Extreme Images

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## Problem

Given a non-singular 2 by 2 matrix *M*, when is the angle between the source vector and the image vector greatest and smallest? When is the distance between them greatest and smallest?

## Reasoning

If *v* is an eigenvector of *M*, then the angle between *v* and *w* = *Mv* is of course either 0° (if the eigenvalue is positive) or 180° (if the eigenvalue is negative). But when is the angle at its maximum? What happens when there is only one or no eigenvector?

After numerous unsuccessful attempts to work with general matrices it finally occurred to me to consider special cases, at which point it became obvious to use the Jordan Normal Form over the reals. Let *P*–1*MP* = *J* where *P* is an orthogonal matrix and *J* takes one of the following Jordan Normal forms:

which is a rotation with a uniform scaling.

 which is a shear with a uniform scaling;

 which is a non-uniform scaling in orthogonal directions;

The cosine of the angle between *w* and *v* is given by *vTw = vTMv* = *vTPJP*–1*v*. As *v* moves around the unit circle, so does *u* = *P*–1*v*, and since *P*–1 = *PT*, *vTw* = *uTJu*, so the extreme angles are given by solving the three special cases for extreme cases at vectors *v*, and they occur at the vectors *Pv*.

Let *v* be on the unit circle, with coordinates [cos(*t*), sin(*t*)]*T*.

#### Rotation and Uniform Scaling

The angle between *v* and *w* is constant as is the distance between them. If the cosine of the angle of rotation is θ, where , the scale factor is then . The angle between *v* and *w* is then θ and using the law of cosines, the distance between them is found to be .

#### Shears

The shear is easiest to see geometrically.

The minimum angle will be 0 at the eigenvector *u* = [0, 1]*T*.

Since the effect of a shear is to slide the source vector parallel to the eigenvector by a fixed distance amount (the shear), the maximum angle between source and image occurs when the image is the reflection of the source in the vector perpendicular to the eigenvector.

Algebraically, the cosine of the angle is which has a zero derivative when *t* = 0 and when tan(*t*) = -2μ. The cosine of the angle is 1 when *t* = 0*,*

and  when tan(t) = –2μ. The corresponding source vectors are [1, 0]*T* and .

The square of the distance between a source of unit length and its image is (λ–1)2 + (λ–1)sin(2*t*) + sin2*t* which has its extreme values when tan(2*t*) = 2(1–μ). The extreme length values are  and . The square rooted expressions are positive for all real values of μ.

Note that the extreme distances and the extreme angles occur at different vectors except when μ=1/2.

#### Non-Uniform Scaling

Algebraically, the cosine of the angle is which has a zero derivative when *t* = 0, ± 90° and when tan(*t*) = . The cosine of the angle is then 1 when *t* = 0, 0 when *t* = ±90°, and  when *t* = ±90°. Note that μ + ν = 0 would mean that tan(*t*) was imaginary.

The corresponding source vectors are [1, 0]*T*, [0, 1]*T* and .

#### Note on Jordan Normal Form

The general shear matrix  is conjugate to the matrix  via the matrix  and its inverse, .

The general matrix with (*a* + *d*)2 < 4(*ad* – *bc*) has complex eigenvectors. It is conjugate to the rotation and scaling matrix via the matrix

where R = which is real when the eigenvalues are complex

and .

Note that *d*+*c* = *a*+*b* forces R to be imaginary contradicting the assumption that it is real because the eigenvalues are complex.