Effective Use of Examples  
in Teaching Mathematics  
at University

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*Longum iter est per praecepta, breve et efficax per exempla*  
[Long is the way (to learning) by rules, short and effective by examples]  
Seneca Epistles 6

## Introduction

The pedagogical adage that ‘we learn by examples’ and its companion adage, ‘practice makes perfect’ have been around for perhaps 4000 years. Students think of worked examples, while teachers think both of worked examples and of paradigmatic or generic examples of concepts. They think of instances that can best clarify and illustrate (and hence make plain) a rule or a method, or a concept (Sierpinska 1994 p88).

Turning to your own experience,

What did you do with examples as a student?

What do you do with examples when reading a paper, textbook, or when exploring a conjecture?

What would you like students to do with the examples you give them (both worked examples and examples of mathematical objects)?

I conjecture that, being successful at mathematics, you learned to follow through the details of a specific instance but at the same time seeking structural relationships that apply generically. In other words you learned to use the specific to ‘see through the particular to the general’. If you are checking or enriching a concept or technique already met then you may also be doing the reverse: instantiating a generality by ‘seeing the particular in the general’. This applies as a student and as a professional researcher and teacher.

In his monumental works on mathematical thinking, George Pólya (1945, 1962, 1967) stressed the fundamental roles of *specialising* and *generalising*: mathematicians usually specialise not to collect data as such, but to locate relationships which might be instances of general properties. For example, David Hilbert is described by his friend and colleague Richard Courant as

… a most concrete, intuitive mathematician who invented, and very consciously used, a principle; namely, if you want to solve a problem first strip the problem of everything that is not essential. Simplify it, specialize it as much as you can without sacrificing its core. Thus it becomes simple, as simple as can be made, without losing any of its punch, and then you solve it. The generalization is a triviality which you don’t have to pay much attention to. This principle of Hilbert’s proved extremely useful for him and also for others who learned it from him. Unfortunately, it has been forgotten. [Courant (1981 p ref)]

The aim of these notes is to problematize ‘examples’: to raise questions about what they are for, what students are expected to do with them, and how they are constructed, and to make suggestions about how student experience of examples can be enriched.

There are two types of things that are exemplified in mathematics teaching, though the dividing line between them is not always clear cut. First, there are ‘worked examples’. The history of mathematics is entwined with the history of mathematics teaching, and from the earliest of written records it is clear that a major feature of teaching mathematics has always been worked examples. Second, there are examples of mathematical objects which illustrate, prove existence, or show by way of counter-example that some condition is necessary. These objects are also used to locate and formulate conjectures.

## Student & Teacher Epistemological Stances

Before considering the two kinds of mathematical examples in detail, it may be helpful to consider what students’ perceive as their role in the teaching-learning interaction.

### Didactic Contract

Most students act as if they believe that ‘doing the work’ is their side of the contract: for them it is the lecturer’s task to set doable tasks that bring about the requisite learning. This of course flies in the face of what people who have become lecturers in mathematics know about studying and about ‘doing mathematics’. Nevertheless the impression among students persists. At university, the main task may be perceived as to ‘attend lectures, take notes, and attempt assignments’.

Many lecturers act as if their task is to expound the mathematical concepts clearly, laying out definitions, lemmas, theorems, proofs, examples and some applications so that any ‘rational mind’ will immediately see the correctness. In this they have support from Henri Poincaré (1956) who famously expressed surprise that “there are people who do not understand mathematics? If mathematics invokes only the rules of logic, such as are accepted by all normal minds; it its evidence is based on principles common to all men, and that none could deny without being mad, how does it come about that so many persons are here refractory?” [p2041]

These two positions may seem like caricatures, but a little observation reveals that they are not entirely misleading interpretations of the salient behaviour of students and teachers.

“A mathematical demonstration is not a simple juxtaposition of syllogisms, it is syllogisms *placed in a certain order*, and the order in which these elements are placed is much more important than the elements themselves. If I have the feeling, the intuition, so to speak, of this order, so as to perceive at a glance the reasoning as a whole, I need no longer fear lest I forget one of the elements, for each of them will takes its allotted place in the array, and that without any effort of memory on my part.

It seems to me then, in repeating a reasoning learned, that I could have invented it. This is often only an illusion; but even then, even if I am not so gifted as to create it by myself, I myself re-invent it in so far as I repeat it.: [p2043]

The often implicit contract between students and teachers is known as the *contrat didactique* (see Brousseau 1997, Mason & Johnston-Wilder 2004 or Mason 2002 for more details). If this is the extent of student perception of the contract, then they may not appreciate the other things that need to be done in order to learn and do mathematics effectively. One aim of these notes is to suggest other actions that students could undertake to make their learning and doing of mathematics more effective (see also Mason 1998). These can be implemented by lecturers as a way of suggesting them to students as part of their responsibility.

Alongside the didactic contract is the persistent belief that *practice makes perfect,* that somehow ‘doing enough examples’ like the ones in assignments or on previous examinations will be sufficient to pass the examination. The trouble with this is that many students manage to pass examinations without really getting to grips with the ideas, leaving them vulnerable in future courses when they no longer have access to appreciation, comprehension and understanding that the examinations were supposed to be testing. The issue is not with practice as such, but rather with how that practice is undertaken.

## Worked Examples

#### Phenomenon

Students ask for more and more ‘examples’ (meaning worked examples).

The immediate issue for students is to complete their assignments as best they can, because this fulfils their perceived side of the didactic contract, their perception of what is involved in learning and doing mathematics. The reason they want more examples is that they have been enculturated into seeking templates from which a minor change of constants yields the solution they seek. Never mind that the next time they meet a similar question they may not have a template to hand, or that they may not recognise when a template can be used.

For example, given a proof that √2 is irrational, can students adapt the reasoning to prove that  is irrational or that is irrational? Could they articulate what is the same and what is different about the reasoning in these two cases. Or why the same style of reasoning does not prove that √9 is irrational?

Sierpinska (*op cit*) goes on to look at things from the student’s perspective:

The learner is also looking for … paradigmatic examples as he or she is learning something new. The problem is, however, that before you have a grasp of the whole domain of knowledge you are learning, you are unable to tell a paradigmatic example from a non-paradigmatic one. [Sierpinska 1994 p88-89]

So students treat worked examples as templates. Templating can become habitual as a stance, in which examples are used to provide a format for ‘doing’ certain questions, without enquiring into how to reconstruct a technique or strategy for oneself, or appreciating when and how it works. Some students may need to be jogged out of this stance. Although templating is more likely to be found among students who are required to take mathematics courses as part of their studies towards a degree in something else such as engineering, computer science, social sciences or even business studies, it is a stance that can be detected amongst some mathematics majors who have succeeded at school despite this orientation. Students who are addicted to templating may need considerable persuasion that there are more effective ways to pass examinations and more effective ways of learning than trying to memorise procedures.

Wise students ask themselves whether in the future, they could do another question ‘like this’, raising the question of what constitutes another question ‘of this type’. Again following Pólya, it is not so much the ‘doing of yet another example’ as developing an awareness of when and how a particular technique or strategy works, what it consists of, and under what conditions it might be useful. Getting students to address these questions could be of benefit to students who have not yet become aware of what is involved in learning mathematics.

The slogans ‘seeing the general through the particular’ and ‘seeing the particular in the general’ can be useful as reminders for what the lecturer does naturally and for what the student may need prompting or support in doing. They add a little detail to an observation of Alfred Whitehead:

to see what is general in what is particular and what is permanent in what is transitory is the aim of scientific thought [Whitehead 1911 p4].

When lecturers are ‘doing’ a worked example for students, they are most likely seeing the particular in the general. The example is for them an instance of a general class. But students see only the example, so they need to be trying to see the general through the particular. They need to be asking themselves what is specific to the particular, and what is common to all ‘similar’ uses of the technique.

Given only one ‘example’, students are likely to need assistance in discerning what is generic. They need to learn to act like Hilbert, and although some few students will do this naturally, many may need support and encouragement at first. Given two or more examples, students need to be asking themselves ‘what is the same and what is different’ about them. Seeing the general through the particular is not always easy to do, and can give rise to classic misconceptions and misunderstandings (see later). If the lecturer is aware of seeing the particular in the general, they can take steps to direct student attention to pertinent features or parameters that can be changed.

A useful label for what can be varied in an example, and still it remains an example is *dimensions of possible variation*, a term based on work of the psychologist Ference Marton (Marton & Booth 1997) and extended by Watson & Mason (2005). With each dimension of possible variation there is an associated *range of permissible change*. Thus for example most theorems in mathematics can be seen as a statement of something that remains invariant while something else is permitted to change. Very often the invariance is stressed in the statement of the theorem, while what can change and over what range is relegated to quantifiers.

For example, the Jordan Curve Theorem states that:

A simple closed curve in the plane divides the plane into an interior and an exterior.

The indefinite pronoun ‘a’ is all that signals the huge range of possible variation, while the invariance (interior-exterior distinction) gets all the attention. Even the word ‘any’ used in place of ‘a’ can be overlooked by students eager to get to grips with what the theorem means.

As another example consider

Given a cubic, the family of chords, whose midpoints all have the same *x*-coordinate, all pass through a single point of the cubic.

Here there is little stress on what can change (the chords satisfying a certain constraint) and attention is likely to be absorbed making sense of the constraint and the invariant single point. No mention is made of the special case of a tangent at the point on the curve with the same *x*-coordinate.

As another example consider the Squeeze Theorem

Suppose *f*(*x*) and *g*(*x*) are functions from *R* to *R*, and that *g*(*x*) is continuous at *x* = *a*. If a neighbourhood of *x* = a in which |*f*(*x*)| ≤ *g*(*x*), then *f* is continuous at *x* = 0.

Here there is little stress on the scope or range of generality of the functions, compared to the detail about the invariant property of being continuous. Appreciation of what is being asserted depends on awareness of the scope or range of the functions *f* and *g*.

Marton goes further than drawing attention to the importance of appreciating dimensions of possible variation. He proposes that what is available to be learned by students are dimensions of experienced variation. In order to be available to be learned, these have to be experienced in a short period of time. Thus encountering a different object each week is too far apart to be likely to be juxtaposed by learners; encountering many features varying at once is likely to be too much to take in for most students. Judging the pace, space and complexity of examples is a matter of research and experience. It is part of the art of teaching effectively.

If dimensions of possible variation and their associated ranges of permissible change are not explicitly pointed out, at least until students have internalised the process (made it part of their side of the didactic contract) to seek and express to themselves, then they will have to make conjectures themselves about what can be changed in the worked examples they have, and how. The issue of what students are attending to, and in what way(s) is taken up later.

Recognising a worked example as a possible template for an assigned problem is not as easy as it might seem at first. If the wording is the same and no extra-mathematical context is involved, then it might be possible to align parameters, nouns and verbs. However if the context is different, the formulation of the situation is different or if different parameters are given and some of those that were previously given are to be found, the match may not be evident.

### What makes a worked example useful?

Renkl (1997, 2002) and colleagues (Renkl, Mandl & Gruber 1996; Renkl, Stark, Gruber & Mandl 1998; Renkl, Atkinson, Maieer 2000) have carried out extensive research in what makes a worked example useful to students, as have Chi & Bassok (1989). The core of their finding is that what really matters is not what to do next, but how you know what to do next. This aligns perfectly with the observation that it is the actions that come to mind which are triggered or resonated by the situation that can be used; actions that do not come to mind will not be. So what is it about the situation that brings to the lecturers mind to carry out the next action. A variant on this is the question from a student “where did you get that 3?” which is not actually concerned about the 3 as such, but rather how you knew to produce the 3, or how you knew what action to perform that gave rise to the 3.

Renkl and colleagues tried introducing blanks for students to fill in, and places for reasons to justify actions, both of which prompt the students to make sense of the example. Students sometimes work through the example for themselves and then look at the worked example when they get stuck; some look first and then try to reproduce it; some copy it out in their notes as if somehow they consequently posses the know-how. But many students work through the details of an example without drawing back from the action and asking themselves what general strategies were employed; what actions were important; what tricky points they encountered. Without reflection it is very difficult to learn anything, and without pro-flection, that is, imagining yourself in some future situation and having a specific action come to mind, it is less likely that that action will actually come to mind when needed.

#### Example: Application of the Extreme Value Theorem

The Extreme Value Theorem states that a continuous function on a closed and bounded interval in *R* attains its global extrema. Here is an application of the theorem, intended to act as a ‘worked example’ concerning how the theorem is used.

A continuous function *f* on *R* which tends to positive infinity as *x* tends to positive or negative infinity attains a minimum value.

Worked solution:

Since *f* tends to infinity as *x* tends to positive infinity, there exists a positive number *N* such that for all *x* > *N*, *f*(*x*) >*f*(0). Similarly, since *f* tends to infinity as *x* tends to negative infinity, there exists an integer *M* such that for all *x* < –*M*, *f*(*x*) > *f*(0). Confining *f* to the closed and bounded interval [–*M*, *N*], *f* is continuous on the interval and so must attain its local extrema. Consequently *f* attains its minimum value *m* on that interval. Certainly *m* ≤ *f*(0). By the choice of interval, *f* is greater than *m* outside of the interval, and so attains a global minimum value.*//*

What is missing from this worked example is how it ‘came to mind’ to use the Extreme Value Theorem by constructing a suitable closed interval, but this is exactly what students need to find out about in order to be able to use the theorem similarly for themselves, which is what is generally implied by ‘fully understand, appreciate and comprehend the theorem and its uses’.

In a particular group of students at a university well known for its mathematics, only two out of eight were able to make a reasonable stab at the task when set as homework, though neither constructed a suitable interval. Having been shown again how the reasoning proceeds, the group met a variant a few weeks later:

Show that *p*(*x* )= –*x*102 + *ax*101 + *bx*99 + *cx* + *d* attains a maximum value.

None of the students thought to invoke the action construct-a-suitable-interval for use with the Extreme Value Theorem (Scataglini-Belghitar & Mason 2012). A plausible conjecture is that the students had not asked themselves in what ways the theorem might be used, nor paid explicit attention to the device of constructing a suitable interval. In a follow-up tutorial they were asked to construct graphs of functions for themselves meeting various constraints such as being continuous on a bounded but open interval while being bounded above but unbounded below and without a global maximum, and then asked for variants that would still guarantee a maximum. After some slow starts, a range of examples appeared. Yet two of the students, when asked in a tutorial later the same day to reconsider the polynomial question, resorted to ineffective approaches until they were asked to sketch a graph. Eventually the thought came to mind that the situation was similar to that considered in the tutorial.

### A Worked Non-Example

Mistakes or misapprehensions that are common can sometimes be addressed through asking students what is wrong with a particular resolution or bit of reasoning.

#### Example

A student was considering the continuity of the following function:

For *x* a real number, express it as a decimal but without an infinite tail of recurring 9’s.

If *x* = *n* + 0.*d*1*d*2*d*3… where the *dk* are the decimal digits, then *f*(*x*) = *n* + 0.*d*1*d*3*d*5… .

The reasoning went as follows:

For any ε>0 choose *n* such that 10–*n* < ε. Choose δ = 10–2*n–*1.

Then |*x* – *a*| < δ implies that *x* and *a* agree in at least the first 2*n* + 1 decimal digits. This means that *f*(*a*) and *f*(*x*) agree in at least the first *n* decimal digits, so |*f*(*x*) – *f*(*a*)| < 10–*n* < ε. This *f* is continuous at *a*.*//*

This could be seen by students as a format for proving that a function is continuous, especially where it is hard to imagine what the function actually looks like. I could also be appreciated as an example of how outline reasoning can contain errors which only show up when you are more precise (perhaps one of the main reasons for teaching analysis in the first year). Here, intuition might raise an alarm because whenever you fiddle with decimal digits, recurring 9’s can be an obstacle. The reasoning is fine as long as *a* is a non-terminating decimal, for when it terminates, there are numbers with a large finite number of trailing 9’s which also converge to *a* and which for large enough *n*, do not in fact agree with *a* in the first 2*n* + 1 digits. So this reasoning can be used to draw attention to the pitfalls of recurring 9’s in the specification of particular functions.

This proposed reasoning can itself serve as an example of a pedagogic choice, to propose incomplete or erroneous reasoning and to ask students to find the flaw, before drawing attention to what you think the reasoning exemplifies. A good source of ‘worked non-examples’ is of course from students’ work which can be used in subsequent years in this same way.

### Working–Through & Working–On

One possible lesson from research on worked examples is that following a worked example is not in itself sufficient. Rather, it is necessary for students to work-on the example, to consider what can be changed and still the technique works, as well as becoming aware of, and internalising the key ideas of the technique.

The notions of *conceptual insights* (what needs to come-to-mind) and *technical handles* (what needs to come-to-action, that is, the techniques needed to exploit the conceptual insights) can be used to locate the *key ideas* used in a technique or proof (Raman 2003, Sandefur, Mason, Stylianides & Watson 2013). Constructing a narrative that accounts for the effectiveness of the core idea and the range of variation possible is one way to internalise the technique so that something appropriate comes to mind when needed in the future. Another is to develop a personal narrative that can serve as an ‘inner incantation’ to guide you through the process.

A good way to strengthen a conceptual insight is to construct examples of similar problems that call upon the same idea. Constructing (and doing) your own problems encourages working-on the class of problems rather than merely working through a selection of similar ones. Students can be encouraged to go further, and to construct

Some easy problems of the type;

Some hard problems of the type;

Some peculiar problems of the type;

Some general problems of the type;

Some problems that indicate that you know how to do ones of the type;

A statement of how to recognise a problem of the type.

By *peculiar* is meant problems with a twist or unexpected feature that might require extra thought. All this can be topped off by constructing a narrative about things to look out for and where extra care may be needed; what the technique(s) or technical handles accomplish and how. This will contribute to a rich inner incantation that could become available when embarking on a similar problem in the future.

For example, students learning to use the extremal value theorem could benefit from constructing a narrative about how it is often useful to locate a closed and bounded interval to which to apply the theorem, and these situations apply particularly to conditions in which the behaviour of a function is known asymptotically.

For additional depth and breadth of understanding, students can be encouraged to consider what happens when a problem is changed so that some of the givens have to be found while what was to be found is given (an instance of the pervasive mathematical theme of Ding & Undoing). Those who will take to mathematical thinking will find this exciting and stimulating; those who simply want mathematics as a tool may benefit but perhaps only in the long term.

By identifying the core *conceptual insights* and the relevant *technical handles*, a tutor or lecturer can arrange to direct student attention appropriately. It is not, of course, sufficient merely to point out to students what the tutor considers to be *key ideas*. To be effective, students need to be prompted to try to articulate this for themselves. As with much of mathematical exposition, the critical features are whether

the tutor is attending to the key ideas, the conceptual insights and technical handles that make the technique work;

the students are attending to the same thing as the tutor, and in the same way;

the tutor is ‘seeing the particular in the general’, while students are seeing the particular but also trying to see the general through the particular;

the students are considering the scope of generality, considering what can be changed and still the same technique, the same procedure applies.

Being sensitive to these issues, and acting appropriately can support students in becoming aware of what it is that constitutes the type of a problem, or in other words, a *question* or *problem space*. Although the notion of a type of problem appears unproblematic, when you try to automate the selection of an instance from an intuitive question space, unexpected complexities arise concerning the difference between trivial and non-trivial instances, and the issue of how the ranges of permissible change for various parameters interact (Sangwin 2006, 2013). This is what students need to encounter if they are to master a problem space.

## Examples of Concepts (Mathematical Objects)

#### Phenomenon

Students often fail to check that the conditions necessary for a theorem or a technique actually hold in their situation.

or *constructs*, stressing the agentive role of sense-making and meaning construction. Jean Piaget (1970) used the term *genetic epistemology*, while others use the term *constructivism* to refer to the stance that learning requires action, preferably initiated by the individual, within a social context of teacher (or text or website) and peers. Lev Vygotsky (1978) stressed the importance of the social context, leading to distinctions between several kinds of constructivism, which are not really relevant here. Suffice it to say that students learn from each other and from their own experience as well as from being told things. Indeed, to learn from being told things (or from reading things in books) it is necessary to be prepared so as to be in a state in which you can hear what is being said. The examples that come to mind (your accessible example space) plays an important role in the sense that can be made of what you are told.

### Example Spaces

What comes to mind when you hear the word ‘number’ in a mathematics lecture? The context plays an important role of course: in a number theory lecture you are likely to think integer, or perhaps rational; in an analysis lecture you are likely to think ‘real’, and in a complex analysis lecture you are of course likely to think ‘complex number’. Rarely would you have ’quaternion’ come to mind. But students may not be so finely attuned. For many the term ‘number’ will bring to mind a whole number or a rational, probably not even admitting, at least at first, negatives.

#### Another & Another

A useful way to test this is to invite students to write down a decimal number that is as close to 1/2 as possible, does not use the digit 5 but does have 7 as a digit, then ask them to write down another, and once more, another. At first many are likely to be conservative, even fearful that they will then be asked to do something complicate with their number. If it is made clear that there will be no follow-up calculations, students may begin to open up. By being asked for three constructions in a row, many will, by the third, start to be more creative. Getting students to reveal their third choice may open up possibilities that others had not had come to mind, such as using radicals, exponentials, and trigonometric functions.

Here there are three different ways of getting close:

0.4799.., 0.499..99 and using both ideas, 0.499..99799.. .

This leads naturally to a discussion of limits, the difference between a finite and an infinite string of 9s, and whether the 9s really can continue ‘forever’.

The same technique can be used with any mathematical concept. Asking students to write down or construct an object, then another, then another can help to awaken them to a richer scope of possibility than previously.

One variation of another-and-another type tasks is to ask for a simple, a complicated, a peculiar and a generic example. Another variation is to predict unnecessary constraints that students might impose unwittingly, and then ask for a sequence of constructions which block out these assumptions or which extend their awareness of possibilities in some way.

For example, many students have an immediate reaction when they hear ‘graph of a cubic’, so the following sequence of constraints are chosen to block out assumptions they may be making unwittingly, which can assist them to become aware of other possibilities.

Sketch the graph of a cubic.

Now sketch the graph of a cubic that does not pass through the origin.

Now sketch a cubic that has a negative inflection slope (the slope at the point of inflection).

Now sketch a cubic that has a negative inflection slope and three negative real roots; and also one that has only one real root, which is negative.

Now sketch a cubic that has a negative inflection slope and three negative roots but no relative maximum.

The task is constructed so as to direct attention to features that many students include implicitly. Their first thought is *y* = *x*3 if they think in formulae, but their first sketch tends to be of the form *x*(*x*2 – 1). So the second part is to move away from the origin: not a big step, but nevertheless enriching their sense of what cubics can look like. The third constraint reminds students that cubics come in two forms depending on the sign as *x* gets large. The fourth constraint directs attention to the possibility of translating to the left, since most people translate to the right, and also translating up (or down). The fifth constraint is of course impossible, but directs attention to the interconnection between real roots and relative extrema.

Many students have polynomial come to mind when they hear the word ‘function’. So a sequence of constructions such as

sketch a function on the interval [-1, 1];

sketch a function discontinuous at one point on the interval;

sketch a function discontinuous in different ways at two points on the interval.

In how many essentially different ways can a function be discontinuous at a point?

The final question will undoubtedly lead to a discussion of what counts as ‘different’, which can open students to the scope of phenomena that the definition of continuity is trying to block out. The constraints can be adjusted so as to challenge students to encounter functions they might not otherwise have had come to mind. If this richer collection comes to mind in the future, their sense of the import of theorems will be enriched.

If the term ‘continuous function’ brings to mind only polynomials, then students may not appreciate the significance of theorems that are intended for use with rational functions, functions like *x* sin (1/*x*) for *x ≠* 0 and 0 for *x =* 0, or the many other continuous functions that are not readily expressed in terms of elementary functions. Even if their example space includes trigonometric and exponential functions, they are unlikely to appreciate the significance of many theorems in analysis unless they realise the wildness of the behaviour admitted under the general notion of continuous function.

Included in the notion of an *example space* are not only the paradigmatic examples that come to mind, but the less common examples that are then accessible, together with techniques for tinkering with examples to create others.

What is generic about this cubic-construction task is the sequence of constraints, aimed at enriching students’ impoverished example spaces by contradicting implicit assumptions.

One way to stimulate students to explore and enrich their accessible example space is to get them to use familiar examples to meet a collection of constraints in a chart like the following:

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| *Range*  *Domain* | *y* in *R* | *y* in *R* & *y* ≥ 0 | *y* inR & *y* > 0 | *y* in *R* & *y* > 1 | *y* in *R* & |*y*| ≤ 1 |
| *x* in *R* |  |  |  |  |  |
| *x* in *R* & *x* ≥ 0 |  |  |  |  |  |
| *x* inR *& x* > 0 |  |  |  |  |  |
| *x* in *R* & |*x*| ≤ 1 |  |  |  |  |  |
| *x* in *R* & |*x*| < 1 |  |  |  |  |  |

Here the challenge is to construct real valued functions taking the values specified, whose largest possible domain is as specified. Students often pay little head to *domain* and *range* because ithey do not seem to have much significance. By dealing with slight variations in the specification, students may begin to realise the role of the technical terms as constraints. Learning to tinker with familiar examples to create more sophisticated ones can contribute to developing flexibility as well as extending the example space.

One common experience is that as you are trying to construct an object for one cell, you find yourself constructing objects that belong in other cells, stimulating a flexible approach rather than doggedly trying to fill the cells in one at a time.

Sierpinska (*op cit*) quoted earlier regarding templating, went on to consider what can go wrong when students mis-interpret what is paradigmatic about an example:

The learner is also looking for such paradigmatic examples as he or she is learning something new. The problem is, however, that before you have a grasp of the whole domain of knowledge you are learning, you are unable to tell a paradigmatic example from a non-paradigmatic one. So you make mistakes, wrong choices, wrong generalizations (because, of course, you generalize from your examples). Moreover, as the example is normally represented in some medium (enactive, iconic or symbolic) you may mistake the features of the representation for the features of the notion thus exemplified. [Sierpinska 1994 p88-89]

Students sense of definitions may include irrelevant characteristics of a standard diagram or other example, causing difficulties in creating or interpreting diagrams (Yerushalmy & Chazan, 1990). Efraim Fischbein (1993) coined the term *figural concept* to describe conceptual constructions of students based on unintended implicit assumptions arising from canonical diagrams. Thus if all triangles have one edge parallel to the bottom of the page, students pick up the impression (and integrate it into their sense of the definition) that all triangles have this property. Consequently only one edge can be considered to be the ‘base’ when finding the area. Similarly, if all squares are drawn with one edge parallel to the bottom of the page, students distinguish ‘tilted squares’ using words like ‘diamond’. What matters is that they then do not accept a ‘diamond’ as an instance of a square. The term *figural concept* can be extended to any concept for which the examples that students meet have unintended relationships or properties that students assume are included in the definition. A suitable label for this extension might be *unintended features*. Such a label can be useful because as it grows richer and more complex through accumulating multiple instances, the likelihood of it coming to mind improves, so informing interactions with students and the design of teaching to try to circumvent students unwittingly stressing inappropriate features or perhaps ignoring appropriate ones.

### Types of Examples

Edwina Michener (1978) distinguished four types of examples or uses for examples in the practice of mathematicians: *start-up examples*, r*eference examples*, *model examples* and *counterexamples*

*Start-up examples*: from which basic problems, definitions and results can be conjectured at the beginning of learning some theory and which can be ‘lifted’ to the general case but are also understandable on their own (what Pólya called ‘leading’ examples);

*Reference examples*: standard cases which are widely applicable and can be linked to several concepts and results, such as using *R*2 to see how things work in metric spaces, also acting as a possible source for counter-examples;

*Model examples*: generic cases which summarise expectations and assumptions about concepts and theorems (what Pólya called ‘representative examples, and which could be called ‘paradigmatic’ examples);

*Counter-examples*: these are supposed to sharpen the distinctions between concepts and demonstrate the necessity of assumptions in theorems and techniques.

One important feature of leading and reference examples is being aware of what unnecessary or irrelevant features might be salient for students and so unwittingly integrated into their sense of what is being exemplified, along the lines of *figural concepts* and *unintended attributes* mentioned earlier. Similarly, paradigmatic examples only function properly when you are aware of what aspects are indeed paradigmatic, and what aspects are irrelevant in general.

While distinguishing these as different types may not be necessary, being aware of the range of uses and purposes could inform lectures, notes and assessment questions.

Mathematical concepts are usually presented as technical terms associated with definitions. Russell & Whitehead (ref) claimed that in mathematics, definitions are merely succinct expressions standing for long and cumbersome sentences. However it is one thing to formulate a definition for yourself so as to make it easier to think, analogously with using labels to refer to pedagogic constructs and strategies in mathematics education. It is quite another to make sense of someone else’s definition. What is the definition allowing, and what is it excluding?

#### Examples & Non-Examples

The psychologists Bruner, Goodnow & Austin (1956) spent some time trying to tease out the role of examples and non-examples in coming to an understanding of a concept. It seems natural that having some examples and some non-examples could help to refine the boundaries of the generality being instantiated. However they were unable to come up with anything definitive. There is more to concept discernment than simply having examples and non-examples. It is probable that the actions taken, the stance adopted make a considerable contribution.

Martin Gardner (1977) described a game he called *Eureka* in which the leader proffers some sequences of three numbers that meet a criterion. Participants then offer examples, and are rewarded with a smiley or a frowning face. The aim is to work out the criteria. When someone thinks they know the criteria they announce “eureka” and then contribute examples that they think will help other participants to work out the criteria. No-one ever explicitly states the criteria. Gardner mentions that the game mimics natural science, because nature never tells you when you have correctly determined the constraints. Playing the ‘game’ brings to the fore the importance of , as Pólya said (1965) “making a conjecture and then not believing your conjecture”. So when someone thinks they know the criteria, good practice would be to test their conjecture by offering as examples instances which they think are non-examples in case they have missed something.

Students equipped with examples of concepts are usually in a slightly better position, because they have text or were present in lectures where there was talk ‘about’ the concept, with indications of how the examples actually exemplified the concept.

### Sources of Examples

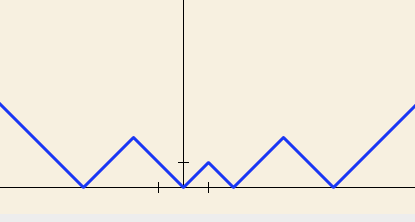
Where do ‘good’ examples come from? Textbooks tend to transmit what become standard examples. Thus |*x*| is commonly used as an example of a function which is continuous on *R* except at one point. But students can easily dismiss this as a *monster* of no importance to them (MacHale 1980), especially if they are studying mathematics in order to develop a tool kit for use in their preferred subject.

Augmenting such an example with a construction task that exploits the basic idea but extends it could enrich students’ example space. For example, the task mentioned earlier is one approach. Another might be to get students to exploit |*x*| by composing it with translations and with itself as in the following.

For example, every course in analysis presents |*x*| as a function that is continuous and differentiable everywhere except at one point. Imre Lakatos (1976) coined the expression *monster barring* to refer to the psychological phenomenon of ignoring (in the case of students) or explicitly excluding (in the case of mathematicians) classes of objects that fail to fit some conjecture. McHale (1962 ref) noted that |*x*| is often monster-barred by students as a pathological case that need not be considered, despite |*x*| being so important in analysis. However, it is rare for students to be invited to develop their example space by considering how |*x*| can be modified. For example, to construct functions differentiable everywhere except at a finite set of points, or a countable set of points. Furthermore, exploring the kinds of graphs that can be obtained by composing several copies of |*x*| with translations enriches students’ example space, by including not only infinite classes of examples but construction techniques for generating those classes. The following task is one example of a task with this intention:

#### ZigZags

The graph of *f*(*x*) = |||*x* – 1| – 3| – 2| is shown below.



What happens if you iterate *f* several or even many times? (see the applet ZigZags[[1]](#footnote-1))

Characterise the types of zigzags that can be constructed in this manner.

What about including constant coefficients other than 1 for the *x*?

The task is constructed to provide contact with the core psychological power to organise and to characterise, that is, to structure a collection of objects. The pedagogical question “What is the same and what is different about …” used with two or three objects is a powerful way to initiate such an action, and of course lies at the heart of many mathematical theorems, so students get experience of a standard mathematical practice. The generic feature of this task is the exploration of classes of objects generated by using one or more simple constructions with a few starting objects, illustrating the general mathematical theme of ‘invariance in the midst of change’.

Objects which are seen as *pathological examples* are easily dismissed by students, and reveals something of the centre of gravity of their accessible example space. In many cases however, they are likely to discover later that even their own discipline has to deal with a wider range of objects than they at first imagined. It is important to appreciate the range of objects admitted under a definition, and not to assume that a definition captures only the kinds of objects you are aware of.

The experienced and reflective mathematician Paul Halmos has written about the role of examples for him:

A good stock of examples, as large as possible, is indispensable for a thorough understanding of any concept, and when I want to learn something new, I make it my first job to build one. (Halmos 1983 p63)

Note that he sees it as his job to construct examples. This aligns with Hilbert’s ‘method’ and with the common mathematical practice of constructing examples in order to see how a proof or a technique works.

## Understanding, Comprehending & Appreciating Concepts

What is implied by the term *understanding* when applied to a mathematical concept? What might it mean to *comprehend* a proof or a technique? What might be implied by the term *appreciating* (the scope and import) of a mathematical concept?

Although mathematics depends on formal definitions of its concepts, it takes more than a memorised definition to make sense of a concept or of some reasoning.

Students’ definitions may include irrelevant characteristics of the standard diagram, causing difficulties in creating or interpreting diagrams (Yerushalmy & Chazan, 1990).

### Boundary Examples

Most concepts are based on and developed from some core or paradigmatic example. To understand a concept is to have a sense of what aspects of an example can be varied and still the object is an example (dimensions of possible variation) and associated range of permissible change. Each aspect that can be varied has an associated *range of permissible change* of that aspect, and of course various aspects (dimensions) and their associated ranges of permissible change may be interwoven.

For example, the fact that the series converges is an instance of several related possibilities. One ‘dimension’ that can be varied is the index. The fact that also converges for ε > 0 but not for ε ≤ 0 presents the harmonic series as a boundary example, in this case for a class of divergent series. The search for series closer and closer to the boundary results in studying series such as , , … all of which are divergent, but act as boundaries because



and are convergent for ε > 0 (Knopp 1951 p122-123).



To have access to a space of examples enriched by a search for slower and slower divergent series on the boundary of being convergent could be experienced as an exciting exploration, or as a dull accumulation of facts to be remembered, depending o how it is presented.

### Concept Images

On the surface, it seems that mathematical concepts are known from their definitions. Russell & Whitehead (ref) claimed that in mathematics, “definitions are merely succinct expressions standing for long and cumbersome sentences” $$$. However it is one thing to formulate a definition for yourself so as to make it easier to think, analogously with using labels to refer to pedagogic constructs and strategies in mathematics education. It is quite another to make sense of someone else’s definition. What is the definition allowing, and what is it excluding?

David Tall and Shlomo Vinner (ref) drew attention to the fact that students’ sense of a definition is not synonymous with the formal definition. Rather, their ‘sense-of’ a technical term consists of the images and instances which come-to-mind in a particular context, referred to as their *concept-image*. This consists of all the associations, images, examples, connections, as well as any dominant feelings and dispositions to behave in certain ways. Curiously, their concept image may or may not include memory of the formal definition, or even the means by which to reconstruct it.

However it is one thing to formulate a definition for yourself so as to make it easier to think, analogously with using labels to refer to pedagogic constructs and strategies in mathematics education. It is quite another to make sense of someone else’s definition. What is the definition allowing, and what is it excluding?

For example, many students arriving at university expect all functions to have a specific formula, and in this they are in good company as Euler was of this disposition, though he admitted equations of any degree in *y* as well. The now common definition of a function evolved over a considerable period of time (Boyer ). One way to bring this to student attention is the following task:

Sketch the graphs of the functions *f*(*x*) = *x*|*x*| and .



Encountering two apparently disparate specifications of the same function contributes to student awareness that it is the functional relationship not the description that matters. There are then opportunities to consider other properties of this function such as whether it has a point of inflection at 0 and whether it has a second derivative, in contrast to *h*(*x*) = *x*3.

### Example Construction

Where do the examples of mathematical objects that students meet come from?

Who constructs them?

Paul Halmos is in no doubt about the pedagogical value of people constructing their own examples:

Every time I learn a new concept I look for examples . . . and non-examples. . . . The examples should include wherever possible the typical ones and the extreme degenerate ones. (Halmos 1983 p62)

Where can we find examples, non-examples, and counter-examples? Answer: the same place where we find the definitions, theorems, proofs and all other aspects of mathematics—in the works of those who came before us, and in our own thoughts. . . . We find them first, foremost, and above all, in ourselves, by creative thinking. (Halmos, 1983, p. 64)

Many are canonical: they appear in textbook after textbook or are part of the folklore. But some students treat examples on a par with definitions and theorems, as things to be ‘learned’. Arriving at university they need to learn what to do with the examples they are given, and they need to become aware of the families of examples of which they are only shown individual instances. Again, the richness of their understanding is in large part indicated by the complexity and richness of their accessible example spaces.

One way to get students to reveal something of their thinking is to get them to construct examples. For example,

Construct some more examples like this where the integral is 0, and say in what way your examples are the same and in what ways different to this integral.



Some students will not think to vary the ends of the interval; some will not think to multiply by a constant; some will stick to polynomials, perhaps even polynomials of degree 1; some will not think to add together examples over the same interval.

Of course absence of evidence is not evidence of absence: just because they do not vary something does not mean that they did not think of it, but once students become accustomed to constructing objects they are likely to begin to display all of the features they are aware of that can be varied, especially if this is brought to their attention as desirable. The structure of the task involving cubics can be used here too: asking for other polynomials of degree 1, of a general polynomial of degree 2, a trigonometric function and so on.

When constructing their own examples, not only do students’ example spaces expand, but they can become aware of tools for tinkering with examples, and they experience mathematics as a creative and constructive activity (Watson & Mason 2005). For example, one of the powerful features of a series of lectures by Helmut Wielandt in the 1960s was an explicit collection of construction techniques for combining finite groups into larger and more complex groups. Students were encouraged to construct their own examples.

The following task provokes students to construct examples and as a result of seeking what is the same about all of them, to realise something that most likely does not come to mind immediately.

What is the locus of points at which the tangent to *f*λ(*x*) = *e*λ*x* passes through the origin?

What then is the locus of points at which the tangent to *g*λ(*x*) = *e*λ*x* + 2*e*3λ*x*

What then is the locus of points at which the tangent to *h*λ(*x*) = λ(*x*2 + 1) passes through the origin?

Trying some specific values of λ (the task can be stated in the particular, asking about *ex*, then *e*2*x*, then *e*3*x* and provoking generality) brings a common feature to attention; working with the general reveals an unexpected, but once seen ‘obvious’ result. However, on the way students encounter the problem of notation for a parameter that they want to vary later and which is usually called *x*, then changing what is varying and what is being held constant.

One of the phenomena mentioned at the beginning was the propensity of students, particularly in service courses, to ignore the conditions which are integral to the application of a theorem or technique. Getting students to construct counter-examples to weakening of the conditions of important theorems is not only a good opportunity for example construction, but also brings home the importance of the conditions. At least, it does this if students become aware of whole families of counter-examples rather than one or two isolated instances.

## What Do Students Do When They Study Mathematics?

At school, the method of studying is usually to practice doing questions from previous exam papers, working against a clock so as to develop facility. But how are students supposed to recognise what technique to use? How are they to learn how to recognise an appropriate technique, to have a relevant technique come-to-action or even come-to-mind?

The best way is to become aware of classes of problems, that is, to the features of problems to which each technique applies. The best way to do this is to construct variations on standard or routine questions so as to become aware of the complexity, of the features that can change (dimensions of possible variation) and the range of permissible change in each of these aspects.

If students spent some of their time refreshing their sense of the question space (Sangwin 2006, 2013), including tools for tinkering, then they are in a much better position to recognise the type of a question or the possibility of using a particular technique.

Textbook exercises can be really useful objects, as long as they are treated as a single mathematical object (Watson & Mason 2006) rather than merely collection of individual tasks. In many Russian textbooks, in some Finnish school textbooks, and in some English textbooks of previous generations, exercises were assembled so as to reveal structural relationships and to suggest families of similar problems. By paying attention to what changes and what remains the same (invariance in the midst of change), students’ appreciation of the scope and range of a technique, or of the relevance and use of a theorem or property can be enriched. Contrast this with a collection of disparate and unrelated tasks presented to students at the ends of sections and chapters.

One study technique that not all students tumble to is sorting: putting end-of-chapter exercises on cards and then sorting them according to method of solution. A deck of such cards can be useful for review, by reminding the student of what technique to use in what situation (Watson & Mason ref).

## Attention

#### Phenomenon

Students do not always pick up on what is being offered to them; it is as if they do not hear what is being said or do not see what is being shown.

This phenomenon raises the question:

What are students actually attending to in lectures, or when going through their notes, reading a text or attempting a problem?

It seems plausible that if students are not attending to what the lecturer is attending to, then communication between them is going to be difficult. This puts a premium on the lecturer themselves being aware of what they are attending to. A little self-observation reveals that it is not just what is being attended to that matters, but the nature of that attention: how that attention is structured.

The notions of ‘seeing the general through the particular’ and ‘seeing the particular in the general’ have been used several times, and appear to lie near the heart of teacher-student interactions. They are particular examples of the role and nature of attention. Observations of my own, are in alignment with observations made since ancient times (see particularly van Hiele 1986): it is possible to distinguish at least five different ‘ways of attending’.

Holding Wholes (gazing, contemplating)

Discerning Details

Recognising Relationships (in a particular situation)

Perceiving Properties

Reasoning on the basis of agreed Properties alone

These forms or states of attention are not sequential. Although one or another can become stable for a period of time (especially when you are stuck on a problem), most often they occur in quick succession as the mind darts about.

The last form or state of attention is what is often referred to as mathematical abstraction, when deductions are made using agreed properties (often as axioms together with proved theorems) ‘in the abstract’, that is, without making use of specific examples. Of course particular examples may be used to suggest possibilities, which is what Pólya meant by his term ‘specialising’, but the reasoning then proceeds ‘in general’.

The first form or state of attention is typical of that first moment when you look at or think about something, waiting for something to come-to-mind or at least come-to-action. It is typical in geometry to gaze at a diagram waiting for some possible relationships to appear.

Discerning details enables parts of a diagram, an argument, a worked example, or indeed any mathematical object to become a ‘whole’ to be gazed at or contemplated. It is often necessary to discern specific sub-objects or aspects, to stress these while ignoring other aspects in order to make progress on a problem, and certainly in order to be able to ‘hear or see’ what is being said or pointed to in a lecture or textbook.

Recognising relationships requires appropriate details to have been discerned, and it is the shift to seeing those relationships as instantiations of properties that may apply more generally which mark the movement t mathematical abstraction.

If a lecturer is talking about and pointing to specific details when learners are still gazing at a whole; if they are referring to relationships while students are trying to discern the details being related; if they are perceiving and talking in terms of properties while students are trying to recognise relationships in the particular, then communication between lecturer and student is at best attenuated.

When a lecturer is aware not only of what features are being stressed, what is being attended to, and how it is being attended to, it is possible to be sensitised to where student attention is focused and its state, and thus to direct student attention appropriately. When a lecturer is caught up in and oblivious to their own way of attending, they are likely to leave students behind, to push them into a state of simply copying down what is said and written in the hope of making sense later.

## Reprise: What are Examples For?

Examples are objects on which to focus attention. They are the foundation for working with attention so that students experience the movements of attention experienced by experts such as the lecturer.

Examples are the objects with which we check our grasp of generality, through seeing the particular in the general. They are also the main way in which we access the general by seeing the general through the particular. This means stressing structural relationships while downplaying particularities. It is just as important to be aware of particularities in examples that are *not* generic as it is to be aware of aspects that can be varied. Where variation is possible (a dimension of possible variation) it is important to become aware of the range of permissible change, and how that interacts with parameter variations.

### Pedagogic constructs and strategies

Dimensions of Possible Variation and Range of Permissible Change

Template and Templating

Didactic Contract (*contrat didactique*)

Mathematical Narratives & Incantations

Seeing the general through the particular; Seeing the particular in the general

Generic Examples

Figural Concepts extended to Unintended Features

Stressing & Ignoring

Another & Another

Can I do another question like this?

What kinds of ‘practice’ make perfect?

Actions that come to mind or to action

Attention and its various forms

Withdrawing from action and reflecting: what general strategies were employed; what actions were important; what tricky points were encountered

Conceptual Insights & Technical Handles

Working–Through & Working–On exercises

Exercises as a mathematical object

Question or Problem Space

Boundary Examples

Mathematical Themes: invariance in the midst of change; Freedom & Constraint; Doing & Undoing

## References

Anthony, G. (1994). The role of the worked example in learning mathematics. SAME papers 1994, p129-143, University of Waikato, Hamilton.

Atkinson, R., Derry, S., Renkl, A., and Wortham, D. (2000) Learning from examples: instructional principles from the worked examples research. *Review of Educational Research.* 70(2) pp. 181-214.

Benbachir, A. and Zaki, M. (2001) Production d’exemples et de contre-examples en analyse: étude de cas en première d’université. *Educational Studies in Mathematics* 47 pp.273-295.

Bills , L. (1996). The use of examples in the teaching and learning of mathematics. In L. Puig and A. Gutierrez (Eds.) *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education* (pp.2-81-2-88)Valencia, Spain.

Bills, L. and Rowland, T. (1999). Examples, Generalisation and Proof. In L. Brown (Ed.) *Making Meaning in Mathematics: Visions of Mathematics 2, Advances in Mathematics Education,* 1, 103-116, York: QED.

Brousseau, G. (1997). *Theory of Didactical Situations in Mathematics: didactiques des mathématiques,* *1970-1990,* N. Balacheff, M. Cooper, R. Sutherland, V. Warfield (Trans.), Dordrecht, Netherlands: Kluwer.

Bruner, J., Goodnow, J., & Austin, A. (1956). *A Study of Thinking*. New York: Wiley.

Chi, M. and Bassok, M. (1989) Learning from examples via self-explanation. In L. Resnick *(ed.) Knowing, Learning and Instruction: essays in honour of Robert Glaser.* Hillsdale, NJ: Erlbaum.

Courant, R. (1981). Reminiscences from Hilbert’s Gottingen, *Math Intelligencer* 1981 (3) 4 p. 154-164.

Dahlberg, R. and Housman, D. (1997) Facilitating learning events through example generation. *Educational Studies in Mathematics* 33. 283-299.

Fischbein, E. (1993). The Theory of Figural Concepts. *Educational Studies in Mathematics*, 24 (2) 139-162.

Gardner, M. (1977). Mathematical Games. *Scientific American*, Oct. p18-25.

Hershkowitz, R., & Vinner, S.: Basic geometric concepts - definitions  and images *Proceedings of the 6th PME Conference,* Antwerp, Belgium, 1982

Hershkowitz. R. (1987). The acquisition of concepts and misconceptions in basic geometry-Or when "a little learning is a dangerous thing." In J. D. Novak (Ed.), *Proceedings of the second international seminar on Misconceptions and Educational Strategies in Science and Mathematics* Vol. 3, p236-251. Ithaca. NY: Cornell University.

Knopp, K. (1951; 1990 reprint). *Theory and Application of Infinite Series*. London: Dover.

Lakatos, I. (1976). *Proofs and Refutations: the logic of mathematical discovery*. Cambridge: Cambridge University Press.

MacHale, D. (1980). The Predictability of Counterexamples, *American Mathematical Monthly*, 87, p752.

Marton, F. & Booth, S. (1997). *Learning and Awareness.* Hillsdale, USA: Lawrence Erlbaum.

Mason, J. (1998). *Learning & Doing Mathematics.* (2nd revised edition), St.Albans: Tarquin.

Mason, J. (2002). *Mathematics Teaching Practice: a guide for university and college lecturers*, Horwood Publishing, Chichester.

Mason, J. & Johnston-Wilder, S. (2004). *Fundamental Constructs in Mathematics Education*. London: RoutledgeFalmer.

Michener, E. (1978). Understanding Understanding Mathematics. *Cognitive Science*, 2, p361-383.

Piaget, J. (1970). *Genetic epistemology*. New York: Norton.

Poincaré, H. (1956 reprinted 1960). Mathematical Creation: lecture to the Psychology Society of Paris, in J. Newman. *The World of Mathematics*. London: George Allen & Unwin. p2041-2050.

Pólya, G. (1945). *How To Solve It: a new aspect of mathematical method*. Princeton, USA: Princeton University Press.

Pólya, G. (1962). *Mathematical Discovery: on understanding, learning, and teaching problem solving* (Combined edition). New York: Wiley.

Pólya, G. (1965). *Let Us Teach Guessing,* (film) Mathematical Association of America, Washington.

Raman, M. (2003). Key ideas: What are they and how can they help us understand how people view proof? *Educational Studies in Mathematics*, 52, 319–325.

Renkl, A. (1997). Learning from worked-out examples: A study on individual differences. *Cognitive Science*, *21*, 1-29.

Renkl, A. (2002) Worked-out examples: Instructional explanations support learning by self-explanations. *Learning and Instruction,* *12*, 529–556.

Renkl, A. Atkinson, R. Maier, U. (2000). From Studying Examples to Solving Problems: Fading Worked-Out Solution Steps Helps Learning. www.ircs.upenn.edu/cogsci2000/ PRCDNGS/SPRCDNGS/PAPERS/RENAT-MA.PDF (accessed may 2005)

Renkl, A. Mandl, H. & Gruber, H. 1996, Inert Knowledge: analyses and remedies, *Educational Psychologist*, 31 (2) p115-121.

Renkl, A. Stark, R. Gruber, H. & Mandl, H. (1998). Learning from Worked-Out Examples: the effects of example variability and elicited self-explanations. *Contemporary Educational Psychology* 23 p90-108.

Sandefur, J. Mason, J. Stylianides, G. & Watson, J. (2012). Reaching for the Familiar: Example Generation in the Proving Process. *Educational Studies in Mathematics*. 83(3) p323-340.

Sangwin, C. (2006). Mathematical Question Spaces. <http://hdl.handle.net/2134/4418> (accessed 2012).

Sangwin, C. (2013). *Computer Aided Assessment of Mathematics.* Oxford: Oxford University Press.

Scataglini-Belghitar, G. & Mason, J. (2012). Establishing Appropriate Conditions: students learning to apply a theorem. *IJSME* 10(4) p927-953.

Sierpinska, A. (1994). *Understanding in Mathematics*. London: Falmer Press.

Vygotsky L. (1978) *Mind in Societ*y: *the development of the higher psychological processes*. London: Harvard University Press.

Watson, A. & Mason, J. (2005). *Mathematics as a Constructive Activity: learners generating examples*. Mahwah: Erlbaum.

Watson, A. & Mason, J. (2006) Seeing an Exercise as a Single Mathematical Object: Using Variation to Structure Sense-Making. *Mathematical Thinking and Learning*. 8(2) p91-111.

Yerushalmy, M. & Chazan, D. (1990). Overcoming visual obstacles with the aid of the supposer. *Educational Studies in Mathematics* 21, 199-219.

1. ZigZags is available for download at ref) [↑](#footnote-ref-1)