Nowhere-zero 3-flows in arc-transitive graphs on solvable groups

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Joint work with Xiangwen Li, Central China Normal University (Wuhan)

circulations

Definition

Let $D = (V(D), A(D))$ be a digraph and A an abelian group. A **circulation** in D over A is a function

$$f : A(D) \to A$$

such that

$$\sum_{a \in A^+(v)} f(a) = \sum_{a \in A^-(v)} f(a), \quad \text{for all } v \in V(D),$$

where $A^+(v)$ ($A^-(v)$, respectively) is the set of arcs of D leaving from v (entering into v, respectively).

We say that f is **nowhere-zero** if $f(a) \neq 0$ for every $a \in A(D)$, where 0 is the identity element of A.

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Theorem

(W. Tutte 1954)
A plane digraph is k-face-colorable if and only if it admits a nowhere-zero circulation over $\mathbb{Z}_k$.

Whether a digraph admits a nowhere-zero circulation over a given abelian group depends only on its underlying undirected graph.

So we can speak of nowhere-zero circulations in undirected graphs.

Four-Color-Theorem Restated:
Every planar graph admits a nowhere-zero circulation over $\mathbb{Z}_4$.

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A nowhere-zero circulation f over $\mathbb{Z}$ in a digraph D is called a (nowhere-zero) k-flow if

$$-(k-1) \leq f(a) \leq k-1, \quad \text{for all } a \in A(D)$$

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A graph admits a k-flow if and only if it admits a nowhere-zero circulation over $\mathbb{Z}_k$.

Four-Color-Theorem Again:
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A graph admits a 2-flow if and only if its vertices all have even degrees.

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A 2-edge-connected cubic graph admits a 3-flow if and only if it is bipartite.

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Tutte's 5-flow conjecture

Tutte proposed three conjectures on integer flows (1954, 1968, 1972).

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(The 5-flow conjecture)
Every 2-edge-connected graph admits a 5-flow.

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Theorem

(M. E. Watkins 1969; W. Mader 1970)
Every vertex-transitive graph of valency d is d-edge-connected.

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Every vertex-transitive graph of valency at least 4 admits a 3-flow.

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3-flows in Cayley graphs on nilpotent groups

Theorem
(P. Potačnik 2005)
Every Cayley graph of valency at least 4 on a finite abelian group admits a 3-flow.

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Every Cayley graph of valency at least 4 on a finite nilpotent group admits a 3-flow.

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Theorem
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Every Cayley graph of valency at least 4 *on a finite abelian group admits a* 3*-flow.*

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an intermediate goal

Prove that every graph of valency at least 4 admitting a solvable vertex-transitive group of automorphisms admits a 3-flow.

As before it suffices to prove this for the case of valency 5.

result so far

Theorem

(X. Li and S. Zhou 2014, Ars Math. Contemp.)
Let G be a finite solvable group. Then every G-arc-transitive graph with valency at least 4 admits a 3-flow.

- Any G-arc-transitive graph is G-vertex-transitive and G-edge-transitive
- Any G-vertex-transitive and G-edge-transitive graph with odd valency is G-arc-transitive

Therefore, our result is equivalent to:

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solvable groups

Definition

$G' := [G, G]$: derived subgroup of G, the subgroup of G generated by all commutators $x^{-1}y^{-1}xy$, $x, y \in G$

$G^{(0)} := G$, $G^{(1)} := G'$, $G^{(i)} := (G^{(i-1)})'$, $i \geq 1$

G is solvable if $G^{(n)} = 1$ for some $n \geq 0$

The least integer n with $G^{(n)} = 1$ is the **derived length** of G.

- Solvable groups with derived length 1 are precisely nontrivial abelian groups.
- Subgroups and quotient groups of a solvable group are solvable.
- Any solvable group G contains a normal abelian subgroup N such that G/N has a smaller derived length.

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multicovers

Definition

Let Γ be a graph and $\mathcal{P}$ a partition of $V(\Gamma)$.

Γ is a **multicover** of the quotient $\Gamma_{\mathcal{P}}$ if for each pair of adjacent $P, Q \in \mathcal{P}$, the subgraph $\Gamma[P, Q]$ of Γ induced by $P \cup Q$ is a t-regular bipartite graph with bipartition $\{P, Q\}$ for some integer $t \geq 1$ independent of P, Q.

Lemma

Let $k \geq 2$ be an integer. If a graph admits a k-flow, then its multicovers all admit a k-flow.

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normal quotients

Definition

Let Γ be a G-vertex-transitive graph, and let $N \trianglelefteq G$.

The set $\mathcal{P}_N$ of N-orbits on $V(\Gamma)$ is a G-invariant partition of $V(\Gamma)$, called a G-normal partition of $V(\Gamma)$.

Denote $\Gamma_N := \Gamma_{\mathcal{P}_N}$.

Lemma

Let Γ be a connected G-vertex-transitive graph. Let $N \trianglelefteq G$ be intransitive on $V(\Gamma)$. Then

(a) Γ_N *is G/N-vertex-transitive under the induced action of G/N on $\mathcal{P}_N$;*

(b) *for $P, Q \in \mathcal{P}_N$ adjacent in Γ_N, $\Gamma[P, Q]$ is a regular subgraph of Γ;*

(c) *if in addition Γ is G-arc-transitive, then Γ_N is G/N-arc-transitive and Γ is a multicover of Γ_N.*

result so far

Theorem

(X. Li and S. Zhou 2014, Ars Math. Contemp.)
Let G be a finite solvable group. Then every G-arc-transitive graph with valency at least 4 admits a 3-flow.

- If $\mathtt{val} = 4$, then the graph has a 2-flow and hence a 3-flow.
- If $\mathtt{val} \geq 6$, then the graph is 6-edge-connected and so admits a 3-flow by LTWZ (2013).
- It is boiled down to the case $\mathtt{val} = 5$.

We prove:

Claim

If G is solvable, then every G-arc-transitive graph with valency at least 4 and not divisible by 3 admits a 3-flow.

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outline of proof

- We may assume G is faithful on $V(\Gamma)$. We may also assume the graphs under consideration are connected.
- Make induction on the derived length $n(G)$.
- If $n(G) = 1$, then G is abelian and so is regular on $V(\Gamma)$. Hence Γ is a Cayley graph on G and the result is true by Potačnik's result.
- Assume for some $n \geq 1$ the result holds for any finite solvable group of derived length n.
- Let G be a finite solvable group with derived length $n(G) = n + 1$.
- Let Γ be a connected G-arc-transitive graph such that $\mathtt{val}(\Gamma) \geq 4$ and $\mathtt{val}(\Gamma)$ is not divisible by 3.
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- Since G is solvable, there exists an abelian $N \trianglelefteq G$ such that G/N has derived length $n(G) - 1 = n$.
- If N is transitive on $V(\Gamma)$, then it is regular on $V(\Gamma)$. So Γ is a Cayley graph on N and admits a 3-flow by Potačnik's result.
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difficulty for vertex- but not arc-transitive graphs

A G-vertex- but not G-arc-transitive graph Γ may not be a multicover of its normal quotients Γ_N.

In fact, in this case blocks of a normal partition are not necessarily independent sets.

This makes a similar induction difficult.

a conjecture on Cayley graphs

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(Alspach and Zhang 1992)
Every Cayley graph with valency at least two admits a 4-flow.

We only need to consider the cubic case due to Jaeger's 4-flow theorem.

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- Every Cayley graph of valency at least 4 on a finite solvable group admits a 3-flow?
- This will generalize both [Alspach, Liu and Zhang 1996] and [Nánásiová and Škoviera 2009].
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thank you for your attention