Belyĭ functions, dessins, Galois actions, and regularity

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Outline

Riemann surfaces and algebraic curves

Bernhard Riemann already knew the following

Theorem

The categories "compact Riemann surfaces" and "smooth complex projective algebraic curves" are equivalent.

He did not say that in this manner, of course, and he had no proof we would completely accept. Moreover, even nowadays it is difficult to make this equivalence explicit.

For example: can we give function theoretic conditions under which a compact Riemann surface X can be defined — as an algebraic curve — by polynomial equations with coefficients in $\overline{\mathbb{Q}}$? Or in short, *can we see somehow that* X *is defined over a number field ?*

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Belyĭ's contribution

Yes, we can! In 1979, Belyĭ proved the

Theorem

The smooth algebraic curve $\mathbf{X}$ *can be defined over a number field* $\iff$ *There is a nonconstant meromorphic function* $\beta : \mathbf{X} \to \hat{\mathbb{C}} = \mathbb{P}^1(\mathbb{C})$ *ramified above at most three points.*

We call functions of this kind Belyĭ functions, and by a Möbius transformation we can always assume the critical values to be $0, 1, \infty$.

Easy examples of Belyĭ functions are

$$\beta(z) := z^m \quad \text{on the Riemann sphere ,}$$

$$\beta(x : y : z) := x^n/z^n \quad \text{on the Fermat curve} \quad x^n + y^n = z^n \text{ .}$$

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The proof for " $\Longrightarrow$ "

consists of a tricky algorithm. If $\mathbf{X}$ is given by the zeros of polynomials with coefficients in $\overline{\mathbb{Q}}$ as e.g. $y^2 = x(x-1)(x - \frac{1}{\sqrt[3]{2}})$ (real 3rd root)

- start with a nontrivial projection $\mathbf{X} \to \hat{\mathbb{C}}$, here e.g. $f_0 : (x,y) \mapsto x$, obviously ramified above algebraic points, here $x = 0, 1, \infty, \frac{1}{\sqrt[3]{2}}$,
- then apply polynomials $\in \mathbb{Q}[x]$ sending these critical points to $\mathbb{Q}$ and having themselves critical values of smaller degree, here $f_1 : x \mapsto x^3$,
- repeat this procedure until all critical values of $f_n \circ \ldots \circ f_1 \circ f_0$ are in $\mathbb{Q}$. Here, $f_1 \circ f_0$ is already ramified over only $0, 1, \infty, \frac{1}{2}$.
- Invent some miracle polynomials decreasing the number of critical (now rational) values step by step until you have only three of them. In our case, already $f_2 : u \mapsto 4u(1-u)$ will do the job!

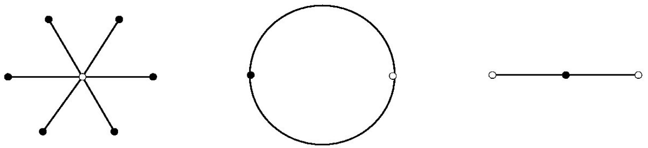
In our case, the resulting Belyĭ function is therefore

$$\beta \;=\; f_2 \circ f_1 \circ f_0 \;:\; (x,y) \;\mapsto\; 4x^3(1-x^3)\,.$$

Visualization via dessins

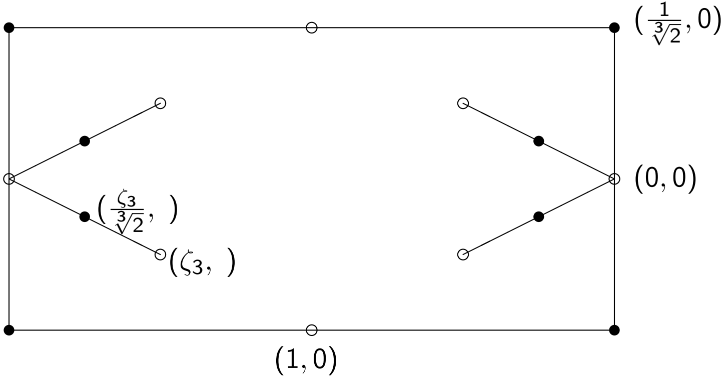
Grothendieck observed in his *Esquisse d'un programme* (1984, not published until 1997 !) that the topological behaviour of the Belyi functions can be visualized using (hyper)maps in *Walsh representation*: the inverse images $\beta^{-1}[0,1]$ of the real $0\,1$–interval ○——● is a bipartite graph, embedded in the Riemann surface X, i.e. cutting the surface into simply connected cells.

For $\beta : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, $\beta(z) := z^6$ or $\frac{(z-1)^2}{-4z}$ or $4z(1-z)$, $\beta^{-1}[0,1]$ looks as follows.



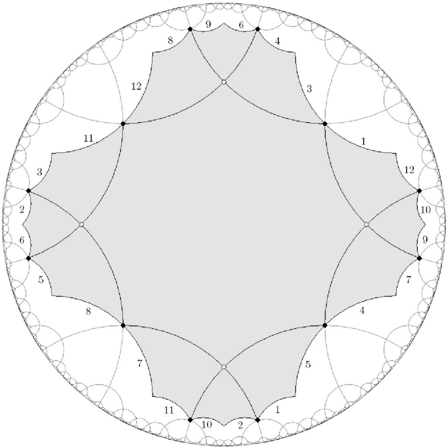
A dessin on a genus 1 curve

for our example $y^2 = x(x-1)(x - \frac{1}{\sqrt[3]{2}})$ with Belyĭ function $\beta(x,y) = 4x^3(1-x^3)$, drawn on the fundamental parallelogram of its covering group, looks like



A Fermat curve

For the genus 3 Fermat curve of exponent 4 , the dessin for $\beta(x:y:z) = x^4/z^4$ can be drawn on the fundamental domain of its universal covering group as



Why so beautiful?

The drawing shows the fundamental domain of its universal covering group, acting on the hyperbolic plane. The numbers on the border indicate the necessary identifications to get the surface. Observe that

- we have 4 cells,
- the cell midpoints are the poles of β,
- the underlying graph is $K_{4,4}$,
- the valencies of the white vertices are the zero orders of β,
- the edges are parts of geodesics.

Another very beautiful dessin in genus 3 lives on

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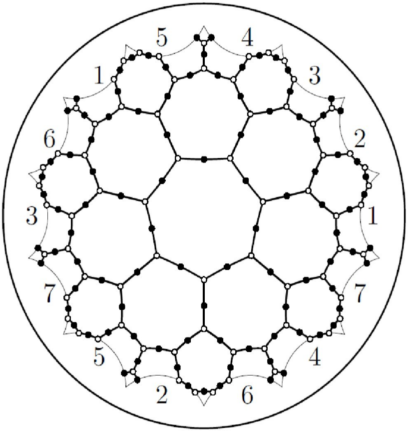
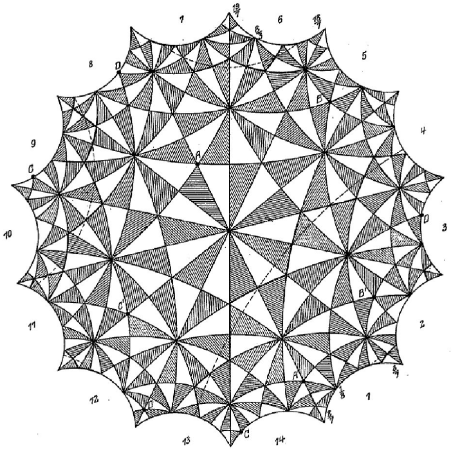
Klein's quartic

whose equation can be written as $x^3y + y^3z + z^3x = 0$



A tesselation

of the fundamental domain of Klein's quartic by fundamental domains of the triangle group $\langle 2, 3, 7 \rangle$ explains why the dessin looks so nice.



The role of the j–function

Suppose the compact Riemann surface $\mathbf{X}$ can be written as the quotient $\Gamma\backslash\mathbb{H}$ of the upper half plane $\mathbb{H}$ by some subgroup Γ of a triangle group $\Delta = \langle p, q, r \rangle$. There is a Δ–automorphic function $j : \mathbb{H} \to \hat{\mathbb{C}}$ (which can be explicitly constructed by means of hypergeometric functions)

- sending each white (open) triangle onto $\mathbb{H}$,
- each black triangle onto $-\mathbb{H}$,
- the skeleton of the tesselation onto $\mathbb{R} \cup \{\infty\}$,
- the fixed points of Δ (vertices of the tesselation) onto $0, 1, \infty$ with orders p, q, r,
- and locally biholomorphic outside these fixed points,
- providing an identification $\Delta\backslash\mathbb{H} \cong \hat{\mathbb{C}}$ by $\Delta z \mapsto j(z)$
- and a canonical Belyĭ function $\beta : \mathbf{X} \to \hat{\mathbb{C}} \quad : \quad \Gamma\backslash\mathbb{H} \to \Delta\backslash\mathbb{H} \quad : \quad \Gamma z \mapsto j(z) \leftrightarrow \Delta z$.

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Belyĭ functions and uniformization

This is already one direction of

Theorem

There is a Belyĭ function on the compact Riemann surface $\mathbf{X}$ *if and only if* $\mathbf{X}$ *can be written as* $\Gamma\backslash\mathbb{H}$ *for a finite index subgroup* $\Gamma < \Delta$ *of a Fuchsian triangle group* Δ *.*

For the other direction, let β be a Belyĭ function on $\mathbf{X}$ and choose $\Delta = \langle p, q, r\rangle$ such that p is a common multiple of all zero orders of β , r is a common multiple of all pole orders, and q is a common multiple of all zero orders of $\beta - 1$. If j is the Δ–automorphic function introduced above,

$$\beta^{-1} \circ j : \mathbb{H} \to \mathbf{X}$$

can be well defined and gives a covering map (ramified in general) with covering group $\Gamma < \Delta$.

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The benefits of regularity

Most examples presented here are (orientably) regular dessins: there is an (orientation preserving) automorphism group of the hypermap (and of the Riemann surface!) acting transitively on the set of edges.
Important because

- every dessin is a quotient of a regular one,
- their Belyĭ functions define normal coverings $X \to \mathbb{P}^1(\mathbb{C})$,
- their Riemann surfaces X are quasiplatonic
- corresponding to very special points in their moduli spaces,
- whose universal covering groups Γ are *normal* subgroups of a triangle group Δ,
- whose quotient mappings $X \to (\mathrm{Aut}\, X)\backslash X$ are Belyĭ functions (if $g > 1$; for $g = 1$, $\mathrm{Aut}\, X$ is infinite!).

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Break

Coffee now!

Coffee over, now the " $\Longleftarrow$ " part of Belyĭ's theorem

Theorem

Let **X** *be a Belyĭ surface, i.e. a compact Riemann surface with a Belyĭ function* β *. Then, as an algebraic curve,* **X** *can be defined over a number field, and so can* β *.*

Belyĭ attributes this part to A. Weil. (??)

Let $\mathrm{Gal}\,\mathbb{C}$ be the group of field automorphisms of $\mathbb{C}$. Caution: $\mathbb{C}/\mathbb{Q}$ is not an algebraic extension, so there is no Galois correspondence available.
Let $\sigma \in \mathrm{Gal}\,\mathbb{C}$ act on the points of the smooth algebraic curve **X** (acting on their coordinates), on its defining polynomials (acting on their coefficients) and on β (acting on its coefficients).

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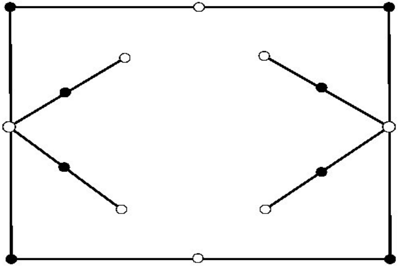
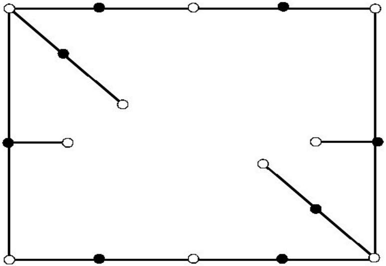
The effect of Galois conjugation, an example:

Take the curve $y^2 = x(x-1)(x-\frac{1}{\sqrt[3]{2}})$ (real 3rd root) with Belyĭ function

$$\beta(x,y) := 4x^3(1-x^3)$$

and its Galois conjugate under $\sigma : \sqrt[3]{2} \mapsto e^{-2\pi i/3}\,\sqrt[3]{2}$. Then – up to homeomorphism – the two dessins look as follows (opposite sides have to be identified to get a torus).





Galois invariants

- the resulting set $\mathbf{X}^\sigma$ is again a smooth algebraic curve and
- the resulting morphism $\beta^\sigma : \mathbf{X}^\sigma \to \mathbb{P}^1(\mathbb{C})$ is a Belyĭ function,
- hence defining an action $\mathcal{D} \mapsto \mathcal{D}^\sigma$ on the set of all dessins.

Moreover, the following data remain invariant.

- The list of valencies of white vertices = zero orders of β,
- the list of valencies of black vertices = zero orders of $\beta - 1$,
- the list of valencies of faces,
- the number of edges = degree of β,
- the genus of $\mathbf{X}$,
- the automorphism group (preserving orientation, and up to $\cong$),
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The moduli field

For a given data configuration consisting of genus, number of edges and list of vertex valencies, there exist only a finite number of dessins. Since dessins determine uniquely the conformal structure of $\mathbf{X}$ (remember David Singerman's talk!),

$$U(\mathbf{X}) \ := \ \{\sigma \in \mathrm{Gal}\,\mathbb{C} \mid \mathbf{X} \cong \mathbf{X}^\sigma\}$$

has finite index in $\mathrm{Gal}\,\mathbb{C}$. Its fixed field $M(\mathbf{X})$, the moduli field of $\mathbf{X}$, is therefore a number field.

Easy: every field of definition for $\mathbf{X}$, i.e. containing all coefficients of the defining polynomials for (a model of) $\mathbf{X}$ contains the moduli field.

Is the moduli field itself a field of definition?

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there are counterexamples due to Earle, Shimura, Dèbes/Emsalem.

On the other hand, based on a criterion due to A. Weil and work of many people (Coombes/Harbater, Wolfart, Koeck, Hammer/Herrlich, G. González) we have

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The moduli field $M(\mathbf{X})$ is a field of definition for $\mathbf{X}$ if

- $g = 0$
- $g = 1$
- $\mathrm{Aut}\,\mathbf{X}$ *is trivial*
- $\mathbf{X}$ *is quasiplatonic.*

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Galois actions on quasiplatonic surfaces

Almost obvious:

If a regular dessin is uniquely determined (up to isomorphism) by its type (p, q, r) and its automorphism group, its underlying quasiplatonic surface is definable over the rationals.

This does not mean that we can write down the equations!

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"Low" genera regular dessins

Theorem (Conder/Jones/Streit/Wolfart 2013)

1. *All quasiplatonic surfaces in genera* $1 < g < 7$ *can be defined over* $\mathbb{Q}$.
2. *Most quasiplatonic surfaces in genera* $7 \leq g \leq 18$ *can be defined over* $\mathbb{Q}$ *up to* 21 *pairs defined over quadratic number fields (which can be explicitly determined) and*
3. *one Galois orbit of length* 4 *in genus* 12 *of signature* $(2, 5, 10)$ *(graph* K_{11}*), automorphism group of order* 110 *and minimal field of definition* $\mathbb{Q}(\zeta_5)$,
4. *one Galois orbit of length* 3 *in genus* 14 *of type* $(2, 3, 7)$ *and automorphism group* $\mathrm{PSL}_2\mathbb{F}_{13}$ *(third Macbeath–Hurwitz group), defined over* $\mathbb{Q}(\cos\frac{2\pi}{7})$.

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Remarks and questions

The theorem gives a wrong impression concerning the long–range–behaviour: a recent result of González–Diez and Jaikin–Zapirain shows that the absolute Galois group acts faithfully on the set of all quasiplatonic surfaces.

The situation is slightly more complicated for the regular dessins (i.e. for their Belyĭ functions) than for their underlying quasiplatonic surfaces. As an example, take the Fricke–Macbeath curve of genus 7 with its maximal dessin of type $(2, 3, 7)$ and automorphism group $\mathrm{PSL}_2\mathbb{F}_8$ of order 504 . It is definable over $\mathbb{Q}$, but it has two (non–isomorphic) dessins of type $(2, 7, 7)$ (the chiral pair of the *Edmonds maps*, regular embeddings of K_8) whose Belyĭ functions are defined over $\mathbb{Q}(\sqrt{-7})$.

Needed: a more complete list of Galois invariants for dessins and a better understanding of the link between Galois action and (hyper)map operations.

Let's hope that the next talk reveals some secrets about this link!

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