What are examples for?
What makes them useful?

Promoting Active Learning

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Sept 2009

These slides will be on the MSOR website and on mine within a few days

Phenomena

- Students often ignore conditions when applying theorems
- Students often have a very limited notion of mathematical concepts …
  - What is it about some examples that makes them useful for students?
  - What do students need to do with the examples they are given?
- Students often ask for more examples …
  - But do they know what to do with the examples they have?

What do students say?

- "I seek out worked examples and model answers"
- "I practice and copy in order to memorise"
- "I skip examples when short of time"
- "I compare own attempts with model answers"
- NO mention of mathematical objects other than worked examples

These are Oxford ugs starting their 2nd year in mathematics

On worked examples

- Templating: changing numbers to match
- What is important about worked examples?
  - knowing the criteria by which each step is chosen
  - knowing things that can go wrong, conditions that need to be checked
  - having an overall sense of direction
  - having recourse to conceptual underpinning if something goes wrong

Examples of mathematical Objects

- What do you do with examples?
- What would you like students to do with examples?
- Unfortunately, students sometimes over-generalise, mis-generalise or fail to appreciate what is being exemplified

Really Simple Examples (sic!)

- To find 10%, you divide by 10;
  so, to find 20%, you ...
- To differentiate x^n you write nx^{n-1}
  so, to differentiate x^x you write ...
- $(x-1)^2 + (y+1)^2 = 4$ has centre at (1, -1) and radius 2
- if f(3) = 5 then f(6) = 10
  - other appearances:
    $\sin(2x) = 2\sin(x)$; $\ln(3x) = 3\ln(x)$
    $\sin(A+B) = \sin(A)+\sin(B)$ etc.

In each case, students have used an example as the basis for generalisation!
Sometimes linearity is only used when the 'scale factor' is large
(accounting for: minimise effort; find something to do)

What would you use as an example of …

- a continuous function on R differentiable everywhere except at a single point

$|x|$

$x^{\frac{1}{3}}$

$$\begin{cases} x\sin\left(\frac{1}{x}\right), x \neq 0 \\ 0, x = 0 \end{cases}$$

- a function whose integral on a finite subinterval of R is 0.

$\int_0^{2\pi} \sin(x)$ $\int_a^b 0\,dx$ $\int_0^2 (1-x)\,dx$

- a function whose integral requires integration by parts?

$x\sin(x), xe^x, \ln(x)$

What are you expecting learners to see being exemplified?
What do want them to see as changeable?
What is exemplary about these?
In how many different ways can you do the first one?

Proposal

- The first time you give an example of a mathematical object (not a worked example)
  - ask students to write down on a slip of paper what they expect to do with the example
- Near the end of the course, when you give (or get them to construct) an example
  - ask students to write down on a slip of paper what they expect to do with the example
- get them to put their initials or some other identifying mark on the papers!
- Send the results to me! j.h.mason@open.ac.uk

Example Construction

- What features need to be salient?
  - contrasting several examples
  - What can be changed (dimensions of possible variation) and over what range (range of permissible change)
- Learners Constructing Examples

Imposing Constraints

- Sketch a cubic
- which does not go through the origin
- and which has a local maximum
- and for which the inflection slope is positive

- Is there a quadratic whose root slopes are perpendicular?
- Is there a cubic whose root slopes are consecutively perpendicular?
- A quartic … ?;
  for what angles is it possible?

Aim is to impose a constraint which is contrary to implicit assumptions made by learners in order to enrich and expand their example space: quadratic & cubic are possible; quartic is not.

In fact, for any polynomial with distinct real roots, the sum of the cotangents of the root slopes is always zero! More generally, if a line of slope m cuts a polynomial of degree d in d distinct points, the sum of the cotangents of the angles between the tangents at the cutting points and the straight line is md.

Conjecture

- When learners construct their own examples of mathematical objects they:
  - extend and enrich their accessible example spaces
  - become more engaged with and confident about their studies
  - make use of their own mathematical powers
  - experience mathematics as a constructive and creative enterprise

Bringing Possibilities to Attention

- Construct an injective function from [-1 1] to R for which
  - the limit as x approaches ±1 is 1,
  - and f(0) = -1
- Possibilities
  - f(x) = x if first decimal digit is odd, else f(x) = -x
  - use rationals and irrationals differently for x>0 and x<0
- Is there a non-constant periodic function on R with no positive minimal period?

Preparing the Ground

- Construct a function F : [a, b] --> R
  - which is continuous and differentiable on (a, b)
  - for which f(a) = f(b)
  - but nowhere on (a, b) is f'(x) = 0

Setting an impossible task in order to prompt possibility and significance of a theorem

Construction Task

- Write down a pair of distinct functions for which the integral of the difference over a specified interval is zero.
  - and another
  - and another
- How did your attention shift so that you could come up with those examples?
- Construct an example of a pair of distinct functions for which the integral of the difference is 0 over finitely many finite intervals; over infinitely many.

When technical terms are used, we are triggered into accessing a space of examples. We want students to have as rich an accessible example space as possible. This means providing them with construction techniques.

$$\int_0^2 (1-x)\,dx = 0$$

- Write down another integral like this one which is also zero
  - and another
  - and another
  - and another which is different in some way
- What is the same and what is different about your examples and the original?
- What features can you change and still it has integral zero?
- Write down the most general integral you can, like this, which has answer zero.

Example of reversing or undoing something which is familiar as a doing

Some learners tried preserving the difference between the limits(!) despite associating integral with area
Some immediately multiplied by a constant, but others did not
Some altered the linear function; some went to sin(x)
some revealed uncertainty between integral and area and negative area.

Have you ever found that students don't recognise when a theorem can be applied (or else apply it when it is not applicable)?

Exemplifying Mathematical Concepts

- Suppose you were about to introduce the notion of relative extrema for functions from R to R.
  - what examples might you choose, and why?
  - would you use non-examples? why or why not?
- How might you present them?

Sorting Task

- Sort the functions in some sensibly mathematical way
- Sort them according to relative extrema
- What is the same and what is different about ...

The Exemplification Paradox

- In order to appreciate a generality, it helps to have examples;
- In order to appreciate something as an example, it is necessary to know the generality being exemplified;
- so,
  I need to know what is exemplary about something in order to see it as an example of something!

What can change and what must stay the same, to preserve examplehood?

Dimensions of possible variation
Ranges of permissible change

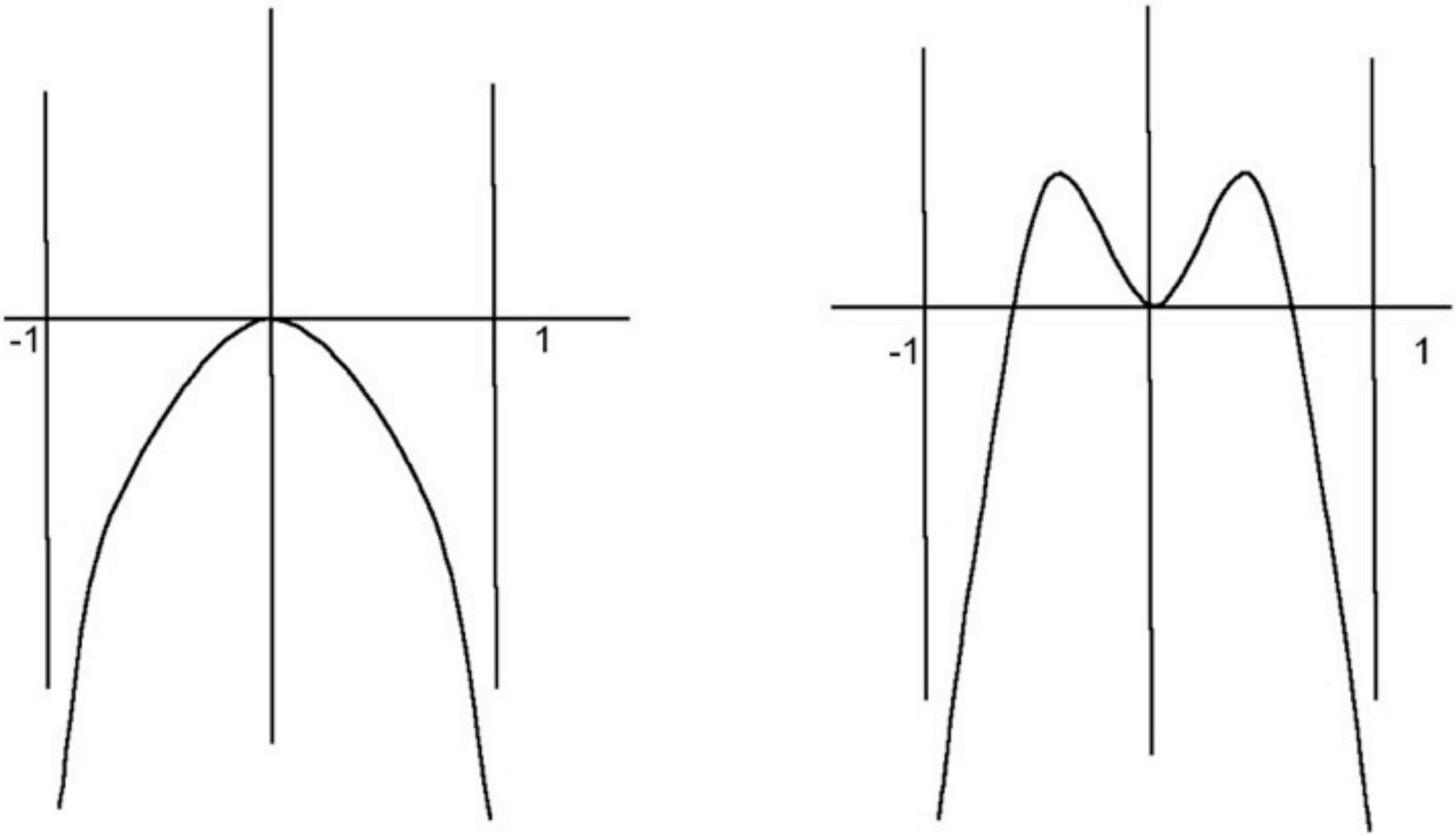
Need to break into this 'shell'
Using the previous 'examples', we can explore some pedagogic tactics

Applying a Theorem as a Technique

- Sketch a continuous function on the open interval (-1, 1) which is unbounded below as x approaches ±1 and which takes the value 0 at x = 0.
- Sketch another one that is different in some way.
- and another; and another
- What is common to all of them?
- Can you sketch one which does not have an upper bound on the interval?

Students had failed to use the theorem about achieving extrema on a closed interval, to apply to this case of an open interval. So tutor asked them to construct examples.

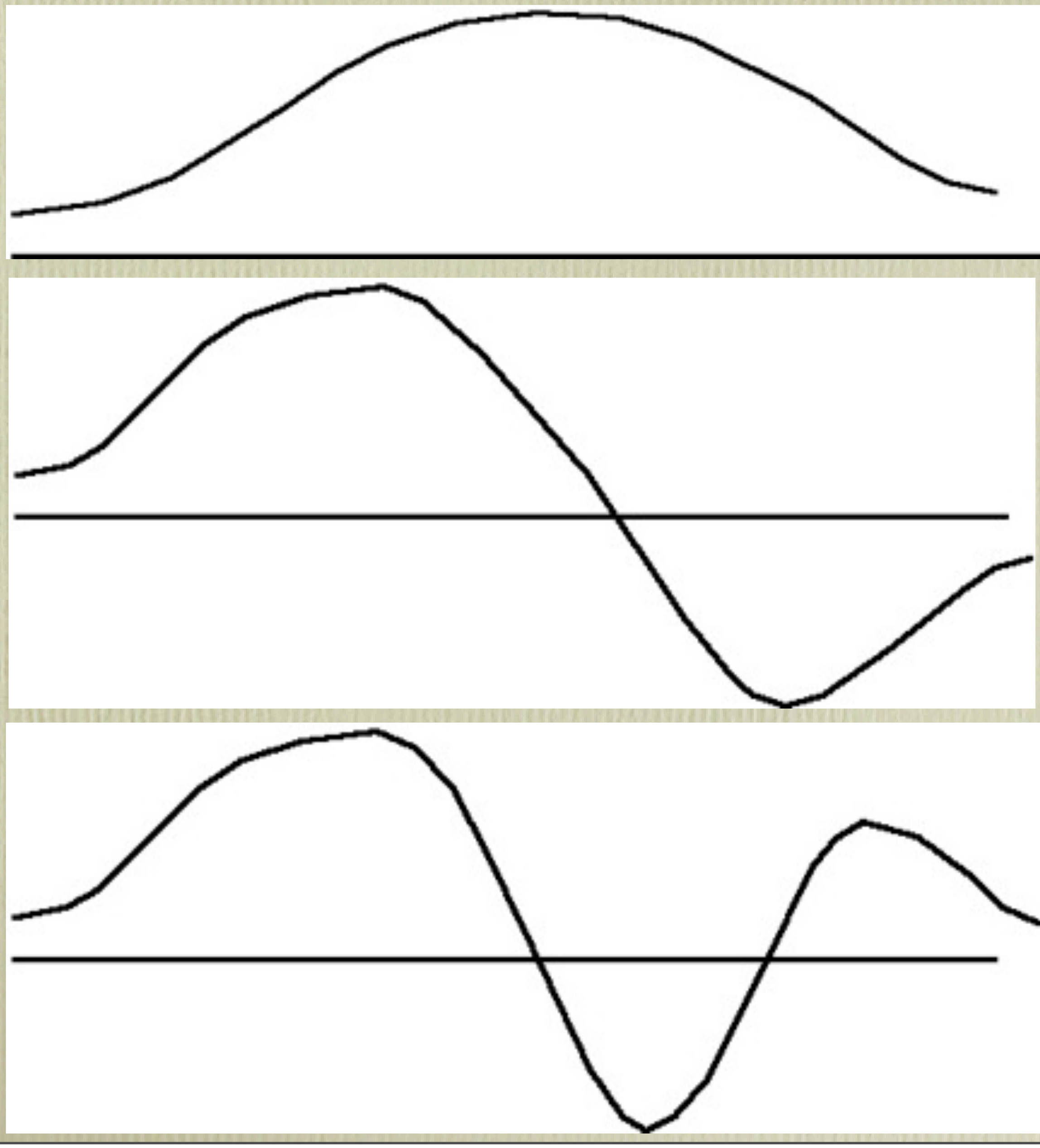
Student Constructions



Because it is continuous was all that students offered.

Only when the tutor mentioned that because the function goes to negative infinity at both ends, there cannot be a maximum ‘close’ to an end, so you can locate a subinterval which is closed, did ‘closed & bounded’ cue appropriate steps.

More Student Constructions



- What can we change in the conditions of the task and still have the same (or similar) phenomenon, i.e. make use of the same reasoning?

Here we want to force the function to have a relative maximum, or a relative extremum.

The tutor then asked what other conditions, for functions defined on other types of domain, would similarly guarantee a maximum, or a minimum.

Undoing a Familiar Doing

familiar doing: Given f(x), find $\lim_{x \to 3} \frac{f(x) - f(3)}{x - 3}$

unfamiliar undoing: What can you say about f if

$$\lim_{x \to 3} \frac{f(x) - 5}{x - 3} = 2$$

f(x) = 2x+1 + h(x)(x−3)^2 for any h(x) with a limit at 3.
f'(3) = 2

Bury The Bone

- Construct an integral which requires two integrations by parts in order to complete it
- Construct a limit which requires three uses of L'Hôpital's rule to calculate it.
- Construct an object whose symmetry group is the direct product of four groups

Alternatives:

construct an object which makes it difficult to see that it is ...
an integration by parts; a use of L'Hopital; that its symmetry group is a direct product; ...

|x|

- Of what is |x| an example?
- What is the same and what different about graphs of functions of the form

$$f(x)=|x-a| \qquad f(x)=||x-a|-b| \qquad f(x)=|||x-a|-b|-c|$$

…

or

$$f(x)=|x-a_1|+|x-a_2|+\ldots+|x-a_n|$$

Characterise these graphs in some way

Using |x| ...

- ... write down a continuous function which is differentiable everywhere on R except at two points;
  - and another
  - and another
  - and another but at three points
- What sets of points can be the set of points at which a continuous function on R is not differentiable?

Another and another generates a desire to be more varied, more sophisticated, more general

Presenting Examples

- ‘give’ or ‘work through’ examples
- use a sorting task
- get students to construct some objects for themselves (another & another)
- tinker publicly, even interactively,
  with simple ones, to indicate other possibilities

Probing Awareness

- Asking learners what aspects of an example can be changed, and in what way.
  - learners may have only some possibilities come to mind
  - especially if they are unfamiliar with such a probe
- Asking learners to construct examples
  - another & another; adding constraints
- Asking learners what concepts/theorems an object exemplifies

What Makes an Example Exemplary?

- Awareness of
  'invariance in the midst of change'
- What can change and still the technique can be used or the theorem applied?
- Particular seen as a representative of a space of examples.

Useful Constructs

- Accessible Example Space (objects + constructions)
- Dimensions of Possible Variation
  - Aspects that can change
- Range of Permissible Change
  - The range over which they can change
- Conjecture:
  - If lecturer's perceived DofPV ≠ student's perceived DofPV then there is likely to be confusion
  - If the perceived RofPCh are different, the students' experience is at best impoverished

Link to Chris Sangwin's Question Space

Pedagogic Strategies

- Pedagogic Strategies
  - Another & Another
  - Bury the Bone
  - Sequential constraints designed to contradict simple examples
  - Doing & Undoing
  - Request the impossible as prelim to proof
- These are useful as study techniques as well!

I have offered you one or two examples of each tactic … is that sufficient to be exemplary? Is your example-space corresponding to each richer than one example in one mathematical domain?

To Investigate Further

- Ask your students what they do with examples (and worked examples, if any)
- Compare responses between first and later years
- When displaying an example, pay attention to how you indicate the DofPV & the RofPCh.
- Consider what you could do to support them in making use of examples in their studying

Proposal Reminder

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For MANY more tactics:

- Mathematics Teaching Practice: a guide for university and college lecturers, Horwood Publishing, Chichester, 2002.
- Mathematics as a Constructive Activity: learners constructing examples. Erlbaum 2005.
- Using Counter-Examples in Calculus College Press 2009.
- j.h.mason@open.ac.uk mcs.open.ac.uk/jhm3
- MathemaPedia (NCETM website)

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