions. What could these other elements look like?

The specifics are of course not important, as long as there is some resonance with struggling with some concept and gradually experiencing the fog lifting.

### Incident 2B

I recall in my early years at the Open University being introduced to the function

 when λ = μ = 1

as an example of a function which is continuous at 0 but has arbitrarily large slope arbitrarily close to 0. I gained familiarity with it by introducing the parameters λ and μ and exploring variations which are differentiable at 0 but arbitrarily large slope arbitrarily close to 0. By showing it to lots of people while rehearsing the reasoning, not only were my concept images for continuity and differentiability enriched, but my intuition about slopes actually changed.

### Incident 2C

Working on the issue of  with some third-year undergraduates from the mathematics department of a prestigious university who were taking a mathematics education course, I recall vividly that one of them said “it may be 1 in analysis, but not out on the street”, and this was agreed to by many of the students.

### Analysis 2

All three incidents speak to ways in which familiarity and confidence concerning a complex or unfamiliar idea can develop, at various speeds, over time. Sometimes a fresh idea is grasped, assimilated, internalised immediately, but most often there is a gradual adoption of a way of thinking through adopting and internalising a way of acting. James (1890, 1925 p201) called this ‘acting-as-if’, which he saw as a very effective strategy for changing one’s state. Incident 2A was quite quick, a matter of a few hours at most, whereas incident 2B took place over several months. Incident 2B relates to being stuck because an unexamined intuition is a form of rut in which it is possible to be trapped, unwittingly. Even when you have become aware of it, the educated intuition or insight may not come to mind in the moment when under pressure, allowing the naïve intuition to hold sway. Fischbein (1987) based his claim that intuitions are not displaced, merely overlaid, on this sort of experience displayed by learners.

The issue of is one that I have found engages undergraduates. It seems that there are deep seated concerns or questions, and that over a period of time, as a teacher, at some point one becomes aware that rather than going through the motions when working with others, the fact that the value is 1 has been internalised and accepted. But this can sometimes take years. Of course the value is 1 if you are working in the standard model of the reals, but in non-standard models, it is not 1, because it differs from 1 by an infinitesimal. This can provide a deep challenge, especially when  has finally been accepted, appreciated, internalised and apparently comprehended.

Gattegno (1987) used the term *awareness* to mean ‘that which enables action’, which might not be conscious (such as somatic actions like adjusting and maintaining breathing and heart rate). He spoke of ‘educating awareness’ meaning the process by which actions became internalised, and he proposed that ‘only awareness is educable’. It seemed to me, drawing on common images from the Upanishads and elsewhere that this is in contrast to ‘only behaviour is trainable’ and ‘only emotion is harnessable’ (Mason 1992a) to which might be added, “only attention can be directed”. Gattegno noticed that very often awareness is educated by “integration through subordination” (Gattegno 1987, 1990; see also Hewitt 1994; Young & Messum 2011), in which attention is deliberately drawn away from the action so that the action can be carried out in the future using only the minimum necessary attention to be carried through. calling upon the intellect only for occasional guidance when something unusual happens. This is typical of expert behaviour: the industrial slogan ‘just in time’ works for efficient action too: only sufficient minimal attention needed for an action need be focused on the action, freeing attention to be directed elsewhere.

A completely different discourse based on the biological metaphor of *assimilation* and *accommodation* was used by Piaget (1971) for much the same idea, though it led him to focus on *reflective abstraction* in which the learner is drawn out of immersion in action in order to become aware (consciously) of the action, with the intention of invoking it in the future. Building on a lecture by Bennett (1976) I articulated the *discipline of noticing* as, among other things, a collection of actions and practices designed to enhance the possibility of having a desirable action come to be enacted in the moment when it is perceived to be needed. Vygotsky (1978, 1981) spoke of ‘internalising higher psychological processes’ as being accessed first through the social by being in the presence of more experienced people manifesting useful actions, and then internalising these. It seems clear that each of these discourses adds a dimension of richness to the process of becoming familiar with, gaining facility with, and integrating into one’s functioning, useful actions for getting oneself unstuck.

Incidents 2B and 2C illustrate a social aspect of coming to grips with concepts which challenge intuition, whether naïve or sophisticated. Participating in a community, whether as leader (teacher) or as participant can contribute to a gradual adoption of ways of thinking and acting. Bruner (1991 p4) asserted that “we organise our experience and our memories of human events mainly in the form of narrations”. It is through narrative construction that we weave together the fragments of incidents, blending them into a single narrated experience. These narratives are the source of our sense of ‘I’, our supposed identity, even though, as Eastern mystics have maintained (Ravindra 2009) and as Norretranders (1998), Kahneman (2012) and others have shown, this is largely an illusion. Although James (1890 p224ff) talked about the ‘stream of consciousness’, it seems that most often not only do we recall experience in disconnected fragments which we try to glue together into a continuous narrative, but experience itself is fragmentary (Mason 1998, 2002). Even our memories of incidents are fictions, glued together from fragments so as to give ourselves a sense of coherence and unity. But trying to articulate to others can make a significant contribution to clarifying for oneself, to comprehending and appreciating connections and relationships. As the adage has it, ‘the best way to learn is to teach’.

It can take a significant period of time during which one acts, as James (1890 Vol 2 p321; 1925 pref) suggested, *as if* something is the case, eventually coming to accept and to internalise that acceptance so that future functioning takes it as being the case. Such internalised intuitions can be even more robust against further challenge than naïve intuitions. Fischbein (*op cit*) maintained that intuitions are never displaced, simply overlaid, and certainly there are times when the previously overlaid intuition seems to surface without the guidance of any subsequently internalised intuition. For example, Paz & Leron (2009) report struggles of teachers coming to grips with functional programming, who, when they applied a function to an object, thought of the object itself as being changed, rather than as the action producing a new object. Even after they were entirely familiar with the issue, and functioning competently, under pressure they reverted to previous behaviours. Unexamined assumptions can contribute to familiar actions not having the desired effect, and getting to grips with a new way of thinking can take time, a sensibility crystallized in the famous grook T.T.T. (Piet Hein’s webref). Throughout the period there may be periods both of waxing and of waning confidence and clarity, all forms of being in transition, and a form of being stuck. The onion model for the growth of understanding which was elucidated by Pirie and Kieren (1989, 1994) nicely captures this ebb and flow.

Vergnaud (1983) used the term *theorem-in-action* to refer to a related phenomenon, in which students *act as if* they know a mathematical result, even though they have no explicit articulation. For example, young children often act as if they know that addition and multiplication are commutative; people accept that is the same as without being able to explain why, and often without even being aware that there is something to explain. This is a subtle form of being stuck because you seem to be able to act, and yet you are ‘stuck in a conceptual rut’, confined by habits of thinking and acting.

## Becoming Unstuck: Pushing & Pondering

Sometimes an action is merely part of being stuck in a rut, but sometimes a fresh action is enacted. Where do these fresh possibilities come from, and how might they be accounted for psychologically and sociologically?

### Incident 3

Early in my first year of high school (age 13), I recall being given a test in which one item consisted of four 4-digit numbers with the decimal point in different places, and being asked to add them up. Having no idea what the dot referred to I aligned all the numbers, added, and then briefly pondered where to put the dot. I don’t now recall where I put it, but I was definitely aware that I was unclear!

### Analysis 3

When we don’t know what to do, we frequently do whatever we can do. This may mean enacting some internalised act, which may or may not get processed through consciousness. For example, some students, when given a task, do or say the first thing that occurs to them, often without ‘thinking’. Mathematics lessons provide an opportunity to work at responding thoughtfully rather than reacting automatically, to work at ‘parking’ initial impulses until they have been thought through or displaced by a better idea, and this could carry over into the awkward period of adolescence when choices are ‘made’ spontaneously and not always with much forethought or criticality.

Brown & van Lehn (1980) noticed that students are prone to enacting fragments of ‘learned procedures’. Sometimes they may be uncertain as to what to do, as in incident 4, and sometimes they don’t get as far as cognitive processing because a possible action is enacted and off they go in whatever direction it takes them. Van Lehn (1989) developed a comprehensive theory for how learned procedures develop bugs leading to both systematic and unsystematic errors in execution.

Bob Davis (1984) accounted for children adding numbers when working on a word problem in which they didn’t know what else to do, by suggesting that they had spent a long time at first learning to add, then less time learning to subtract, even less to multiply and hardly any to divide, by comparison. So naturally people turn to what is most familiar, most secure and confidence inspiring. This aligns with a framework called MGA (Manipulating–Getting-a-sense-of–Articulating) developed in the 1980s at the Open University (Mason & Johnston-Wilder 2004a, 2004b) which arose from a mixture of experience and the notion of spiral learning (Bruner 1960). We proposed that when you are stuck it is natural to turn to something confidence inspiring, like a familiar example, or a special case. Something particular where you can act. The purpose of specialising, of manipulating the particular, is not to get an answer, but rather to see what is going on, to get a sense of underlying relationships. These may then be articulated, and as the articulation gains in facility and succinctness, it itself becomes a confidently manipulable entity, to be turned to in the future.

The incidents offered so far all indicate the difference between reacting and responding (Mason 2002, 2009), which is the essence of the dual system theory propounded by Kahenman (2002, 2012) and Kahneman & Frederick (2005), drawing on a distinction first made by Stanovich and West (Kahneman 2012 p48, p450), and used to account for phenomena in mathematics education in Leron (undated) and Leron & Hazzan (1997). In the theory of dual processes, system 1 (S1) consists of automatic, habitual reactions which are enacted without cognitive processing (James 1890 p128-144). System 2 (S2) is a slower-functioning cognitive critical rationality. S1 provides S2 with conjectures based on whatever limited data is available, in line with past experience, a process characteristic of abduction (Eco 1983). It is often essential to act immediately, without considering pros and cons, but it is sometimes valuable, even essential, to respond rather than react, to park initial impulses.

The etymological roots of *reacting* and *responding* are instructive, since *re-acting* refers to a cognitively unprocessed act that is initiated without thought. It is the functioning of automaticities, either those rising genetically from the species, or those internalised, integrated into functioning.

Without habits we could not cope with the myriad of incoming stimuli, but at the same time, some habits are less than helpful and actually interfere with making a suitable response. By contrast, the etymological roots of *respond* are in the Italian verb *respondere* meaning ‘able to justify’, signalling involvement of the cognitive dimension of the psyche, spinning a story to account for actions already taken or at least prepared.

The difference between reacting and responding is critical in education because it can be used to account for a good deal of student behaviour, for example, when someone known to ‘know’ nevertheless makes a silly mistake, or when a state of being stuck arises because immediate habitual reactions lock students into an unhelpful line of thinking (see also Paz & Leron 2010 p36).

To illustrate system S1, Kahneman & Frederick (2005) use the following example:

### Incident 4

A baseball bat and ball cost together one dollar and 10 cents. The bat costs one dollar more than the ball. How much does the ball cost?

This ‘incident’ is not so much about being stuck in not knowing what to do, but rather about reacting with a ‘conjecture that requires modification’, that is, being stuck in a rut of inappropriate action. Almost everyone reports an initial tendency to answer ‘10 cents’ because the sum $1.10 separates naturally into $1 and 10 cents.

### Analysis 4

Frederick (quoted in Paz & Leron 2009 p23) found that many intelligent people yield to this immediate impulse: 50% (47/93) of Princeton students and 56% (164/293) of students at the University of Michigan gave the wrong answer (Kahneman 2002, p451; see also Kahneman and Frederick, 2005, p273). Kahneman uses this as evidence to support a distinction between S1 and S2. They correspond quite well to the common sense notions of intuitive and analytical thinking, and are in alignment with the ancient psychology of the Bhagavad Gita and the Upanishads where perception and cognition operate at different speeds and in different ways. Kahneman (2012) sees S2 as a kind of lazy monitor (p105), fed proposals by S1 and disinclined to be critical. He uses the distinctions to urge a greater rationality based on educating intuitions concerning statistics and being apprised of the many ways in which S1 deceives us through actions such as priming (p52-58), availability and a range of other factors.

There is neurological evidence that these modes are activated by different parts of the brain, and that they have different evolutionary origins. S2 is evolutionarily more recent, largely reflecting culturalevolution. S1 acts are somewhere between perception and (analytical) cognition, and there are important consequences for how empirical findings in cognitive psychology are interpreted, including application to mathematics and computer science education research. (Paz & Leron 2009 p23)

However, when a more refined model of the human psyche is used, drawing on aspects of ancient psychology there are indications of a whole spectrum of modes, and of ways in which these modes interact. Starting from the notion that we function differently in different situations, the human psyche can be thought of as consisting of multiple selves, as indicated earlier, with characteristic ways of transforming and using energy. Each self is a particular interconnection between cognition, affect, enaction and attention/will, and each of these ‘centres’ itself has cognitive, affective and enactive components (Ouspensky 1950). For example, each self is associated with particular dispositions (an affective dimension) which itself comprises a particular relationship between cognition, affect, behaviour and will. As James (1925 p100) noted, attention to be active or passive, and requires constant reactivation in order to maintain its focus (James 1925 p104). Thus, as the next incident indicates, sometimes a self with the disposition to persevere is over-ruled by another self which wants to get unstuck and get on with something else, and because of stronger claims or stronger energy flow, it dominates temporarily.

### Incident 5

On the look out for situations exploiting symmetry, I recently noticed the following result in the solutions to problems section of a journal (Dolan 2014):

.

Try as I might I could not see how to prove it. I tried particular examples and I tried generalising from the detailed workings of particular examples; I became aware of a connection with the Euclidean algorithm; I drew a diagram that gave the result but which I could not see how to justify. Nothing ‘worked’.

Eventually I resorted to looking at the solution (I had enough other problems I was working on and did not want to divert my attention!). I followed and reproduced the reasoning. But later, thinking through the reasoning while on a bus without access to the article, I got stuck again! I could not reproduce the elementary observation on which the reasoning depended. Partly it was because I could not remember ‘the solution on the page’, and partly because my unhelpful previous thinking kept coming back and getting in the way. I puzzled and puzzled, trying to reproduce what I had (thought I had) previously appreciated and comprehended.

The key idea which I kept missing is that  counts the number of integer points above the *x*-axis and on or below the line *y* = *mx* at *x* = *k*.

Although I had tried to count lattice points myself it had been without success. Having written this paragraph, I went back to the problem and suddenly saw how it worked. I re-produced the reasoning for myself from scratch, and it feels as though won’t forget it in a hurry!

### Analysis 5

For me this was a good example of being stuck in a rut, of one particular ‘self’ with a fixed perception, attention structure and action dominating, trying to connect algebra to a diagram but thinking in an ineffective way. I was focusing on what proved to be unfruitful, yet returning to it in the absence of any other ideas coming-to-mind, like the person searching under the light! In order to learn from the experience it is necessary to reflect back and to consider what might have been effective actions. I did try examples, but only arithmetically; when I tried to use a diagram, I did not hit on an effective way of displaying the problem. To get out of a rut, it may not be sufficient to ‘get out and push’: sometimes you have to ‘get out and ponder’, waiting for inspiration, clearing your mind so as to be prepared for a fresh juxtaposition, as Hadamard (1945) describes. The biblical parable of the wise and foolish virgins comes to mind from Matthew 25: 1-13. Being prepared meant not being fully immersed. It means having an inner witness or observer (Mason *et al* 1982/2010), what Schoenfeld (1985) called an ‘inner executive’. This is the role of S2. The witness is what is available to be awake to unexpected situations, connections and insights. The richer the repertoire of successful actions, the more likely it is that something useful will come to mind.

Frustrated, I turned to help from outside, but found that that ‘help’ was not sufficiently robust to inform my future activity. I subsided back into the rut. Here is another incident related to ‘seeking help’, and how that can inform future activity.

### Incident 6

I recall about the age of 12 or 13, running into a logic problem in which information was given about various people, mostly in relationships, and you had to work out who lived where, did what and was married to whom. Eventually I gave up, looked in the back of the book, and discovered logic tables: two-way tables listing different attributes related in the given information. I used the technique several times and then stopped doing that kind of problem because I felt I had mastered it. It was no longer a challenge (cf the adage of the broken sword on the battlefield: Shah *op cit*)

I recall about the age 14 or 15 pondering the problem of covering a chessboard with a pair of opposite corner square missing, the dominos each exactly covering two squares. Again I couldn’t see what to do, but once I picked up a clue from starting to read a solution, I was able to use the technique to work on variations of the problem.

### Analysis 6

An extreme stance is that expected of students entering an R. L. Moore type course (Jones 1972; Coppin *et al*. 2009), where, in order to get admission, you had to answer the question “what do you do when you get stuck?” with some version of “persist”, together with “not go to the library, not look up on the internet, not ask someone else, but persevere”. An equally extreme stance is to copy out and submit solutions from someone else, but there is a spectrum of various acts of seeking and learning from outside help, whether from peers or relative experts. Even Moore admitted that when stuck, he would carefully reveal proofs or examples to himself, line by line, seeing only enough to suggest a direction to develop. His inner witness, the disposition of his dominant self had sufficient control of energy flow to resist looking ahead. But again, getting unstuck this time is of little value beyond the immediate goal of completing a task. What matters is learning from the situation.

The issue, as always, is not what you do, but what you do with what you do: do you learn from experience, or are you satisfied to have located the answer this time? Getting answers is not learning mathematics; informing future practice through reflection on recent experience is. How is recent experience combined with past experience and integrated into your functioning; how do you learn from the experience. Basically, it is to construct your own narrative, your own story for the concept or the technique. For example, McGowen (1998) found that among students in a two-year college in the USA taking a compulsory mathematics course who were called upon to make frequent concept maps of topics, those whose topic maps grew in sophistication, stuck out the course, whereas those whose topic maps showed no sign of development and looked different each time were the ones who dropped out or failed.

## Reflection, Post-Paration and Pre-Paration

Getting stuck is not in itself a recipe for learning, nor indeed is a succession of getting stuck and then getting unstuck again. At one extreme, when you are stuck you can consult a list of heuristics (I have seen as many as 99 assembled in one place) and somehow choose one that might be of help in your situation. At the other extreme, some action ‘works’ and you find yourself unstuck and forging ahead, without trying to learn from the experience. As I have said many times (Mason 1992)

One thing we don’t seem to learn from experience,
is that we don’t often learn from experience alone.

Something else is required. Worse,

A succession of experiences does not add up to an experience of succession.

This turns out to be a version of the assertion by James (1892 p628) that a succession of feelings does not add up to a feeling of succession. In other words, more is required than simply a succession of experiences if we are going to learn from them.

Schön (1983) distinguished between *reflection in action* and *reflection on action*. He used the former to mean being sensitive to possibilities in the moment rather than being driven entirely by habits and automated-internalised action. But it could also mean an inner witness which is able to observe and comment without being embroiled in the action. The Rg Veda has a stanza which speaks to this:

Two birds, close-yoked companions, cling to the self-same tree.
Of these one eats of the sweet fruit,
The other, nothing eating, looks on intent. (Zaehner 1966 p210)

Sometimes interpreted concerning mortality and immortality, the stanza can also be read as a description of a state sometimes referred to as cosmic consciousness (Burke 1905), of mindfulness (Langer 1997), or self-observation (Ouspensky 1950). The second bird is a witness or observer which can ask questions (why are we doing this?) but does not itself initiate action associated with the specific activity.

Reflection-on-action comes later, in the post-paration phase (as distinct from pre-paration, and from *paration* (literally, put aside) which I use to mean noticing in the moment, as in ‘reflection-in-action’). Post-paration can be turned into pre-paration through imagining yourself in some future situation as vividly and completely as possible, enacting some action which helped you get unstuck this time (Mason 2002). This is how experience informs learning.

When a student is stuck and a teacher intervenes, there are all sorts of possibilities. John Holt (1964 p24-25) describes how Ruth takes him for a ride, biding her time until he asks a really simple question that she can answer, but with no idea of how that relates to the original difficulty. Bauersfeld (1995) used the term *funnelling* for a sequence of questions in which are ever more explicit, specific and pointed until the student can give a response without thinking. Consequently the student is unlikely to be aware of the origins of the teachers’ questions, so there is no cognised sense of a sequence of actions that the student might use for themselves in the future. Leron & Hazzan (1979) provide a possible inner monologue of a student in a funnelling situation, in which the students’ attention is as much on the process of the question sequence as on the questions themselves. They suggest that the student is coping with the situation, trying to make sense and trying to meet expectations, but not being helped to formulate a narrative. Coping with the immediate situation seems to amplify spontaneous reactions (or else silence) rather than promoting thoughtful, considered response.

Nelissen and Tomic (1996) review some 20 years of Russian research into the nature and development of reflection. They see reflection as inner dialogue and as such, as an instance of a ‘higher psychological process’ in the sense of Vygotsky (*op cit*). Through dialog people find their own ideas challenged by others, and learn to challenge themselves. But reflection is also the process by which scientific concepts emerge, what Gattegno (*op cit*) referred to as ‘educating awareness’. Davydov (1990) challenges the classic Western sequence of ‘enactive–iconic–symbolic’ in which ideas are met first with material objects, then diagrams and images, and then symbolic expressions. Rather, Davydov proposes that conceptualisation comes about most effectively when ‘well established ideas (ideal models that can be seen as propositional representations) are used to understand reality’ (Nelissen & Tomic 1996 p39). Later researchers have blurred the distinction and found it more useful to see the empirical and the theoretical as co-emergent. In a range of experiments with young children, evidence has been garnered to support the notion that young children can display reflective behaviour (explicit awareness of actions) and that children can be enculturated into a reflective stance, with good results (Nelissen & Tomic 1996 p39-41).

There is considerable evidence from post-Davydovian projects that promoting self-narrative contributes to learning such as Schmittau (2004) and Dougherty (2008). Others, such as Chi (1989); Chi, Bassok, Lewis, Reiman & Glasser (1998); and Chi, de Leeuw, Chiu, & LaVancher (1994) have developed a similar idea independently. Self-expalanation has been applied explicitly in mathematics education in various places, notably Hodds, Alcock & Inglis (2014).

Further outcomes of the Russian research has led to distinguishing between ‘personal reflection’ which involves formulating meaning to their own actions, and ‘intellectual reflection’ which focuses on inner questions about the nature of the problem at hand, typically, of the type promoted by Pólya (1962). *Personal reflection* is itself seen as taking three forms: situational pertaining to the particular situation; retrospective; and perspective-generating. The latter arises particularly sharply when there is some conflict, some dissonance between effect and desire, such as the ineffectiveness of some proposed strategy (Nelissen & Tomic 1996 p46). Finally, Russian researchers distinguished between ‘productive’ and ‘reproductive’ reflection. The former involves high levels of intellectual and perspective-seeking personal reflection; the latter is more like the spectrum of ‘being stuck but without a specific action being enacted’ as discussed previously.

### Incident 7

As a possible example, consider the logo of our conference, and consider adjusting it to show a weave pattern , and then generalising to more strands:

Computer drawing packages provide layers, but no ways to interweave. How can this drawing and its extensions be drawn in a drawing program?

My first action was a reaction: I know how to interpose white rectangles so as to hide unwanted lines and so give the illusion of interweaving. So I did. It was not until a much later draft of this paper that I experienced a desire to achieve the drawing more generically. An inner voice asked whether there might be a better way. I was seeking an alternative perspective. I suddenly saw the configuration differently, and how to draw it without recourse to white rectangles. Although I see the logo as made up of strips of ribbon, for drawing purposes the objects did not need to be full length strips.

### Analysis 7

For me this is another example of how letting go and returning freshly can help overcome the automatic firing of reactions, enabling a more considered and creative response to be available. It is not simply a matter of monitoring the automaticities thrown up by S1, as Kahneman (2012 p86) seems to indicate. Rather it is about opening a channel to the flow of a different form of energy, processed by a different self. It is about awakening the second bird, the inner witness, and paying attention to its questioning. It is about self-awareness and reflection.

## Coming-to-Mind: action, emotion, intellect and will

The cliché or stock phrase in English for getting unstuck is ‘coming-to-mind’ or ‘having something come-to-mind’. For example Kahneman & Frederick (2005) use the phrase to mean an action that stems from the automaticities of the speaking part of the brain:

The effect of concurrent cognitive tasks provides the most useful indication of whether a given mental process belongs to system 1 or system 2. Because the overall capacity for mental effort is limited, effortful processes tend to disrupt each other, whereas effortless processes neither cause nor suffer much interference when combined with other tasks [...] People who are occupied by a demanding mental activity [...] are much more likely to respond to another task by blurting out whatever comes to mind [...]. (Kahnemann & Frederick 2005 p268)

It seems that the term *mind* is used by many people to encompass a variety of aspects of the human psyche. Relatively recently neuroscientists have been adjusting their theories of mind to acknowledge what has been known for centuries in the East, that the intellect does not always initiate action. Rather action and perception occur together, with action, particularly re-action, preceding conscious thought. For example, Guillery (2014), based on Sherman & Guillery (2013) proposes a reinterpretation of the thalamus as a gateway or bifurcation point where instructions to motor neurons (to act) are copied to the cortex: it is not the cortex that initiates the signal to the motor-neurons. Norretranders (1998) cites neuro-scientific evidence that the assumption that consciousness is the generator and director of action is an illusion. Rather, consciousness is a post-facto phenomenon. Mandler (1989) proposed what was well known in Eastern psychology, that especially under duress, the body reacts first, emotions become aroused next, and thoughts of the intellect-cognition emerge later, a slow third. This can be verified by close observation of oneself, but most clearly when there is a sudden noise: the body acts, then energy starts to flow in and through emotions before any coherent account is available to the intellect. From Eastern psychology also comes the notion that human beings construct narratives which help to self-calm (Ouspensky 1950), in order to maintain the user-illusion that the intellect is in charge. This aligns perfectly with the notion promoted by Bruner (*op cit*) of human beings as narrative animals.

If ‘mind’ is reconfigured to consist of action, emotion, intellect and will, then reaction refers to the activation of the characteristic functionings of a particular self, while response refers to engagement of one or more selves with all aspects fully contributing to the direction of action, including will.

It would be useful to have some words concerning affect which parallel the terms *reaction* and *response* as behaviours. Curiously, I don’t think there is a word in English that signals an intermediate state of emotion or affective ‘reaction-response’. I am finding it useful to speak therefore of *coming-to-act*, *coming-to-heart*, *coming-to-thought* and *coming-to-will* as four distinctive aspects of what is usually clustered under the expression *coming-to-mind*. That affect is too easily overlooked is evidenced by the dualism underpinning system S1 and S2. Coming-to-heart refers to the triggering of emotions, often through idiosyncratic metonymies which release energy through the arising of specific emotional states. Coming–to-head refers to the arising of thoughts characteristic to the particular self. Coming-to-will refers to the production of enough energy to enable initiative to be taken, and attention to be intentionally directed rather than wallowing in habitual ruts. Because of the relative speeds, an action may be enacted before anything else happens; emotions may then be triggered, permitting flows of energy in characteristic ways, and these may stimulate narratives to reaffirm the dominance of the intellect and series or intentions to take initiative.

All of these can take place automatically, through habit, through well established pathways, in S1. They can also contribute to the functioning of S2, to intentional, participative involvement in considering possibilities. It may be more informative therefore to see S2 as a spectrum of different selves contributing to the ‘common presence’ of the individual within the socio-cultural-historical norms into which they have been enculturated.

If we took our time over every action we would of course stop breathing, and even at the macro level we would instantly become dysfunctional. But it is also important to continue to question the efficacy of habits, to be in a state of being challenged about the actions we enact, the emotions we evince, the thoughts and ways of perceiving that we experience and the ways of attending that have become second nature.

In terms of being stuck and getting unstuck, in order for automaticities to get us unstuck it is necessary to have developed a repertoire of actions and of experience of successful use of those actions so that possibilities come to action, to heart, to head and to will. But in order to be creative, to experience insight, and to make connections between apparently disparate situations, it is valuable to foster mindfulness in which we participate fully in making choices by considering possibilities and consequences before acting, before emoting, and before launching into customary patterns of thinking, perceiving and initiating.

The notion of mindfulness (Langer 1989) has been developed recently into a Western approach treating depression and other clinical ailments, but also for general living (Williams & Penman 2011). Of course this is of no surprise to anyone acquainted with Buddhist teachings. Here there is a state beyond ordinary consciousness, when an inner witness is present, observing, but not engaging (Mason 2002). Roots for this go back as far as written records, as in the stanza from the Rg Veda quoted earlier.

The other lesser-known aspect of coming-to-mind, namely coming-to-will deserves more attention. How is it that I can be aware that an action is required, even be consciously aware of what is required, and still not enact that action? Examples abound, at least in my own experience. It is as if there is an energy threshold which needs to be exceeded before action is initiated. Note that this is in complete contrast to re-action which is immediate and unconsidered.

### Incident 8

That there is no algorithm for getting unstuck is not only clear from experience, but within mathematics, it can be proved mathematically. As a final example, here is a problem I posed for myself in 1970, and which I am still stuck on! It could provide an opportunity for readers to try to catch the ebb and flow of energy, the coming to the fore and the receding of dispositions to engage or not, the coming to action, heart head or will of possibilities.

#### Circular Dissection

Dissect the first circle below into 4 congruent pieces.

    

Dissect the second circle into 4 congruent pieces differently.

Dissect the third circle into 4 congruent pieces differently again.

Comment

This task-exercise illustrates the amazing powerful pedagogic strategy of asking people to do the same thing three times. Often it releases some creativity, or at least when discussion takes place, people become aware of possibilities they had not previously thought of (Watson & Mason 2005). Here the usual mathematical acts of specialising and generalising offer little or no assistance; specialising to polygons ends up appearing to be merely misleading.

The issue at first is what counts as different. At first people draw a cross, sometimes tilted, sometimes not. Then they try tilting their cross. Sometimes they then get stuck, because the possibility of using a curve rather than a straight line as the edge does not come to action or to head. Feelings of being stuck, or of ‘having done all that is possible’ arise. Once you become aware of ‘curved edges’, there are some interesting things to explore: just how extreme can the curve be?

  

You may have noticed that your view of what counted as different developed, as you saw similarities where previously there had been difference. Can you characterise nicely which paths from the centre to the boundary are acceptable? How might you specify over the phone to someone how to draw an acceptable curve that can be copied and rotated so s to form the boundaries?

You may have found yourself asking whether it is possible with different numbers of pieces. Consider 12 congruent parts.



Surprisingly, when the curve chosen is a circular arc with the same radius as the circle, there is more than one way: some of the pieces can be rearranged, due to symmetry. Furthermore, going back to a division into six pieces and observing a different symmetry produces a completely different dissection.

  

Notice however what is common: the centre is on the boundary of at least six pieces. For me this raises the question of whether there is a dissection of a circle into congruent pieces with the centre not on the boundary of any piece. I have tackled this every so often over the last 40 years but without success (Mason 2010).

The act of problem posing seems to involve recognising a possible variation and asking whether some relationships are or can be preserved. Getting students to pose their own questions, some of which might then turn into ‘problems’, acquaints students with where tasks come from, and alerts them to how mathematics can be used to address problems outside of mathematics itself (Lomon, Beck & Arnbetter 1975; Skovsmose 2009).

## Social Aspects of Being Stuck

The current fad of recommending collaborative group work is based on the observation that often when people work together they can feed off each other’s ideas so that collectively they can achieve more than any individual. Certainly when you are stuck it can be very helpful to seek out a colleague and to try to articulate your current thinking: where you are stuck and perhaps even why.

### Incident 9

Stuck on a problem I go down the corridor and ask a colleague if I can explain my problem. The colleague sits there, doing nothing beyond appearing to give me some attention. At the end, I thank them and return to my office with renewed vigour and a fresh view of my problem.

The mere act of trying to articulate can bring clarity. But this is not usually what happens when people work in groups.

Many people require some private time in order to internalise the task or problem, and to allow some things to come not just to action but to head, so that they can be considered. Only then does it make sense to engage in collective discussion and idea generation. Even so, group work requires individuals to edit their own thoughts, and suppress their own ideas so as to be able to listen carefully and considerately to others. Having generated ideas, a period of personal work, or perhaps paired-work may be needed in order to consider the ramifications of the suggestions. Continuing in a group may lead to the loudest or quickest taking charge while others struggle to keep up or drift off. Every time someone fresh offers a comment, everyone else has to suspend their own thinking in order to take on board the new contribution.

When a group gets stuck, with no fresh ideas being introduced but perhaps old ideas being re-circulated (again, the story of searching for the key mentioned earlier, comes-to-head) it is time to return to individual work (or work in pairs) to wait for ideas to settle and something fresh to appear.

## Conclusion

Being stuck is a state in which either there are no actions to carry out, nothing to do, or where there are actions, but these are either repetitions of previously unsuccessful actions, or the working out of habitual acts, which mean that situations proceed as normal: you are stuck in a rut. Through inspecting experience it seems that there is a spectrum of states of being stuck from being blissfully unaware, through neing ‘sort of’ aware but unable to act, to being explicitly aware.

You can be stuck in a rut, endlessly going over the same thoughts, trying the same actions, without success. In eastern traditions this is sometimes referred to as the *chattering monkeys*, or in the West, the *broken record*: a habit keeps being enacted, a thought keeps recurring, a worry keeps surfacing. You keep looking under the light rather than in a relevant place or in a pertinent manner.

To get unstuck, you need some act to enact, to come-to-action, but if it is based on habit, unprocessed by cognition or thought, it may not actually be helpful, hence the need for the development of an inner witness, of mindfulness, of a monitor or executive which brings possible actions and nascent emotions to cognition before they are enacted or released.

Past experience associated with failure may trigger emotional response (coming-to-heart) which may then block possibilities for coming-to-head or coming-to-will, or, conversely, may trigger a desire to comprehend, producing a burst of energy to take initiative.

A repertoire of possible acts is necessary so that there are actions on which to fall back. These actions may be associated with particular mathematical concepts, with the use of mathematically oriented natural powers, or with themes that pervade mathematics. But actions alone are not enough. The development of a mathematical witness or monitor is invaluable: something that is not caught up in the action and so can draw attention away from the action, enabling more reflective, contemplative thoughts coupled with appropriate feelings. These form what might be thought of as mathematical selves who can take over control, who can process energies in mathematically useful ways, feeding the will so that initiative can be taken to persevere, whether by pausing and contemplating, or by pursuing some fresh direction.

An obvious conclusion, in alignment with experience at the Open University over several years, is that before dropping learners into unfamiliar ‘problems’ it is vital to give them carefully designed tasks which invoke mathematical actions that are going to be useful in the future. For example, imagining and expressing; specialising (trying examples) & generalising; conjecturing & convincing; being systematic and being adventurous; working forwards and working backwards, and so on. When some or most learners have used such an action naturally, their attention can be drawn to it, a label can be provided or negotiated, and then that label can be used to prompt use of that action in the future. Over time, the prompts can be made decreasingly explicit, using prompts like “what question am I going to ask you?” in a process of scaffolding and fading (Seeley Brown, Duguid and Collins 1989; Love & Mason 1992) until learners have internalised the action and developed from initiating it for themselves (consciously, attentively, intentionally) to experiencing it enacted without their explicit volition. They can then be said to have internalised a higher psychological process (Vygotsky), educated an awareness (Gattegno); accommodated (Piaget), integrated an action into their repertoire of functionings.

# Acknowledgements

I would like to thank Anne Watson for comments on an early draft, Michal Ayalon for some stimulating discussions about these ways of thinking, and Jo Nelissen for drawing my attention to the Russian school of reflection research.

# References

Bauersfeld, H. (1995). ‘Language Games’ in the Mathematics Classroom: their function and their effects. In P. Cobb & H. Bauersfeld (Eds.), *The Emergence Of Mathematical Meaning: interaction in classroom cultures* Hillsdale, NJ: Lawrence Erlbaum Associates, p271-289.

Bennett, J. (1964). *Energies: material, vital, cosmic*, London: Coombe Springs Press.

Bennett, J. (1976). *Noticing*. The Sherborne Theme Talks Series 2, Sherborne: Coombe Springs Press.

Brown, J. & van Lehn, K. (1980),. Repair theory: A generative theory of bugs in procedural skills,. *Cognitive Science,* 4, 379-426.

Bruner, J. (1960). *The Process of Education*. Cambridge: Harvard.

Bruner, J. (1991). The Narrative Construction of Reality. *Critical Inquiry* 18 (1) p1-21.

Bucke, R. (1905). *Cosmic Consciousness*. Philadelphia: Innes & Sons.

Chi, M. & Bassok, M. (1989). Learning from examples via self-explanation. In L. Resnick (Ed.) Knowing, learning and instruction: essays in honour of Robert Glaser. Hillsdale, NJ: Erlbaum.

Chi, M. Bassok, M. Lewis, P. Reiman, P., & Glasser, R. (1989). Self-explanations: How Students Study and Use Examples in Learning to Solve Problems. *Cognitive Science.* 13 p145-182.

Chi, M. T. H., de Leeuw, N., Chiu, M. H., & LaVancher, C. (1994). Eliciting Self- Explanations Improves Understanding. *Cognitive Science, 18,* 439–477.

Coppin, C. Mahavier, W. May, E. Parker, G. (2009). *The Moore Method: a pathway to learner-centred instruction*. Washington: Mathematical Association of America.

Davis, R. (1984). *Learning Mathematics: the cognitive science approach*, London: Croom Helm.

Davydov, D. 1990, *Types of Generalisation in Instruction*, Soviet Studies in Mathematics Education Vol 2, NCTM, Reston.

De Geest, E. (2006). *Energy Transactions in the Learning of Mathematics*. Unpublished PhD Thesis, Milton Keynes: Open University.

Dolan, S. (2014). Solution to Problem 2013.6. *Mathematical Gazette.* 98 (541) p144-145.

Dougherty, B. (2008). *Algebra in the early grades*. Mahwah: Lawrence Erlbaum.

Dweck, C. (2000). *Self-theories: their role in motivation, personality and development*. Philadelphia: Psychology Press.

Eco, U. (1983). Horns, Hooves, Insteps: some hypotheses on three types of abduction, in U. Eco & T. Sebeok (Eds.) *The Sign of Three: Dupin, Holmes, Peirce*. Bloomington: Indiana University Press, Bloomington, p198-220.

Fischbein, E. (1987). *Intuition in Science and Mathematics: an educational approach*. Dordecht : Reidel.

Gattegno, C. (1987). *The science of education part I: Theoretical considerations*. New York: Educational Solutions.

Gattegno, C. (1990). *The Science of Education*. New York: Educational Solutions.

Guillery, R. (2014). A Neuroscientist’s view of Action, Perception and the Self. *Oxford Magazine.* Fifth week, Hilary Term p10-13.

Hadamard, J. (1945). *An Essay on the Psychology of Invention in the Mathematical Field.* Princeton: Princeton University Press.

Hamilton, E. & Cairns, H. (Eds.) (1961). *Plato: the collected dialogues including the letters*. Bollingen Series LXXI. Princeton: Princeton University Press.

Hein, P. (webref). <http://en.wikiquote.org/wiki/Piet_Hein>. (accessed March 2014).

Hewitt, D. (1994). *The Principle of Economy in the Learning and Teaching of Mathematics*, unpublished PhD dissertation, Milton Keynes: Open University.

Hodds, A. Alcock, L. & Inglis, M. (2014). Self-Explanation Training Improves Proof Comprehension. *Journal for Research in Mathematics Education.* 45(1) p62-101.

Holt, J. (1964). How Children Fail. Harmondsworth: Penguin.

Honsberger, R. (1976). *Mathematical Gems II*. *Dolciani Mathematocal Expositions 2*. Washington: American Mathematical Association. p23-28.

Hudson, L. (1968). *Frames of Mind*, London: Methuen.

James, W. (1890 reprinted 1950). *Principles of Psychology*, Vol 1. New York: Dover.

James, W. (1925). *Talks to Teachers on Psychology and to Students on Some of Life's Ideals.* New York: Henry Holt.

Jones, F. (1977). The Moore Method. *American Mathematical Monthly* 84 p273-77.

Kahneman D. & Frederick S. (2002). Representativeness Revisited: attribute substitution in intuitive judgement. In T. Gilovich, D. Griffin & D. Kahneman (Eds.) *Heuristics and Biases: The Psychology of Intuitive Judgment*. p49–81. Cambridge: Cambridge Univ. Press.

Kahneman, D. (2002). Maps of bounded rationality: A perspective on intuitive judgment and choice (Nobel Prize Lecture), in T. Frangsmyr (Ed.). *Les Prix Nobel*. Accessed Nov 2013 at http://www.nobel.se/economics/laureates/2002/kahnemann-lecture.pdf

Kahneman, D. (2012). *Thinking Fast, Thinking Slow*. London: Penguin.

Kahneman, D. & Frederick, S. (2005). A Model of Heuristic Judgment. In K. Holyoak & R. Morrison (Eds.) *The Cambridge Handbook of Thinking and Reasoning*. Cambridge: Cambridge University Press p267-293.

Langer, E. (1997). *The Power of Mindful Learning*. Reading: Addison Wesley.

Leron, U. (undated). Application of Dual-Process Theories in Mathematics Education (and vice versa). <http://www.open.ac.uk/Arts/dualprocess/abstracts/Leron_paper.pdf> (accessed April 2014).

Leron, U. and Hazzan, O. (2006). The Rationality Debate: application of cognitive psychology to mathematics education, *Educational Studies in Mathematics*. 62(2) p105-126.

Lomon, E. Beck, B. Arnbetter, C. (1975). Real Problem Solving in USMES: Interdisciplinary Education and Much More. *School Science and Mathematics*. 75(1) p63-74.

Love, E. & Mason, J. (1992). *Teaching Mathematics: Action and Awareness*. Milton Keynes: Open University.

Mandler, G. (1989). Affect and Learning: Causes and Consequences of Emotional Interactions. In D. McLeod & V. Adams (Eds.), *Affect* *and Mathematical Problem Solving:new perspective.* London: Springer-Verlag, p3-19.

Mason, J. (1985). What Do You Do When You Turn Off The Machine?, preparatory paper for ICMI conference March, *The Influence of Computers and Informatics on Mathematics and its Teaching*, Inst. de Recherche Sur L'Enseignement des Mathematiques, Strasburg, p251-256.

Mason, J. (1988). Fragments: the implications for teachers, learners and media users/researchers of personal construal and fragmentary recollection of aural and visual messages, *Instructional Science* 17, p195-218.

Mason, J. (1992). Images and Imagery in a Computing Environment. In *Computing the Clever Country*, Proceedings of ACEC 10, Melbourne: Computing in Education Group of Victoria.

Mason, J. (1992a). Researching Problem Solving From the Inside, in *Mathematical Problem Solving and New Information Technology: research in Contexts of Practice,* Ponte J., Matos J., Matos J. & Fernandes D.(Eds.) Nato ASI Series F #89, London: Springer Verlag, p17-36.

Mason, J. (2002). *Researching Your Own Practice: the Discipline of Noticing*, RoutledgeFalmer, London.

Mason, J. (2002a). *Mathematics Teaching Practice: a guidebook for university and college lecturers*, Chichester: Horwood Publishing.

Mason, J. (2009). Teaching as Disciplined Enquiry. *Teachers and Teaching: theory and practice* 15 (2-3) p205–223.

Mason, J. (2010). Expressions of Desire: (re)flections on (re)presentation. *Mediterranean Journal for Research in Mathematics Education.* 9 (1) p55-72.

Mason, J. & Johnston-Wilder, S. (2004a). *Designing and Using Mathematical Tasks*. Milton Keynes: Open University; (2006 reprint) St. Albans: Tarquin.

Mason, J. & Johnston-Wilder, S. (2004b). *Fundamental Constructs in Mathematics Education*. London: RoutledgeFalmer.

Mason, J. Burton, L. & Stacey, K. (1982/2010). *Thinking Mathematically* (Second Extended Edition). Harlow: Prentice Hall (Pearson).

McGowen, M. (1998). *Cognitive Units, Concept Images, and Cognitive Collages*. Unpublished PhD thesis, Warwick University, Coventry.

Minsky, M. (1986). *The Society of Mind.* New York: Simon and Schuster.

Nelissen, J. & Tomic, W. (1996). Reflection in Russian Educational Psychology. *Educational Foundations.* Winter p35-56.

Norretranders, T. (1998). (J. Sydenham Trans.). *The User Illusion: cutting consciousness down to size.* London: Allen Lane.

Ouspensky, P. (1950). *In Search of the Miraculous: fragments of an unknown teaching*, London: Routledge & Kegan Paul.

Paz, T. & Leron, U. (2009).The Slippery Road From Actions On Objects To Functions And Variables. *Journal for Research in Mathematics Education* 40 (1) p18-39.

Pirie, S. & Kieren, T. (1989). A Recursive Theory of Mathematical Understanding, *For the Learning of Mathematics*, 9 (4) p7-11.

Pirie, S. & Kieren, T. (1994). Growth in Mathematical Understanding: how can we characterise it and how can we represent it? Educational Studies in Mathematics, 26 (2-3), 165–190.

Pólya, G. (1962) *Mathematical discovery: On understanding, learning, and teaching problem solving* (combined edition). New York: Wiley.

Piaget, J. (1971). *Biology and Knowledge*. Chicago: University of Chicago Press.

Ravindra, R. (2009). *The Wisdom of Patañjali’s Yoga Sutras: a new translation and guide*. Sandpoint: Morning Light Press.

Schmittau, J. (2004). Vygotskian Theory and Mathematics Education: resolving the conceptual-procedural dichotomy. *European Journal of Psychology of Education.*  19(1) p19- 43.

Schoenfeld, A. (1985). Mathematical Problem Solving. New York, NY, USA: Academic Press.

Schön, D. (1983). *The Reflective Practitioner: how professionals think in action*. London: Temple Smith.

Shah, I. (1970). *The Dermis Probe*. London: Johnathan Cape.

Sherman, S. & Guillery, R. (2013). *Functional Connections of Cortical Areas: a new view for the thalamus.* Cambridge: MIT Press.

Skovsmose, O. Valero, P. & Christensen, O. (2009). (Eds.). *University Science and Mathematics Education in Transition*. Berlin: Springer.

van Lehn, K. (1989). *Mind bugs*. Cambridge, USA: MIT Press.

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.) *Acquisition of mathematics concepts and processes,* p127-124. New York: Academic Press.

Vygotsky L. (1978). *Mind in Societ*y: *the development of the higher psychological processes*. London: Harvard University Press.

Vygotsky, L. (1981). The Genesis of Higher Mental Functions. In J. Wertsch (Ed) *The Concept of Activity in Soviet Psychology*. Armonk: Sharpe.

Watson, A. & Mason, J. (2005). Mathematics as a Constructive Activity: learners generating examples. Mahwah: Erlbaum.

Williams, M. & Penman, D. (2011). *Mindfulness: a practical guide to finding peace in a frantic world.*  London: Piatkus.

Young, R. & Messum, P. (2011). *How We Learn and How We Should Be Taught: an introduction to the work of Caleb Gattegno*. London: Duo Flamina.

Zaehner, R. (Trans. & Ed.) (1966). *Hindu Scriptures*. London: Dent & Sons.