Tunja Sequences
as examples of employing
students’ powers to generalise

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# Introduction

Students come to lessons initially ready to specialize and to generalize, to imagine and to express, to conjecture and convince. It takes them only a few days in school to discover whether those powers can be used there. Since the use of these powers by each student is essential if mathematics is going to be appreciated rather than simply trained, it is vital to seek ways in which, in every lesson, these powers can be employed.

In this note, I hope to demonstrate how students’ power to recognize and express patterns and generality can be exploited in a variety of different ways connected with both arithmetic and algebra. I call this approach ‘Tunja sequences’ (pronounced Toon-ha) because they were first developed for a session in Tunja, Colombia, and then elaborated slightly. Tunja sequences stimulate children to use their powers of pattern detection and generalization to learn to multiply bracketed expressions and negative numbers, and also to factor simple quadratics. They also help children experience quadratic expressions as their own expressions of generality in numbers, and not just as ‘arithmetic with letters’ that have been proposed by others.

# The First Tunja Sequence

The first Tunja sequence is presented line by line, in silence, with a pause at the beginning of each line (making it evident that I am thinking about what comes next), a pause after the equality sign (to signal I am thinking about what is to come), and a pause at the end of each line (to indicate I am checking the arithmetic).

1 x 6 + 6 = 3 x 4

2 x 7 + 6 = 4 x 5

3 x 8 + 6 = 5 x 6

4

At this point I turn to the audience and declare “ I know, that you know, what comes next; am I correct?” and people would signal that yes, they do know what is coming next. When I ask what is next, most people will say “9”, but if I wait long enough, someone will indicate “times”. I thank them and write the “x” sign. Otherwise I suggest that I do not agree, and will write the times sign with emphasis. This precision of calling out each of the symbols and not just the numbers becomes a little ‘joke’ between me and the audience, which adds lightness to the session, but also reinforces the virtue of precision in mathematical notation. Furthermore it emphasizes that some features of the expressions are invariant while other features are changing.

By now they ware ready to call out the fourth line sign by sign:

4 x 9 + 6 = 6 x 7.

At this point I would suggest that they had just experienced the basic power of detecting a pattern and expressing that pattern, and that they probably had a sense of generality, of how the pattern would continue.

The power to generalize depends upon and is connected to the power to group and order, that is, to stress (and consequently to ignore), to focus attention in different ways. Most people will have detected the invariance of the + 6, as well as the two multiplication signs and the equality sign, but they may not have been explicit to themselves about these. Rather they simply see them as being present in every line. However, there is a very significant difference between being ‘vaguely aware’ and being ‘explicitly aware’. Some students will see ‘vertical sequences’ of increasing numbers:

Figure 1 Vertical sequences as seen by some students

Others will see relationships between the numbers in one line. I want to force attention to be on relationships in a single line, but the vertical sequences may be very useful in seeing those relationships.

More lines can be proposed in sequence until all students are confident in what comes next in each successive line. So far, students are responding to the simple increasing sequence. But they can be jolted a bit by introducing a larger number, perhaps 12, as a starting number for a line:

1 x 6 + 6 = 3 x 4

2 x 7 + 6 = 4 x 5

3 x 8 + 6 = 5 x 6

4 x 9 + 6 = 6 x 7

12 x …

If they are confounded, then I get different students to read out loud the lines we already have. I get them to emphasize the first number of the line, and one other, say the second number, or the fourth, or the fifth, or even the third. Perhaps someone will make a conjecture: if so, then I can challenge others to see if they agree with the conjecture, and to explain why or why not. Even if the conjecture is incorrect, I can praise the student for making a conjecture, and if a student changes their conjecture, I can praise them for real mathematical thinking. Mathematicians make lots and lots of conjectures that turn out to be false. But when they make a conjecture they do not just accept it. They test it, to see if it needs modifying. The purpose of making a conjecture is to get it outside of you so that it can be examined dispassionately and modified if necessary.

If no conjecture is forthcoming, then students can be invited to continue the sequence of lines themselves until they get close to the 12, to try to see what the line should be. Depending on their experience with this sort of mathematical thinking (detecting pattern, conjecturing, justifying their conjectures, and expressing generality), I might offer a harder starting number (say 37, or even 137), or I might jump to the following train of thought: I am thinking of a number to start a line; I cannot tell you what it is, but I am going to denote it by , a little cloud or thought-bubble. I then add a new line underneath the work so far:

1 x 6 + 6 = 3 x 4

2 x 7 + 6 = 4 x 5

3 x 8 + 6 = 5 x 6

4 x 9 + 6 = 6 x 7

12 x 17 + 6 = 14 x 15

37 x 42 + 6 = 39 x 40

 x

and get them to puzzle over what it should be. It is amazing how many students, even those in upper primary school, will be able to offer a proposal, or at least to articulate how to express the other terms in the line. A useful line of questioning which students will soon become familiar with is “what is invariant, line by line, and what is changing?”, and then “What relationships between different numbers in a line are invariant line by line, and which are changing (if any)?”.

It is a surprisingly easy step to

 x (+5) + 6 = (+2)(+3).

The first time students participate in such an activity I might be content to leave it at working their way up to line 7 or 12. Each time I used such a sequence, I would push them a little further, until they were confident about expressing the generality using a little cloud. As soon as we do get to an expression of generality I would prompt students to test it against the previous examples, and I would do this whether the conjectured expression was indeed correct, or not. Wherever possible, I want the students to be responsible for deciding whether a conjecture is correct. I do not want them waiting for me to confirm or deny it. At each stage I am trying to do for the students what they may not yet be able to do for themselves;. If I do everything for them, they will soon expect me to do the work if they wait long enough (itself a generality!).

Getting students to make up their own lines, and especially to get them to use a ‘peculiar’ number (one that is unusual in some way, or which they think no one else in the room will think of using) is a good way to provoke them to an awareness of the range of choices, the scope of change which the sequence encompasses (Mason & Watson 1998). Some students like to try very large or very small numbers, or long decimal places, and they can use a calculator to check or do the actual calculations. In this way they take ownership of generating examples, while experiencing the scope of generality being expressed. Such experiences are the psychological roots of a sense of the meaning of variables.

## Development of The Sequence

Once students have become familiar with expressions of generality of one particular type, it is possible to juxtapose several such sequences side by side, and to ask “What is invariant and what is changing amongst these examples?”, that is, “What is the same and what is different about them?”. For example see figure 2

Figure 2 Students look for what is the same and what is different

Students will readily detect that the product of the two numbers in the brackets on the right is the invariant addition term, and that their sum is the number in the bracket in the first term of each line. At this point they are ready to discuss whether there might be rules for working with these ‘clouds’ as if they were numbers (after all, they do stand for some as yet unspecified number). Students will quickly work out at least some rules. If they have trouble because there are two sets of brackets, then you can construct a collection of sequences which employ each rule in turn.

You can also generate sequences of equalities with the product on the left:

3 x 4 = 1 x 1 + 5 + 6

4 x 5 = 2 x 2 + 10 + 6

5 x 6 = 3 x 3 + 15 + 6

6 x 7 = 4 x 4 + 20 + 6

with the ‘cloud form’

(+2)(+3) = x+ 5x + 6

Students will soon spot the patterns, and may even suggest collapsing the  x to  2.

# General structure of Tunja Sequences

A Tunja sequence is a sequence of particular cases of an algebraic expression. The generality which students are provoked into detecting and expressing is the statement that two different expressions are equal *for all values of*  , because one expression can be manipulated into the form of the other. Thus students encounter the rules for manipulation, as necessary, so that the expressions on which they work, and the rules which they use, are their own expressions of generality, and not simply rules given by the teacher or text. For example:

*x*(*x* + *a* + *b*) + *ab =* (*x* + *a*)(*x* + *b*) for specified values of *a* and *b*, first both positive, then one negative, then both negative;

(*x* + *a*)(*x* + *b*) = *x*(*x* + *a* + *b*) + *ab.* Same for this order of the equality.

 (*x* + *a*)(*x* + *b*) = *x*2 + (*a* + *b*)*x* + *ab*. Note that the term (*a* + *b*)*x* can be evaluated to a single number without blocking students powers to discern the pattern, or it can be presented as the sum of two distinct terms *ax* and *bx*;

*x*2 + (*a* + *b*)*x* + *ab* = (*x* + *a*)(*x* + *b*). Same for this order of the equality.

Note that there are choices as to which order to use for two equivalent general expressions, since the order will direct attention differently, either to expansion or to factoring. There are also choices to be made as to which terms to leave expanded (e.g. *x* x *x* particularized as , say, 3 x 3, as 32, or as 9, and *ax* + *bx* to be particularized as two terms or as one).

It is not necessary for the teacher to design the expressions each time. Some students who discover out what is going on can make their own sequences; indeed at some point every student can make their own, by starting with a generality involving brackets and some expansion, and then particularizing to form a portion of a sequence.

# What About Negative Numbers?

At some point, depending on the interests and concerns of the students, the sequences can be presented in declining order.

5x 10 + 6 = 7 x 8

4 x 9 + 6 = 6 x 7

3 x 8 + 6 = 5 x 6

2 x 7 + 6 = 4 x 5

1 x 6 + 6 = 3 x 4

and then continued:

0 x 5 + 6 = 2 x 3

(–1) x 4 + 6 = 1 x 2

What does this tell us about the meaning of (–1) x 4? It must mean ‘subtract 4’, or ‘negative 4’, otherwise the arithmetic will not continue to work. Continuing, we have!

(–2) x 3 + 6 = 0 x 1

(–3) x 2 + 6 = (–1) x 0

(–4) x 1 + 6 = (–2) x (–1)

I t seems that (–2) x (–1) must be 2 for the calculation to be correct. Carrying on further continues to validate this way of multiplying negatives.

(–5) x 0 + 6 = (–3) x (–2)

Eventually you reach two versions of the product of negatives in one statement:

(–6) x (–1) + 6 = (–4) x (–3)

Having students discuss what rules for manipulating negative numbers are needed in order to preserve the arithmetic pattern is likely to help most children to internalize those rules, to see why they are needed, and to enjoy making up the most complicated expressions they can using negative numbers!

## Factoring Quadratics

All the work done on sequences may easily have provoked students to observe how factoring works, but in case not, let us begin by assuming that students can expand brackets. Now factoring is the reverse operation of multiplying. The way I express it in my sessions is that whenever you have something you can do, a technique or action, you can reverse the process and ask, “if this was the answer to a question of a certain type, then what was the question?”.

For example, 3 x 4 = 12 is a ‘doing’. Form the ‘undoing’: 12 = ? x ?. Notice that there is more than one answer. Notice also that there is considerable, if not complete freedom in the choice of the first number, but once the first number is chosen, the freedom is restricted to exactly one possibility: 12 = 6 x ? forces a 2, while 12 = 17 x ? also forces a unique, if messy answer.

Similarly, to factor a quadratic is to undo the multiplication of linear terms. So I give students the following type of tasks:

Tell me how to expand (*x* + 2) (*x* + 3).

OK. What if someone gave me an expression such as *x*2 + 7*x* + 12 and told me that it came about by expanding brackets. How could we check? How could we be sure they weren’t lying? (I might choose to put up two or three expanded quadratics as targets, including one which does not factor.)

I want you to make up examples of pairs of terms, multiply them out, and try to see a pattern in the answers. Do enough examples so that you can tell me how it works, and how to undo these sorts of expansions.

After a while I remind students that they had better try putting in negative numbers in place of the 2 and the 3 as well. Most students will quickly see the sum and product property. Many will be able to articulate the technique that you look for two numbers whose sum and product are the two coefficients.

# Summary

In having students take responsibility for the particular examples they choose, they are exercising their power to particularize for a purpose (to detect a pattern) while at the same time rehearsing their skills of bracket expansion. People who have facility in expanding brackets do not need to give it much attention. If follows that if you want to gain facility in a skill, you want to try to *reduce* the amount of attention you give to it (Hewitt 1994). Just doing a set of exercises constructed by someone else invites full attention to doing each exercise in turn, and not to trying to see how it all works, that is, to generalize the technique from the examples. Having a greater goal attracts attention to that goal as the purpose of particularizing (which is to detect a pattern on the way to a generalization). Attention is thus usefully diverted away from routine practice of the skill.

Furthermore, doing other people’s exercises is one stage worse than doing other people’s algebra! It is much more interesting to make up your own examples, and as you make them up, you begin to develop strategies for what sorts of examples are most illustrative, most informative. Using (*x* + 37) and (*x* –19) may be fun to a few, but it is not particularly informative and simply detracts from the overall task of finding out what is going on.

Finally, when might one move from  to letters like *x* andy, *a* and *b*? My preference is to let the students decide this. They will soon get tired of drawing a little cloud. But the cloud is useful for denoting unknown numbers, perhaps which someone else knows or is thinking of, in other contexts (see Mason *et* *al* 1985, 1999), so I tend to use it until students object and want to use something more compact. Perhaps each lesson a different student could be in charge of deciding what symbol was used during that lesson to denote an as-yet-unspecified number, a generality.

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FIGURES

|  |  |  |  |
| --- | --- | --- | --- |
| 1 | 6 | 3 | 4 |
| 2 | 7 | 4 | 5 |
| 3 | 8 | 5 | 6 |
| 4 | 9 | 6 | 7 |
| 5 | 10 | 7 | 8 |

Figure 1 Vertical sequences as seen by some students

|  |  |  |
| --- | --- | --- |
| 1 x 6 + 6 = 3 x 42 x 7 + 6 = 4 x 53 x 8 + 6 = 5 x 64 x 9 + 6 = 6 x 75 x 10 + 6 = 7 x 837 x 42 + 6 = 39 x 40 x (+5) + 6 = (+2)(+3)  | 1 x 7 + 8 = 3 x 52 x 8 + 8 = 4 x 63 x 9 + 8 = 5 x 74 x 10 + 8 = 6 x 85 x 11 + 8 = 7 x 937 x 43 + 8 = 39 x 41 x (+6) + 8 = (+2)(+4) | 1 x 8 + 12 = 4 x 52 x 9 + 12 = 5 x 63 x 10 + 12 = 6 x 74 x 11 + 12 = 7 x 85 x 12 + 12 = 8 x 937 x 44 + 12 = 40 x 41 x (+7) + 12 = (+3)(+4) |

Figure 2 Students look for what is the same and what is different