WHAT IS EXEMPLIFIED IN MATHEMATICS CLASSROOMS?

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May 2005

ABSTRACT

In this paper I use observations from three mathematics lessons in a secondary school to draw out something of the variety of ways in exemplification takes place in mathematics lessons. Beyond the familiar 'worked examples' illustrating mathematical techniques there are many other ways in which mathematical practices are or can be exemplified in lessons.

The observations refresh the question of what makes such proffered examples exemplary for learners. The question of what learners take to be examples, and what they think is being exemplified is part of an ongoing enquiry into what learners attend to, and how. The project aim is to locate ways of making use of examples which maximise their effectiveness for learners.

BACKGROUND

Comments made about the observations make use of Ference Marton's notion of *dimensions of variation* as extended by Anne Watson and John Mason (see the section following the observations). Observations were in three lessons at Wheatley park School in Oxfordshire, one with Stephen Alexander and the other two with Jackie Fairchild.

OBSERVATIONS

LESSON ONE

The following observations were made in one lesson involving twenty-seven top-set 14-15 year olds (year 10), during the sixth week of the school year. They focus on the use of examples.

Sequence One

After initiating the lesson and announcing the topic to be the expansion of brackets, a sequence of tasks was displayed with the announced intention of reminding learners of what they had done previously. The sequence ran

Expand the brackets: $3(t - 7)$
Multiply out: $x(x + 5)$
Remove the brackets: $3x(2x + y)$
Rewrite without brackets: $8(3m – n + 7)$
Simplify: $3(k – 4) + 2(2 + k)$
Multiply out: $(x + 3)(x – 5)$
Write as a sum of terms: $(n + 4)^2$

Learners responded to each prompt in turn by writing down their answer on a small white board and then collectively revealing it to the teacher when asked .

Comment

On the one hand these are revision tasks or exercises, but from another perspective they are being offered as examples of the sorts of tasks the teacher expects the learners to be able to deal with.

Notice the dimensions of variation within the sequence: the language of the tasks, the letters used, and the complexity. At a more subtle level, the use of the white boards and the practices which go with it exemplify a way of working which will be used throughout the school year.

Sequence Two

A collection of tasks was put on the board and learners were asked to write down the question and the answer in their notebooks.

$2(a-3)+3(a+1)$	$(2y+1)(y-4)$
$3(x+2)-2(x+1)$	$(x-2)^2$
$(d+5)(d-1)$	$(2t+1)^2$
$(c-3)(c+1)$	$(n+3)(n^2-7n-2)$

After a period of work on these with the teacher and a support teacher circulating around the room, a pack of cards with names on was shuffled and used to call upon individuals to propose their answers.

In the event, the second one in the first column yielded four different answers, and eventually the correct one was identified, while each of the others was accounted for by a misuse of negative signs. The next obstacle appeared at the squares in the 6^{th} and the 7^{th}: first a negative sign to be distributed, then the product of two negatives, and then the presence of a coefficient for the letter.

For the later tasks some learners when called upon reported that they either did not know or had not got that far, and this was accepted and respected. They were invited to pay particular attention to what others could contribute.

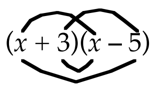
Comment

One thing being exemplified was a way of dealing with not knowing: acknowledging openly, and taking it as an opportunity to learn from others. Another thing was that as each task was worked through, learners could attend to what others has done and what the teachers was 'apparently looking for or expecting'. Thus mathematical practices were being exemplified (such as using brackets to avoid ambiguity, writing equals signs between successive transformations and aligning them when placed underneath each other in a list).

In discussion with the teacher after the lesson the question arose as to whether there was time for learners to experience the 2^{nd}, 6^{th} and 7^{th} as types, or whether their exemplariness might be lost in the flow of the reporting back of answers. In other words, although the teacher chooses the tasks carefully to exemplify possible errors or complexities, it is not clear what needs to be done to maximise the effect on learners. Put another way, how might attention be directed so that learners appreciate some feature as a dimension of possible variation, and furthermore to appreciate the range of permissible change in that dimension?

Incident One

In the midst of working through a bracket expansion the teacher reminded them of the 'smiley face' device of linking all pairs of terms to be multiplied by arcs, producing two eyebrows, nose, and chin, with the outside brackets as ears.



Comment

Here the teacher is offering an example of a way of seeing how to remember to calculate all the terms of the product. Enactively, learners are expected to see through the particular instance to a 'method' which can be used in other expansions. Affectively, learners may see the 'smiley' method as a means for not having to think about what they are doing, but to leave the 'remembering to calculate all the terms' to the device. Cognitively, learners are invited to see the 'smiley' as one way of ensuring that all possible pairs are multiplied, the essence being that each term in one bracket must be multiplied by each term in the other.

This general 'essence' came to the fore in the last task of the sequence, although only some of the learners had had time to tackle it. So a shift from specific to verbal generality in describing a method was being offered.

Sequence Three

Under the heading of demonstrating how the expansion of brackets could be used, the teacher set a final sequence of tasks:

Why is $2n - 1$ always odd?

What is the next odd after $2n - 1$?

Why is an odd number squared always odd?

Why is the product of two consecutive odd numbers always 1 less than a perfect square?

Comment

Here the teacher is exemplifying a use of expanding brackets in writing down a proof of a generality algebraically, as a contribution towards the theme of reasoning in mathematics. In the process, various other things are being exemplified, among them

the use of an algebraic expression to denote a class of numbers, a generality (odd numbers);

a style of potential examination task;

the nature of mathematical reasoning, moving beyond 'odd squared is odd' as an empirical fact;

how such proofs are conventionally written down (emerging in comments on what learners proposed after having worked for a time on the task with the teacher circulating and guiding or commenting).

LESSON TWO

The following observations were made in a lesson of thirty 12-13 year olds (year 8) in the sixth week of the school year. Again they focus mainly on the use of examples.

Sequence One

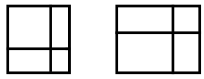
Individuals were invited to take turns announcing a multiplication fact in which at least one of the numbers was a decimal. The teacher offered the first (0.2 x 0.8 = 0.16), then selected someone who was willing to have a go (by throwing them a softball; those not wishing to have a go were asked to avoid eye contact with the person about to throw it). After a while she asked the receiver to restate the previous fact as a product of fractions (0.6 x 0.9 = 0.54, 3/5 x 9/10 = 27/50). Some difficulties over decimal points arose and were ironed out.

Comment

The format had become familiar to the class from previous lessons, so a single 'fact' from the teacher served as an example to follow subsequently. Here learners were expected to make up their own examples, enabling them to challenge themselves or to play safe. All were expected to listen to and check what others said.

Sequence Two

Learners were asked to draw a ten by ten square on dotty paper, then to insert any two lines parallel to the edges, one horizontal and one vertical, crossing the square along lines of dots. The lengths of the edges of the square were specified to be one unit. The teacher showed what was wanted by drawing one herself as she set the task, using lines dividing the edges into 0.6 and 0.4 both horizontally and vertically. Learners were then asked to calculate the area of the four rectangles and of the total, and to check that these agreed. Learners were to work in pairs, with one doing decimal calculations and the other using fractions but on the same diagram. There was an invitation for those who wished to divide the square up into more sub-squares if they wanted to challenge themselves.



Comment

One or two learners counted ten dots rather than ten spaces on their dotty paper, so when they discovered a problem they redrew their squares. The aim of the task was to provide practice in using decimals and fractions, and opportunity to see them as different forms for the same numbers, while at the same time providing an image for what addition and multiplication of fractions means, and furthermore, providing a grounding for a move into algebra, by generalising the lengths.

The structure of the task included opportunities for learners to make choices, both in where the lines were to be drawn, and whether to make the diagram more complicated. Naturally, some chose a very simple division into halves , while others drew diagrams that were far too complicated to be sensible. Those who chose simplicity realised later that they could have been more adventurous, and those who were too adventurous were invited to do a simpler one for the present, and the more complicated one for homework. By exposing the class later to the range of possibilities, a contribution was made to learners' sense of what they were doing as an example of a potentially more general phenomenon.

On the overt level, the teacher provided an example of the lengths of the sides of the sub-rectangles, working through details as she described what they were to do. One or two pairs 'chose' to use her values. Were they aware of the 'worked example' as an example, that is of the possible choices that could be made? Would they become aware of that as the lesson developed? More subtly, the teacher provided an example of 'what they were supposed to do', and also of the possibility of making choices for themselves.

Sequence Two Developed

Learners were then invited to imagine that the width of the square was actually some other number (the teacher chose 8) rather than 1, so that the diagram became schematic rather than an accurate picture. Again they were asked to calculate the areas of the sub-rectangles (one using decimals, one using fractions).

Comment

Many did not know how to deal with the change of scale, but this was their first exposure to the idea, and so they became aware of a possibility as a problem to be resolved in subsequent lessons. It was reported that far more had picked up the idea than had been evident at first. The difference between a scaled drawing and a schematic figure seemed to be a new one, so as might be expected, several tried to redraw their diagrams with the new scale, only to realise that it would not fit easily on their page. The teacher's use of 8 served as an exemplar for many who chose other lengths, like 7. A popular 'choice' was 8, and another was 10, which introduced conceptual difficulties because of the original square being 10 units by ten units.

Sequence Three

The lesson ended with a collective worked example using one of several 'diagrams' used by different pairs of students and displayed on the board. It happened to divide the initial square into four equal sub-rectangles. The teacher worked through the details using question and answer from the class. She then changed the measurement of the width of the square to a and again prompted the class in plenary mode to decide on the lengths of sides and the areas of the four pieces, getting a for the whole area, and $a/4$ for the pieces. She then changed the measurement of the height to b and got ab for the whole area and $ab/4$ (being one quarter of the whole) and $a/2 \times b/2$ (using the edge lengths) for the pieces.

Comment

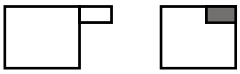
The aim of this task was to expose learners to generalisation as a means for experiencing the arithmetic of fractions. In follow-up lessons it was possible to consider a diagram with the square divided into four rectangles, to rehearse the effect of changing the measurements with the move to generality, to reveal similarities and differences between decimal and fraction arithmetic, and again to move to expressing generality using the diagrams as support for calculations. Through inviting learners to choose details of their diagrams and then displaying these later, learners are exposed to multiple examples which vary in one or two features, supporting their growing sense of generality through having attention drawn to dimensions of possible variation.

LESSON THREE

The following observations are from a lesson with thirty 11-12 year olds (year 7) in the sixth week of the school year.

Sequence One

The lesson began with a rectangle drawn on the board with a stated area of 12, and the question of what possibilities there might be for the lengths of the sides. The rectangle was then augmented in two ways, where the small rectangle has a height of 1:



Again the question was put: what are the possibilities if the total area (shading means a piece missing) is 12.

After a period of individual work, various possibilities were offered such as 5 x 2 + 2 x 1, 3 x 2 + 3 x 2, 4 x 4 – 4 x 1. Arising from this was that the diagram was not to scale, so the little rectangle dimensions were not in scale to the big rectangle dimensions, but that this did not matter. The question was then asked as to what was fixed and what could vary.

The two diagrams, one on each side of a piece of paper inserted in a plastic wallet were then handed out as 'writing frames', with marker pens for writing on the plastic cover. Learners were asked to work in pairs, to choose one side or the other, to work out some possible dimensions and to calculate the area, but then to write on their 'frame' all but one of the dimensions. They were to use a letter for the missing dimension and to write an expression for the area.

The teacher illustrated these instructions with an example of each type, with the help of the class. She used 4 x 6 + 3 x 1 = 27, yielding $4x + 3 = 27$, and 3 x 6 – 5 x 1 = 13 yielding $3x - 5 = 13$, with mention that other letters could be used.

Comment

The aim of the lesson was to use the two-rectangle format with which the learners were becoming familiar, as a supportive image and context for producing equations which could then be solved.

Sequence Two

Pairs then exchanged their diagrams and challenged each other to find the missing value. Most used x for their missing value, but at least one pair used an n. After a period of work on these, seven diagrams were stuck on the board, and the class were invited to try to solve some of them. Equations generated included

$9x - 15 = 57$, $4x + 2 = 14$, $3x + 4 = 25$, $5x + 8 = 53$, $17x + 5 = 90$

These were then worked through using suggestions from the class. The teacher emphasised that she was interested in 'discovering a method for doing these'.

Proposals from the class made reference to adding and subtracting the area of the small rectangle to recover the area of the large rectangle. The teacher mentioned in passing that they would be working later on how to write all this reasoning down, and she put some attention to the solving of multiplications such as $9x = 45$ by using division, again to be developed later.

Comment

Here the examples used in front of the whole class come from the learners, with little or no selection by the teacher. Opportunities are taken to signpost future work in details which emerge from what arises. One potential for exemplification is therefore that mathematical ideas can arise, be noted, and put to one side for further exploration later. It is alright not to know how to do something.

With multiple examples from the learners themselves there are opportunities to get them to draw back and consider what is the same and what different about the various examples, in order to appreciate the broad class being exemplified.

Sequence Three

Learners were asked to put pens down and pay close attention. Attention was drawn to the two structures they had been using. Learners were then invited to make up their own 'template' (to go in the 'writing frame' plastic wallet). Possibilities were asked for, and triangle was offered. A right-angled triangle was drawn on the board with perpendicular edges of 4 and 7. The area was asked for and given. Then one edge measurement was removed and replaced with x to yield $x \times 7 \div 2 = 14$ and also $4 \times x \div 2 = 14$. Someone else suggested 3D. The proposal was that one person does a template, inserts that in the 'writing frame', then passes that to someone else who does the calculations and the sets the equation before passing that to a third person to try to recover the missing length.

A final example was worked in plenary, starting from a learner's proposal of a square.

Some learners drew shapes that were too irregular to permit the calculation of areas, but for homework even they chose shapes whose areas could be readily calculated. One learner used a drawing of two cubes, and this was used to finish the lesson: what lengths could resolve $x^3 + m^3 = 28$ (the 28 was provided by the teacher). Various numbers were tried before alighting on $x = 3$ and $m = 1$.

Homework was to make up their own shape, do the area calculation, make an equation, and see how to recover the missing value.

Comment

The teacher reported 'pushing the envelope' to see what might be possible, due to the opportunities presented by the presence of the observer.

The overt example of the right-angled triangle provided a worked example. When confusion mounted (often due to overly complicated shapes being chosen), the squares example was picked up and worked through on the board.

The teacher's aim is always to embed the overt task in greater complexity so that learners would be exposed to a variety of ideas simultaneously. Often this complexity is due to choices being made by learners. Thus mathematics as a way of making sense of complexity is being exemplified, even if the learners are not overtly aware of it. Mathematics as comprising interconnected ideas is also being exemplified by exposing learners to extreme cases which emerge from the class, such as the cubes, not so that they master the cubes, but so that they are aware that there are opportunities to take the ideas further.

EXEMPLIFICATION

All mathematics teachers and learners are familiar with the notion of a 'worked example'. Indeed most textbooks have included these, from Babylonian tablets, Egyptian papyry and Chinese manuscripts to the present day. A common request from adult learners is for 'more examples'. Socio-culturalists and socio-cultural constructivists are familiar with the way in which people pick up social practices by being immersed in and exposed to the use of those practices in social settings (here, the mathematics classroom). There is nothing new about teacher behaviour providing a model for learners to emulate.

The use of worked examples has been the subject of study be several authors, including Charles (1980), and Chi & Bassok (1989), but these usually restrict attention to the provision of worked examples of techniques. The preliminary observations reported here point to a much broader use of exemplification in mathematics classrooms, with cognitive, affective, enactive, aspects subtly intertwined with each other and with the milieu. As a preliminary attempt to describe this complex web, the following uses of examples arose in these three lessons.

A worked example of a calculation to be performed on other similar instances.

A worked example of a sequence of instructions to be carried out.

An example of a structure (in diagrammatic form) on which a whole class of examples may be based. A frequently used structure represented diagrammatically bridges the gap between the particular and the general.

An example of a task with which learners are expected to be familiar, as a warm-up.

The development of a calculation (or task type) which indicates a dimension of possible variation.

An instance of an expected response when a learner is uncertain or is stuck.

An instance of displaying overt fairness in choosing learners to be asked to respond.

Choices made by other learners as indication of dimensions of possible variation and associated range of permissible change.

Learners are immersed in a milieu (Brousseau 1984, 1997) which includes 'ways of working on mathematics', as social practices, but also something to do with self-confidence and self-image within those practices. Through immersion in practices learners assimilate (some aspects of) those practices into their own functioning or at least adjust-accommodate their behaviour to take account of the situation. More succinctly, learners may adopt or adapt to patterns of teacher and teacher-class behaviour. Notice the melding of a partly Vygotskian perspective cast with Piaget's terms based on a biological metaphor for learning.

ISSUES ARISING

Ference Marton has been exploring (Marton & Booth 1997, Marton & Trigwell 2000, Marton Runesson & Tsui 2004) learning as the extending of awareness of dimensions of variation. In other words, learning consist of recognising further extensions of what can be varied and still something remains an example of something more general. Recognising that what learners are aware of is not always what teachers are aware of led Watson & Mason (2004, in press, see also Mason & Johnston-Wilder 2004) to extended this notion to *dimensions of possible variation*, stressing that what some people see as possible to vary may not be seen by others, so that it is the 'awareness of possible variation' that matters, while at the same time there can be differences in what variations are considered permissible in any particular dimension. This led to the notion of *range of permissible change* associated with each dimension of possible variation associated with a concept.

One of the reasons that these notions seem so powerful is that in mathematics they mirror and support the notion of *seeing the general through the particular*, and of *seeing the particular in the general* (Whitehead 1932, Mason & Pimm 1984). Thus an example only exemplifies if it signals or triggers awareness of features which, although particularised in the example, can be generalised. Thus awareness of a dimension of possible variation is essentially awareness of generality. Put another way, awareness of dimension of possible variation is essential for people's example spaces to be richly representative of concepts (Watson & Mason in press). Appreciating a concept in mathematics requires awareness both of what aspects, features, relationships or properties are invariant, while at the same time other features or aspects are allowed to vary. Thus the theme of *invariance in the midst of change* pervades all of mathematics (Polya 1962, Mason 2002, Mason & Johnston-Wilder 2004).

Teachers and textbook authors may believe that they are offering learners examples of practices at many levels. What matters however is what learners become aware of (even subconsciously) as being exemplified. For example (*sic*) exposure to one 'example' may not provide learners with enough variation to trigger an important dimension of possible variation. Several 'examples' encountered may not be recognised as sufficiently similar to be encompassed as examples of some common property. There may be so many things varying at once, that learners are at best subliminally aware of the many aspects that are possible to vary, much less the range of change which is perceived as actually being permissible in each dimension. On the other hand, to try to reduce teaching to pedestrian revelation of dimensions of possible variation one by one would be to ignore the power of human beings to assimilate and thrive in complexity. Indeed many of the difficulties being experienced in schools might be explained as arising because of attempts to 'do the work for learners', to simplify to the point of tedium so that there is nothing to challenge learners to use their natural powers (Houssart 2004).

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