Galois invariants of weighted trees

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Alexander Zvonkin
(University of Bordeaux, France)

Joint work with
Fedor Pakovich
Ben-Gurion University of the Negev, Be'er Sheva, Israel
Nikolai Adrianov
Moscow State University, Russia

SIGMAP–14
West Malvern, United Kingdom
10 July, 2014

1965: B. J. Birch, S. Chowla, M. Hall Jr., A. Schinzel

Let A and B be two coprime polynomials, $A, B \in \mathbb{C}[x]$. What is the minimum possible degree of $R = A^3 - B^2$ (if $A^3 \neq B^2$)?

Example (N. Elkies, 2000)

$$\begin{aligned} P &= (x^{10} - 2x^9 + 33x^8 - 12x^7 + 378x^6 + 336x^5 + 2862x^4 \\ &\quad + 2652x^3 + 14\,397x^2 + 9922x + 18\,553)^3, \\ Q &= (x^{15} - 3x^{14} + 51x^{13} - 67x^{12} + 969x^{11} + 33x^{10} + 10\,963x^9 \\ &\quad + 9729x^8 + 96\,507x^7 + 108\,631x^6 + 580\,785x^5 + 700\,503x^4 \\ &\quad + 2\,102\,099x^3 + 1\,877\,667x^2 + 3\,904\,161x + 1\,164\,691)^2, \\ R &= P - Q \\ &= 2^6\, 3^{15}(5x^6 - 6x^5 + 111x^4 + 64x^3 + 795x^2 + 1254x + 5477). \end{aligned}$$

Remark. The fact that in this example the coefficients are rational numbers is a great chance. Usually the coefficients are algebraic.

Two conjectures (1965): Let $\deg A = 2k$, $\deg B = 3k$; then

1. $\deg(A^3 - B^2) \geq k + 1$;
2. this bound is sharp.

In the previous example $k = 5$.

1965: The first conjecture proved by H. Davenport.

1981: The second conjecture proved by W. W. Stothers.

1995: The problem is generalized by U. Zannier:

Let two partitions of an integer n be given:

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p), \qquad \beta = (\beta_1, \beta_2, \ldots, \beta_q),$$

$$\sum_{i=1}^{p} \alpha_i = \sum_{j=1}^{q} \beta_j = n,$$

and let P and Q be two coprime polynomials of degree n with complex coefficients, such that

$$P(x) = \prod_{i=1}^{p} (x - a_i)^{\alpha_i}, \qquad Q(x) = \prod_{j=1}^{q} (x - b_j)^{\beta_j}.$$

Denote $R = P - Q$.

Question: What is the minimum possible degree of R?

Two assumptions:

1. The greatest common divisor of $\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q$ is 1.
2. $p + q \leq n + 1$.

Theorem (U. Zannier, 1995)

1. $\deg R \geq (n + 1) - (p + q)$.
2. This bound is attained for any pair of partitions $\alpha, \beta \vdash n$ satisfying the above assumptions.

2010: F. Beukers, C. Stewart: Search for polynomials A and B such that

1. The degree of the difference $A^k - B^l$ attains its minimum;

2. $\underline{A \text{ and } B \text{ are defined over } \mathbb{Q}}$.

Reminder of the notation: $P - Q = R$.

Consider the rational function

$$f = \frac{P}{R};$$

Note that

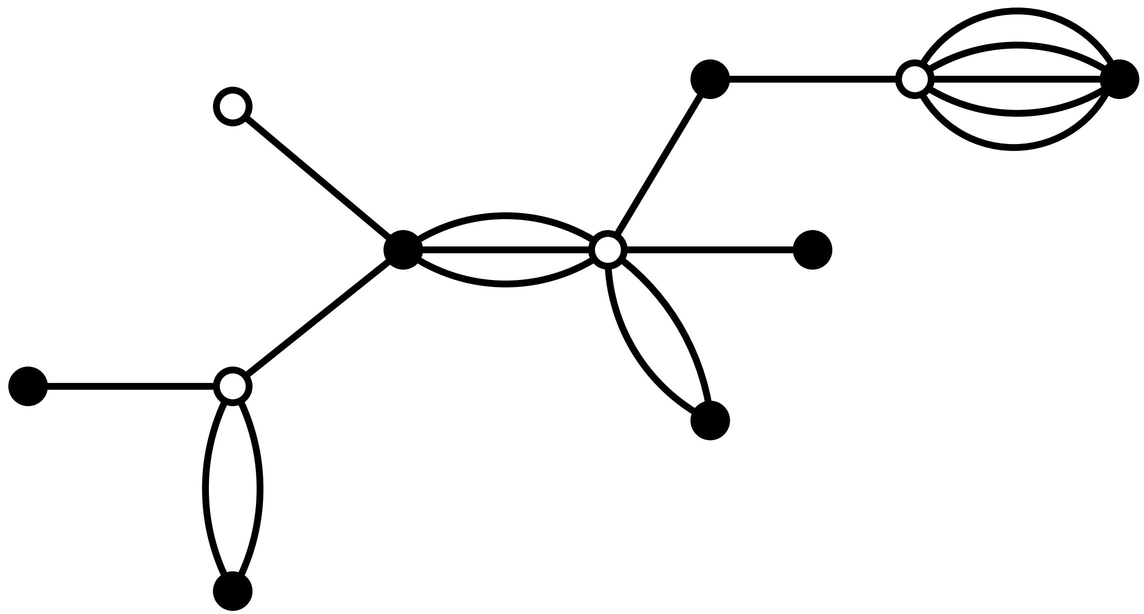
$$f - 1 = \frac{Q}{R}.$$

Theorem: $\deg R = (n+1) - (p+q)$ if and only if f is a Belyi function for a *bicolored* plane map with n edges, such that:

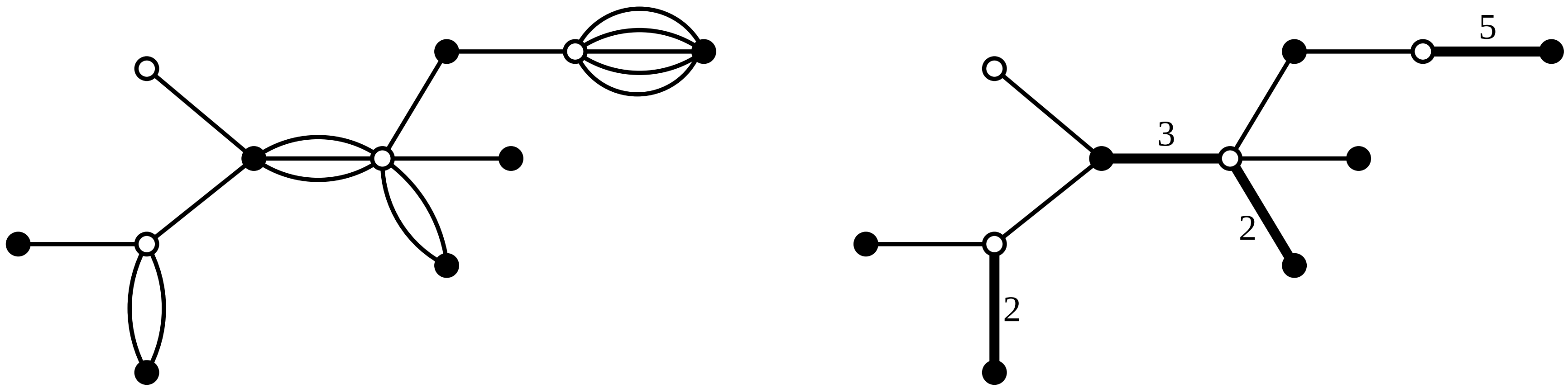
1. The black vertex degrees are $\alpha_1, \ldots, \alpha_p$.
2. The white vertex degrees are $\beta_1, \ldots, \beta_q$.
3. All faces except the outer one are of degree 1.

Face degree is **half the number** of surrounding edges.

Here is how such a map looks like:



It is much easier to handle the corresponding **weighted trees**:



The **degree of a vertex** is the sum of the weights of the edges incident to this vertex.

First result (A. Z.) A great simplification of Zannier's proof.

> For a given (α, β), the existence of a tree implies the attainablity of the lower bound for $\deg R$.

For number theorists it took 30 years: 1965 . . . 1995.

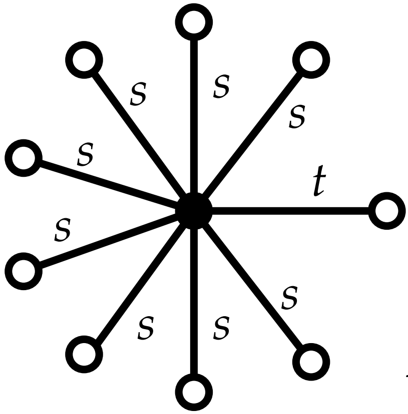
Proposition (obvious): If for a given (α, β) the corresponding tree is unique then the polynomials P, Q, R are defined over $\mathbb{Q}$.

We call such trees **unitrees**.

Second result (F. Pakovich, A. Z.): A complete classification of unitrees. There are:

- 10 infinite series, and
- 10 sporadic trees.

A very long and cumbersome proof. **Pictures follow. . .**



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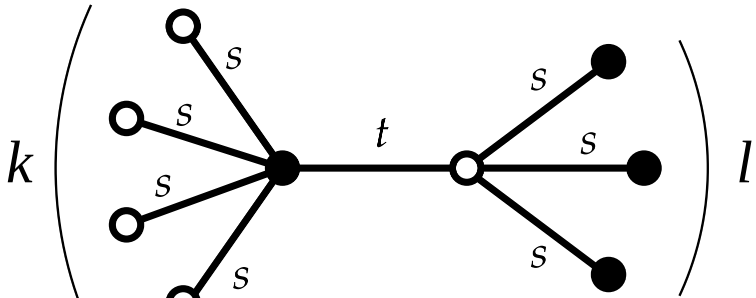

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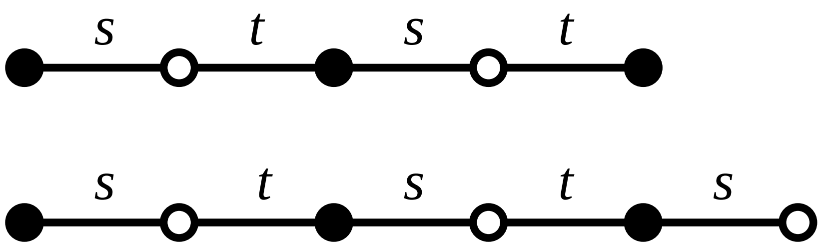

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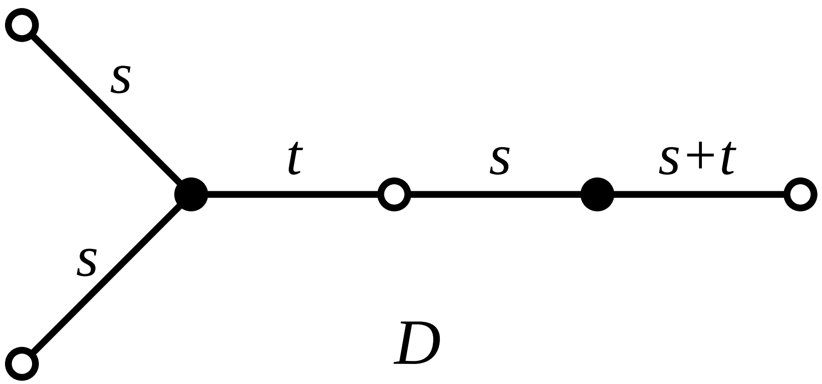

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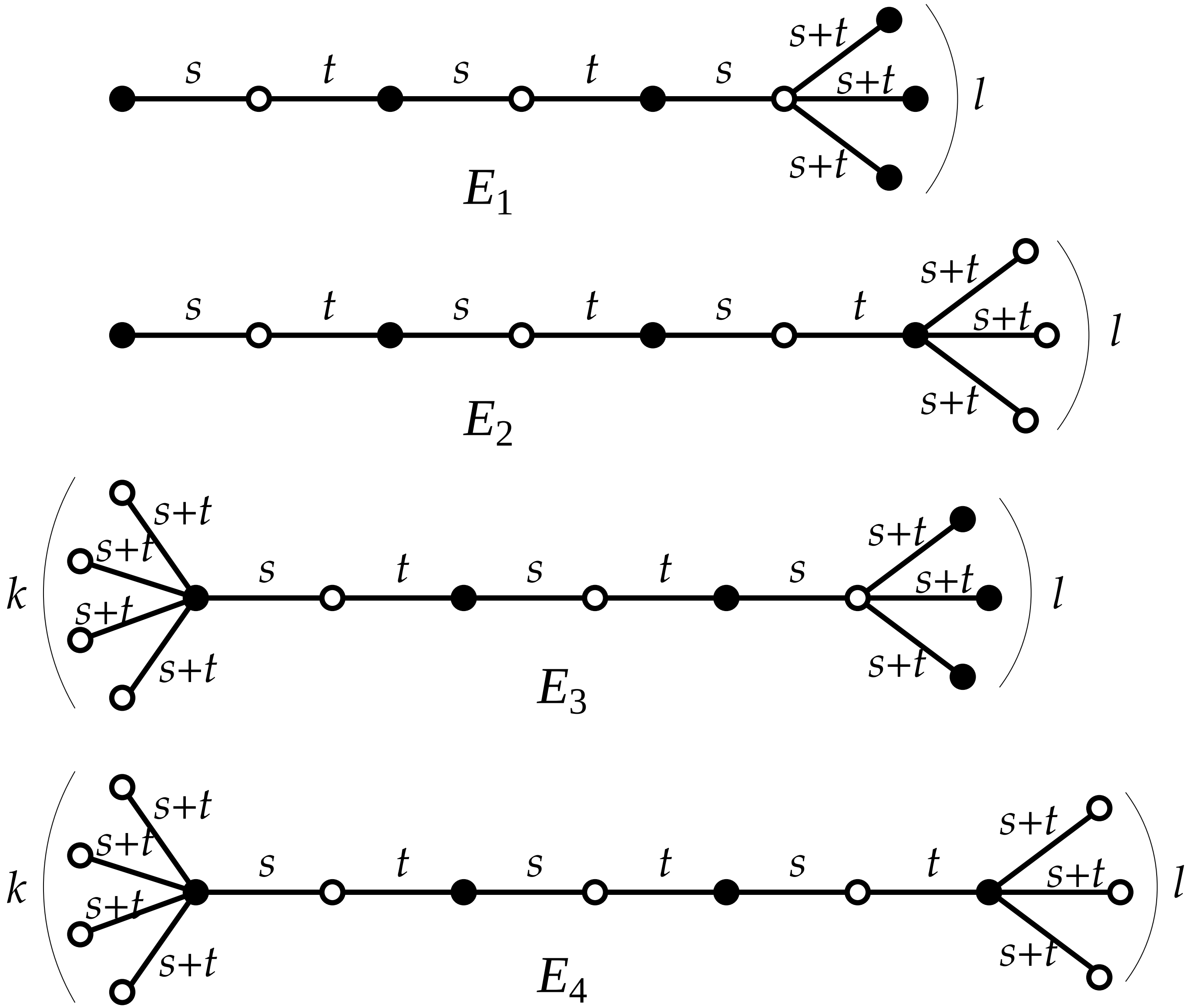
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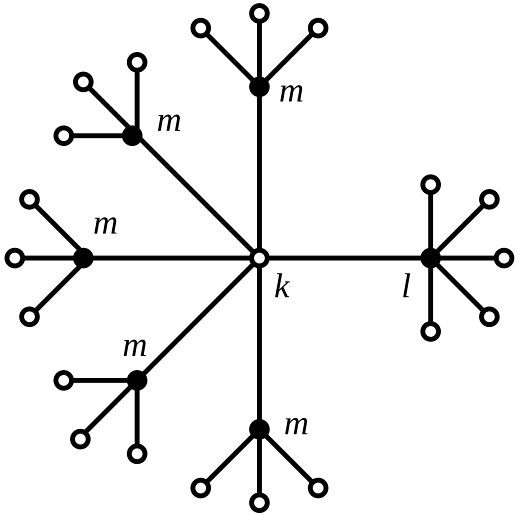

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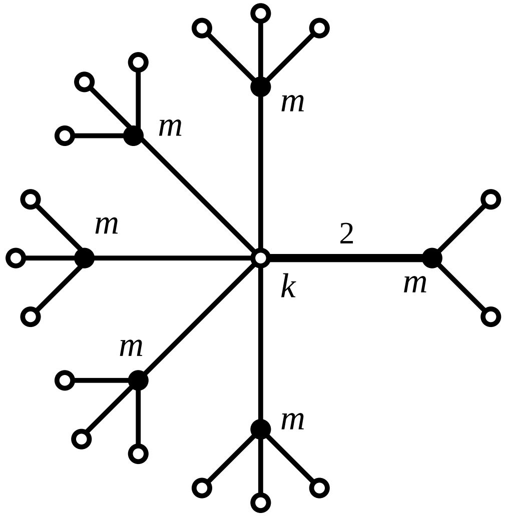

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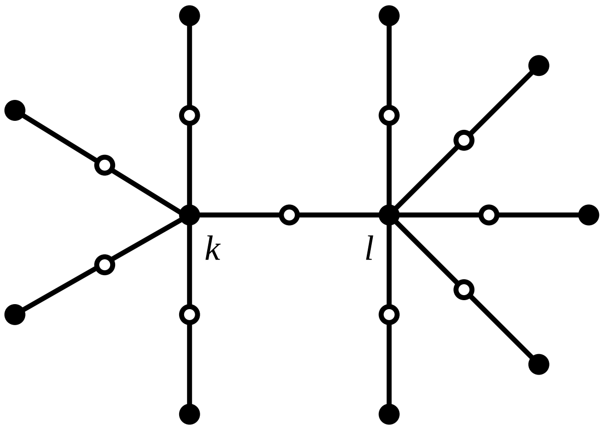

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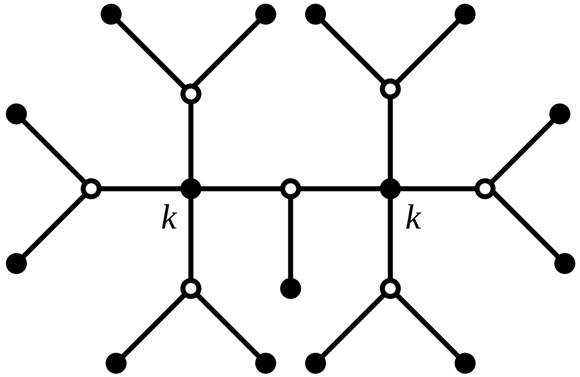

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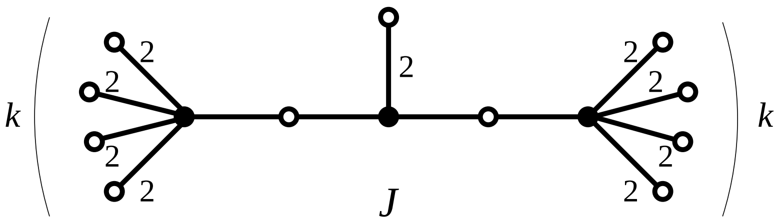

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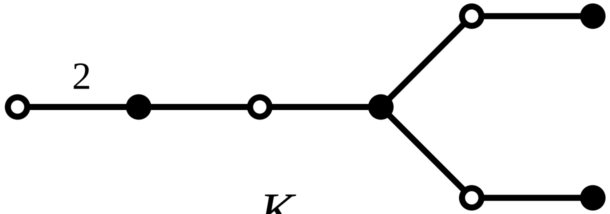

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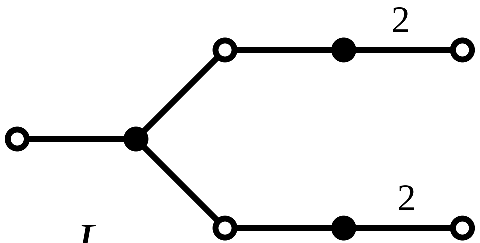

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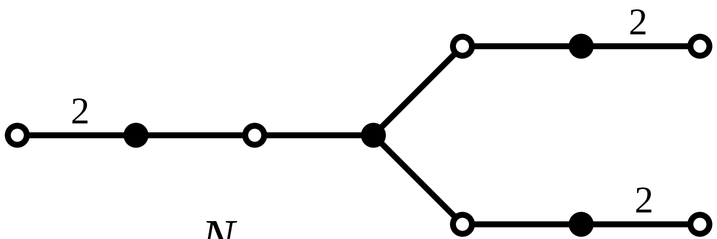

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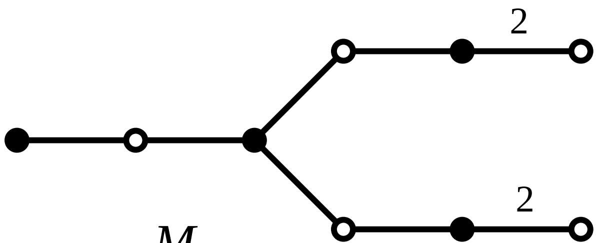

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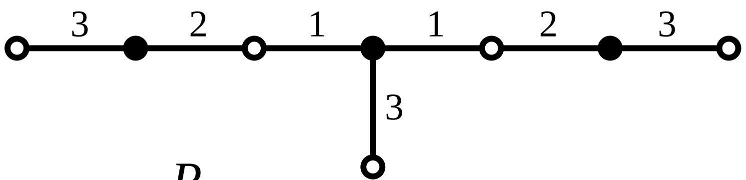

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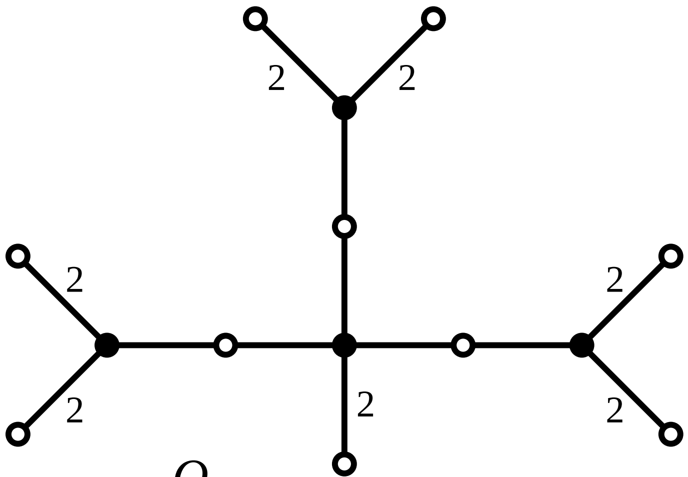

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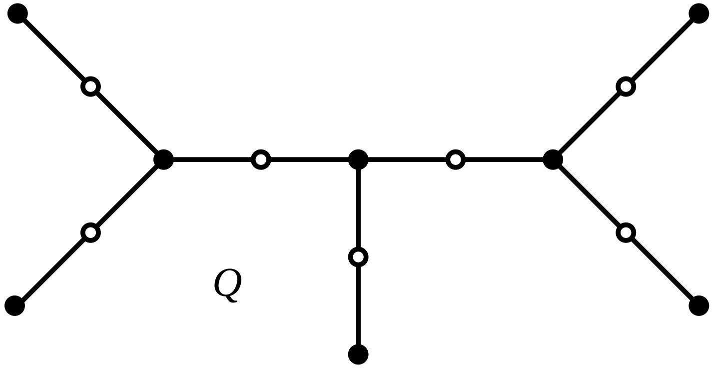
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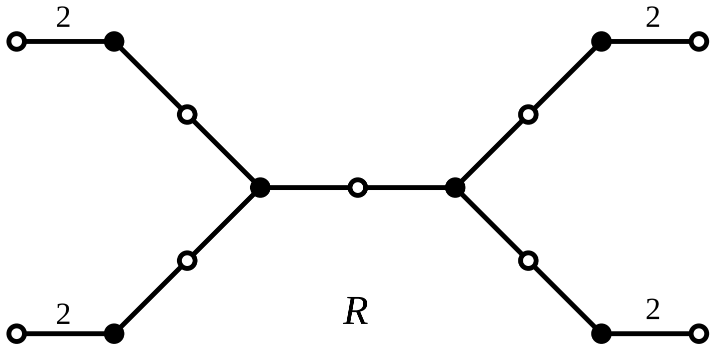
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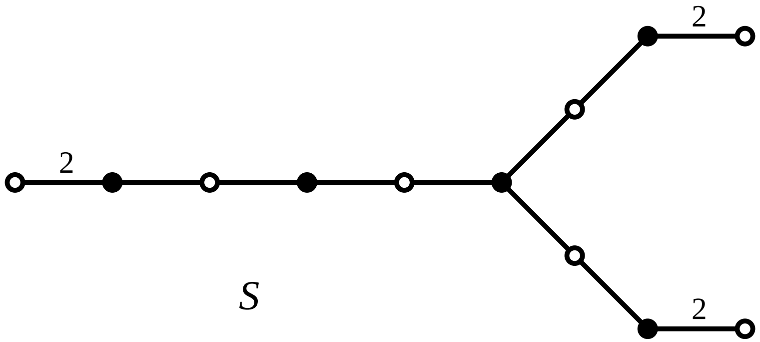
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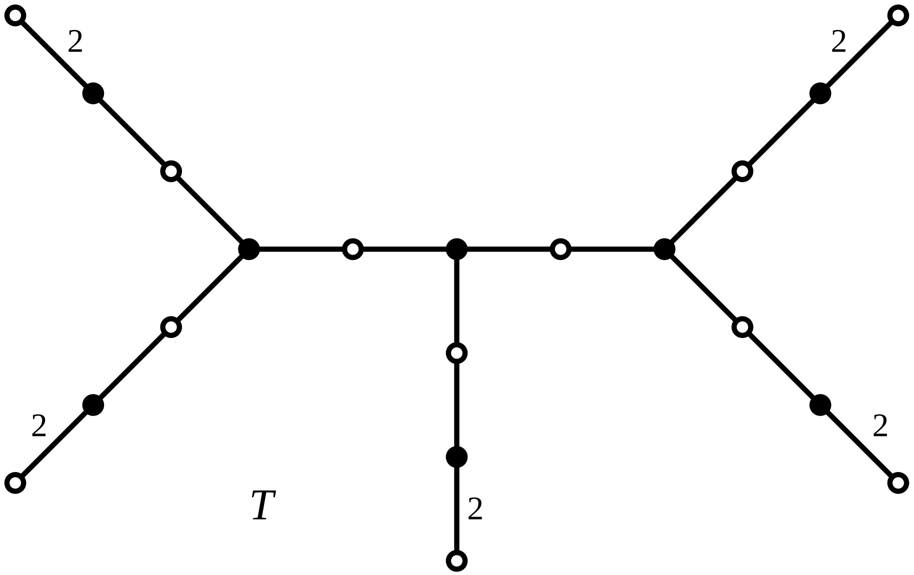
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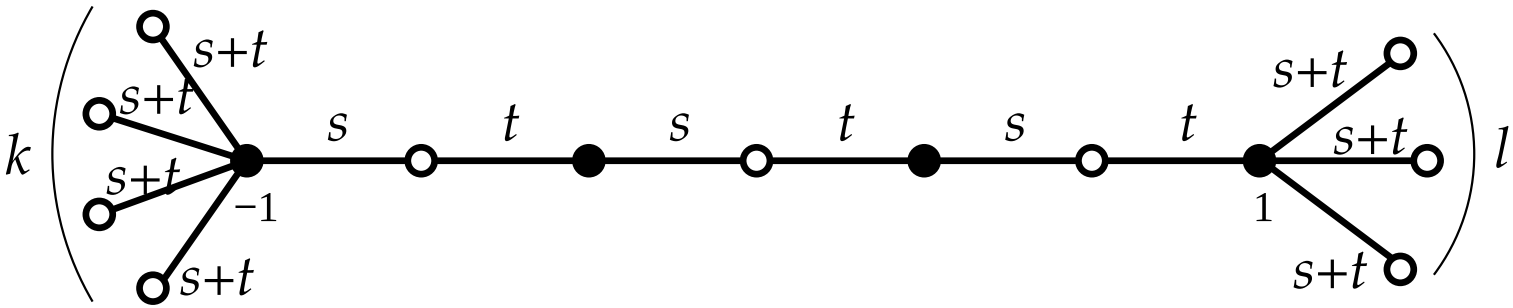
T

Third result (F. Pakovich, A. Z.): Belyi functions for all unitrees are computed.

To give but one example. . .



$$
\begin{array}{lcl}
m_1 & = & l(s+t)+t \\
m_2 & = & k(s+t)+s \\
p & = & \text{number of black vertices of degree } s+t \\
q & = & \text{number of white vertices (all of them are of degree } s+t) \\
a & = & l + t/(s+t) \\
b & = & k + s/(s+t)
\end{array}
$$

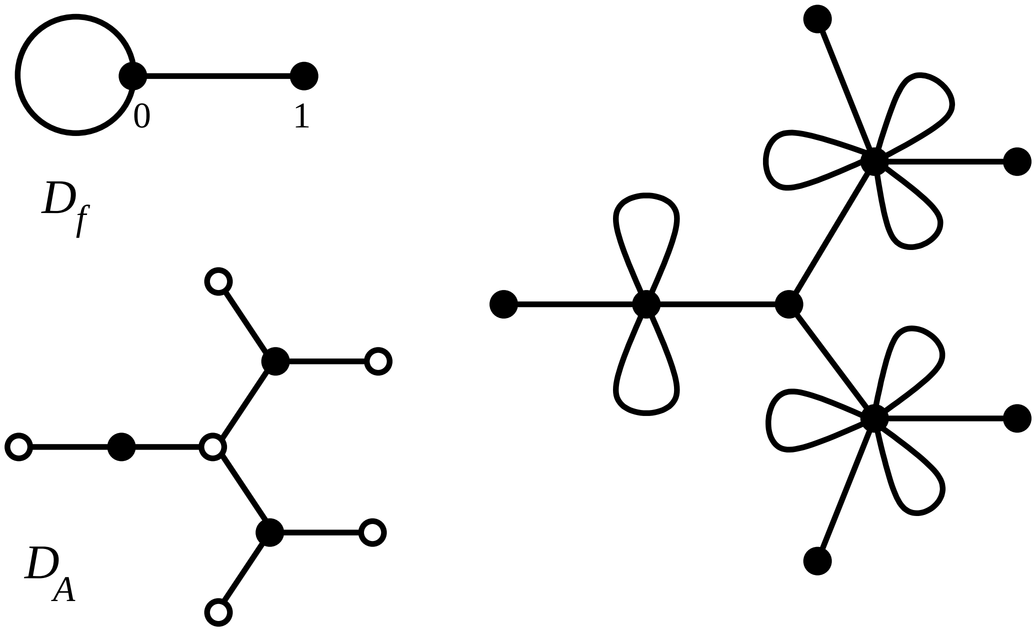
$$
P = \left(\frac{x-1}{2}\right)^{m_1} \cdot \left(\frac{x+1}{2}\right)^{m_2} \cdot J_p(a,b,x)^{s+t}
$$

$$
Q = J_q(-a,-b,x)^{s+t}
$$

Here J_p, J_q are Jacobi polynomials of degree p and q respectively.
Notice the negative parameters $-a$ and $-b$ in J_q.

Remark: The above condition (the uniqueness of the tree) is sufficient but not necessary.

Example: Composition.



D_F where

$F(t) = f(A(t))$

$$f = -\frac{64\,x^3(x-1)}{8\,x+1}, \qquad A = \frac{1}{5^5}\cdot(t^2+4)^3(3\,t+8)^2\,.$$

It is well-known that the monodromy groups of compositions are imprimitive.
What can be said about primitive groups?

Fourth result: (N. Adrianov, A. Z.) Complete classification of primitive monodromy groups of weighted trees:

- 184 trees (up to a color exchange);
- 85 Galois orbits;
- 34 groups;
- the highest degree of a group is 32.

Theorem (Gareth Jones, September 2012) Let G be a primitive permutation group of degree n, not equal to S_n or A_n and containing a permutation with cycle structure $(n-k, 1^k)$. Then one of the following holds:

1. $k = 0$ and
   (a) $\mathrm{C}_p \leq G \leq \mathrm{AGL}_1(p)$ with $n = p$ prime;
   (b) $\mathrm{PGL}_d(q) \leq G \leq \mathrm{P\Gamma L}_d(q)$ with $n = (q^d - 1)/(q-1)$ and $d \geq 2$ for some prime power q;
   (c) $G = \mathrm{L}_2(11)$, M_{11} or M_{23} with $n = 11$, 11 or 23 respectively;

2. $k = 1$ and
   (d) $\mathrm{AGL}_d(q) \leq G \leq \mathrm{A\Gamma L}_d(q)$ with $n = q^d$ and $d \geq 1$ for some prime power q;
   (e) $G = \mathrm{L}_2(p)$ or $\mathrm{PGL}_2(p)$ with $n = p+1$ for some prime $p \geq 5$;
   (f) $G = \mathrm{M}_{11}$, M_{12} or M_{24} with $n = 12$, 12 or 24 respectively;

3. $k = 2$ and
   (g) $\mathrm{PGL}_2(q) \leq G \leq \mathrm{P\Gamma L}_2(q)$ with $n = q + 1$ for some prime power q.

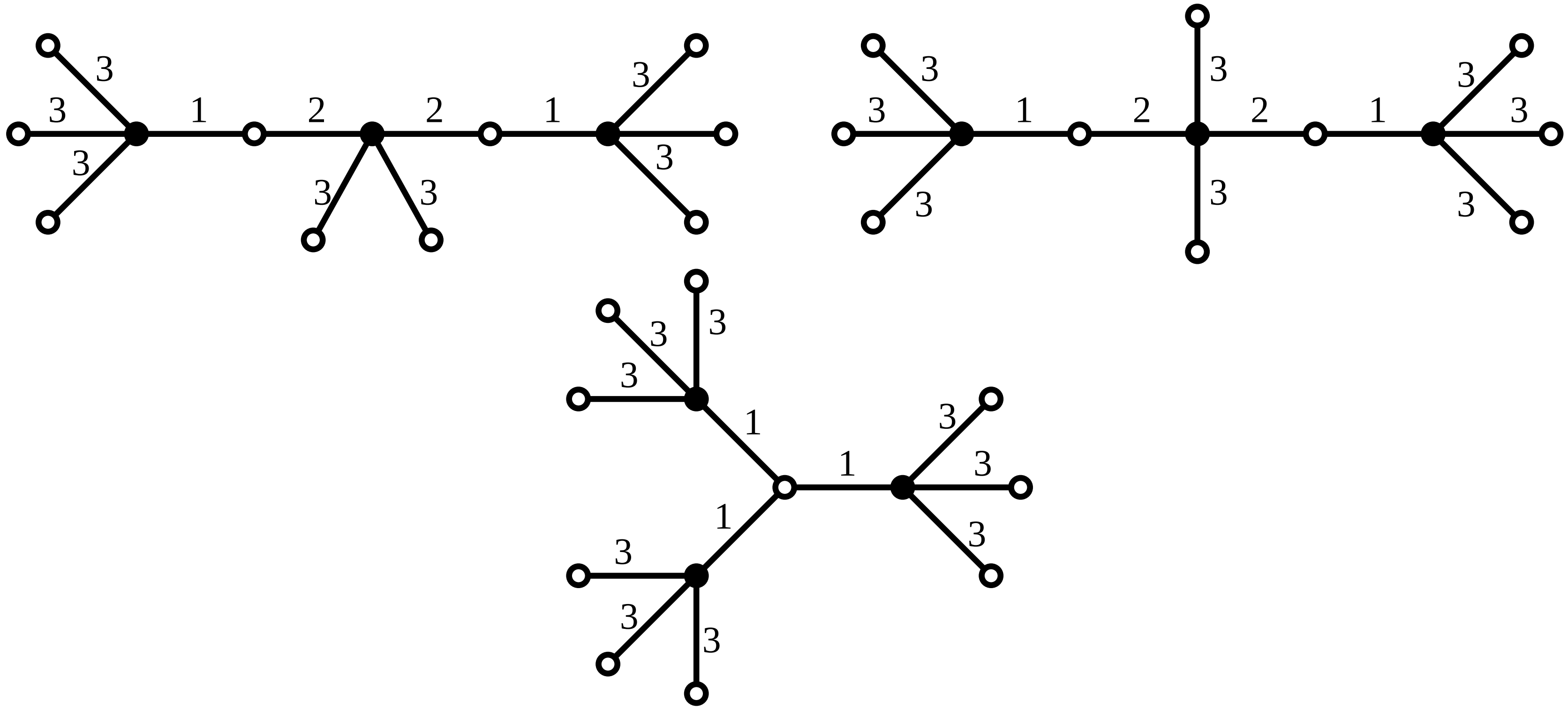
Weight	Group	Order	Orbits	Trees
5	$AGL_1(5)$	20	1	2
6	$PSL_2(5)$	60	2	2
	$PGL_2(5)$	120	7	7
7	$AGL_1(7)$	42	1	2
	$PSL_3(2)$	168	2	4
8	$A\Gamma L_1(8)$	168	1	4
	$PSL_2(7)$	168	2	2
	$PGL_2(7)$	336	6	7
	$ASL_3(2)$	1344	6	14
9	$A\Gamma L_1(9)$	144	1	2
	$AGL_2(3)$	432	2	4
	$PSL_2(8)$	504	3	3
	$P\Gamma L_2(8)$	1512	4	10
10	$PGL_2(9)$	720	3	3
	$P\Gamma L_2(9)$	1440	2	2
11	$PSL_2(11)$	660	1	2
	M_{11}	7920	1	2
12	$PGL_2(11)$	1320	2	4
	M_{11}	7920	3	10
	M_{12}	95040	9	20

Weight	Group	Order	Orbits	Trees
13	$PSL_3(3)$	5616	3	12
14	$PSL_2(13)$	1092	1	1
	$PGL_2(13)$	2184	2	4
15	$PSL_4(2)$	20160	3	6
16	$A\Gamma L_2(4)$	5760	1	2
	$AGL_4(2)$	322560	4	12
17	$PSL_2(16)$	4080	1	1
	$PSL_2(16) \rtimes C_2$	8160	1	1
20	$PGL_2(19)$	6840	1	3
21	$P\Gamma L_3(4)$	120960	1	2
23	M_{23}	10200960	1	4
24	M_{24}	244823040	5	18
31	$PSL_5(2)$	9999360	1	6
32	$ASL_5(2)$	319979520	1	6
Total	**34**	—	**85***	**184**

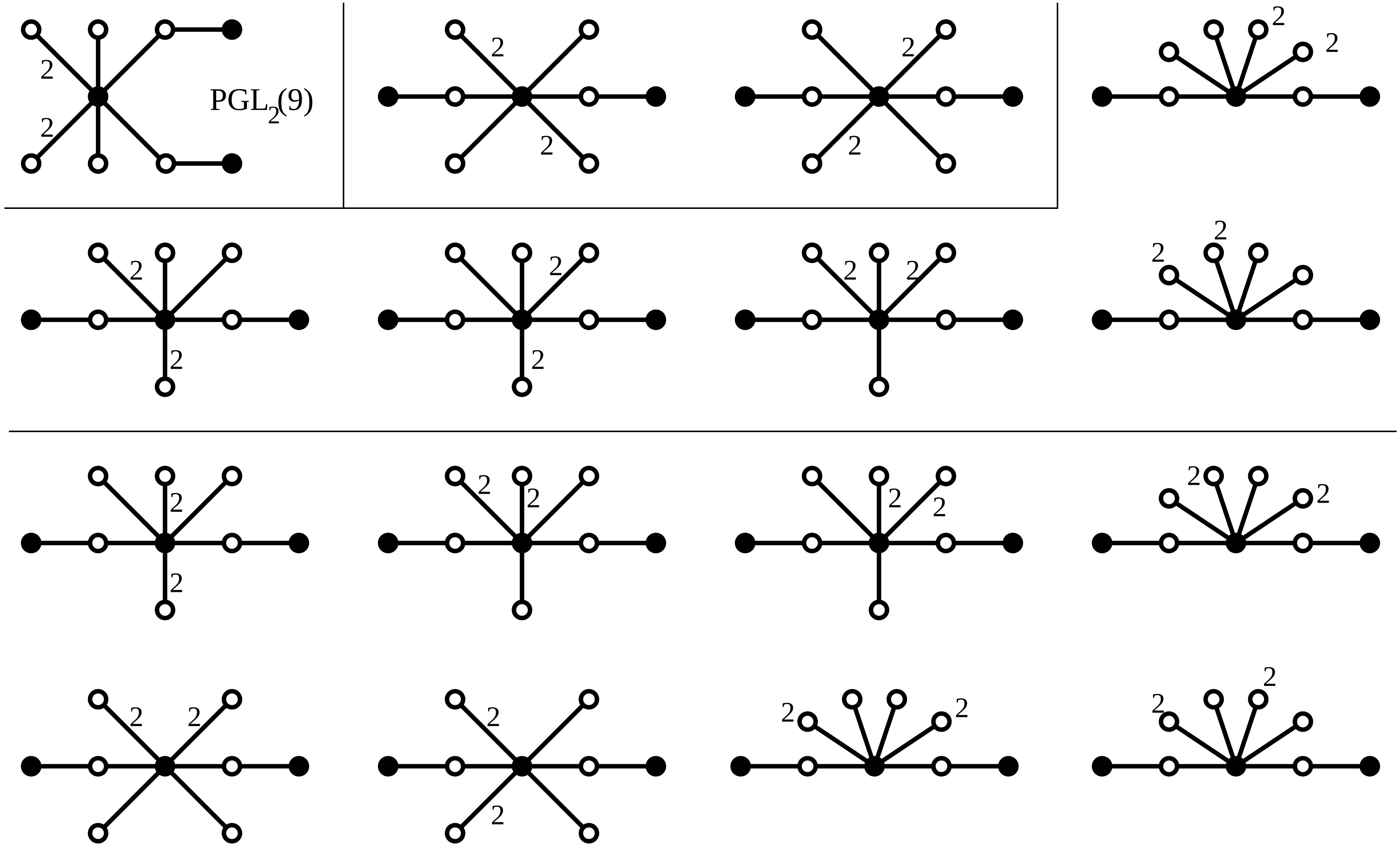
*For certain orbits we are not entirely sure that the "orbit" in question is indeed a single orbit and not a union of several orbits.

Fifth result: A number of funny examples. A small sample follows.

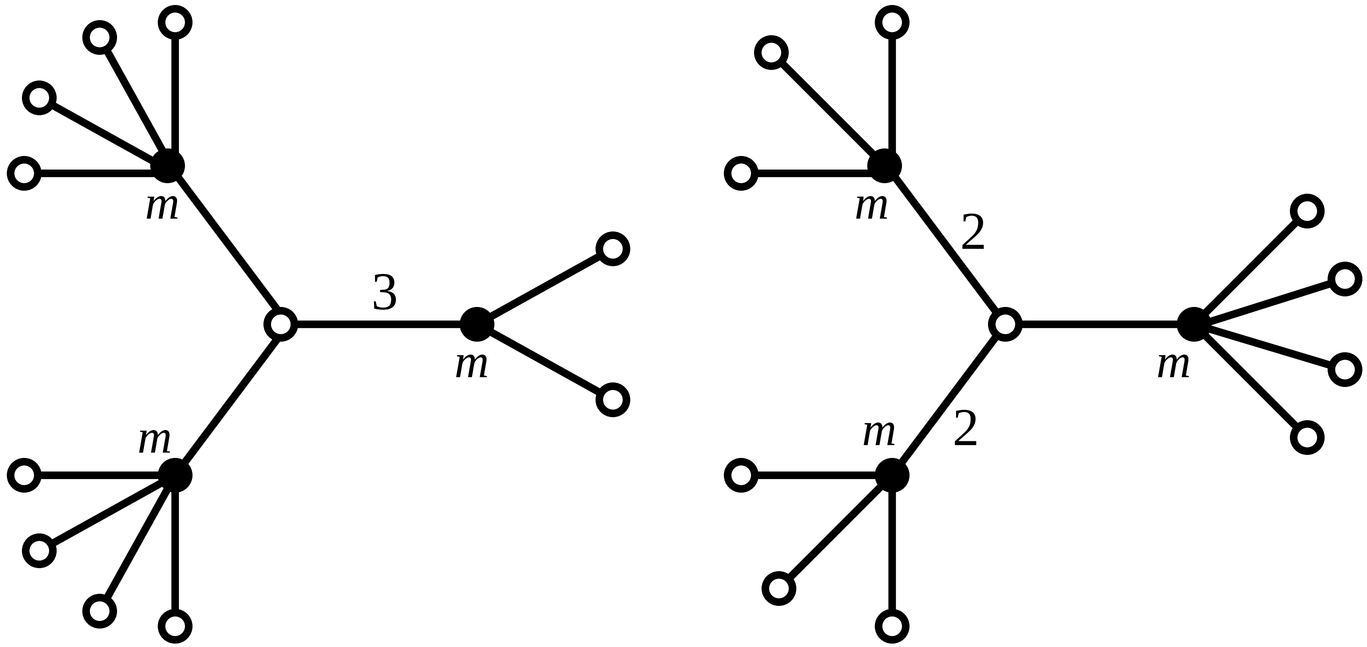
<u>Here all three dessins are defined over $\mathbb{Q}$:</u>



Note that all black degrees are equal to 10 and all white degrees are equal to 3. Therefore, this example corresponds to the minimum degree problem for $A^{10} - B^3$.



Weight $n = 10$, passport $(8^1 1^2, 2^4 1^2, 8^1 1^2)$: 16 trees, 4 orbits.
The sizes of orbits: 1, 2, 5, 8. **Why 13 splits into $5 + 8$?**
Five are self-dual, eight are not.



Passport: $(m^3, 5^1 1^{3m-5})$

- either one orbit over a real quadratic field;
- or two orbits over $\mathbb{Q}$.

Computation gives the field $\mathbb{Q}(\sqrt{\Delta})$ where

$$\Delta = 3\,(2m-1)\,(3m-2).$$

Question: can $\Delta = 3\,(2m-1)\,(3m-2)$ **be a perfect square?**

1. $2m-1$ and $3m-2$ are coprime:

$$\begin{aligned} 3m-2 &= 1\cdot(2m-1)+(m-1), \\ 2m-1 &= 2\cdot(m-1)+1. \end{aligned}$$

2. Only $2m-1$ can be divisible by 3.

3. Hence, $3\,(2m-1)$ and $3m-2$ must both be squares.

4. Denoting

$$6m-3=a^2, \qquad 3m-2=b^2$$

we get

$$a^2-2b^2=1.$$

Pell equation ! (Plus the condition of a being a multiple of 3.)

Pell's name was attributed to this equation by error. . .

- Pythagoras (VI before J. C.): $a^2 - 2b^2 = 0$
- Brahmagupta (VII)
- Bhaskara II (XII)
- Narayana Pandit (XIV)
- Brouncker (XVII)
- Fermat, Euler, Lagrange, Abel, . . . (XVII–XIX)
- Dirichlet (XIX)

Infinitely many solutions

First values of the parameter m (vertex degree):

$1\,634,\ 1\,884\,962,\ 2\,175\,243\,842,\ \ldots$

Growth exponent: $(17 + 12\sqrt{2})^2 \approx 1154$.

Sixth result: Enumeration (A. Z.)

Let a_n be the number of rooted trees of weight n, and let $f(t) = \sum_{n \geq 0} a_n t^n$. Then

$$f(t) = \frac{1 - t - \sqrt{1 - 6t + 5t^2}}{2t}$$
$$= 1 + t + 3t^2 + 10t^3 + 36t^4 + 137t^5 + 543t^6 + 2219t^7 + \ldots$$

Recurrence:

$$a_0 = 1, \quad a_1 = 1, \quad a_{n+1} = a_n + \sum_{k=0}^{n} a_k a_{n-k} \quad \text{pour} \quad n \geq 1.$$

Asymptotic: $a_n \sim \frac{1}{2}\sqrt{\frac{5}{\pi}} \cdot 5^n n^{-3/2}$.

Sequence A002212 of the "On-Line Encyclopedia of Integer Sequences".

Let $b_{m,n}$ be the number of rooted trees of weight n with m edges, and let $h(s,t) = \sum_{m,n\geq 0} b_{m,n} s^m t^n$. Then

$$\begin{aligned} h(s,t) &= \frac{1 - t - \sqrt{1-(2+4s)\,t+(1+4s)\,t^2}}{2st} \\ &= 1 + st + (s+2s^2)\,t^2 + (s+4s^2+5s^3)\,t^3 \\ &\quad + (s+6s^2+15s^3+14s^4)\,t^4 + \dots \end{aligned}$$

Explicit formula for $b_{m,n}$:

$$b_{m,n} = \binom{n-1}{m-1} \cdot \frac{1}{m+1}\binom{2m}{m}.$$

An enumeration problem you are allowed to work on:

Count the number of weighted trees corresponding to a given pair of partitions (α, β)

and

to do that without inclusion-exclusion.

An enumeration problem you are allowed to work on:

Count the number of weighted trees corresponding to a given pair of partitions (α, β)

and

to do that without inclusion-exclusion.

$$* \quad * \quad *$$

Oh when dessins go marching in
Oh when dessins go marching in
Oh how I'd like to learn their number
When all dessins go marching in!

Thank you !

Conference "Embedded Graphs"

Saint-Petersburg, Russia

Last week of October (27–31 October)

Peter the Great

Leonhard Euler

The conference will be held at the

Euler International Mathematical Institute