Chirality

in

Polyhedra, Polytopes and Thin Geometries

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joint work with

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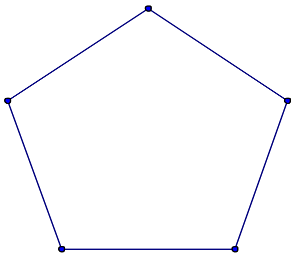
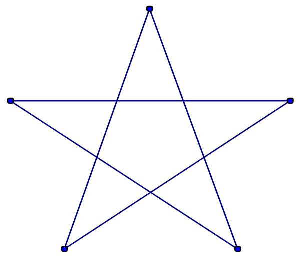
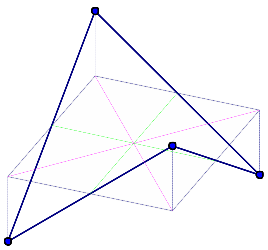
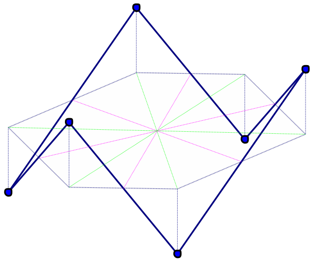
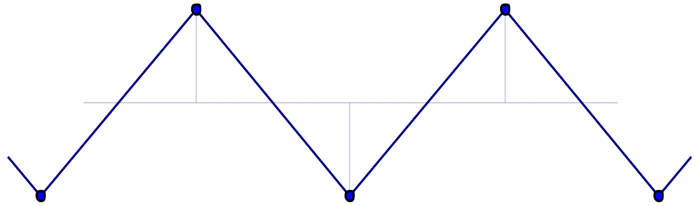
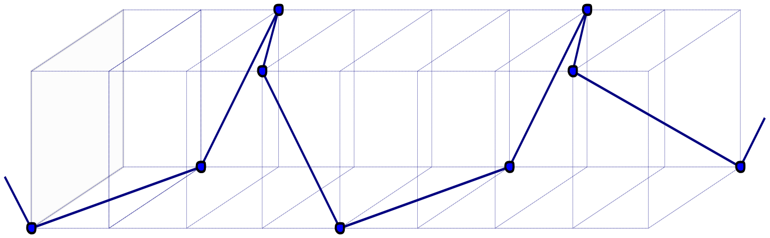
Geometrically regular polytope is the polytope whose symmetry group (the group of isometries keeping the polytope invariant) is flag transitive while the group of a geometrically chiral polytope has two orbits on the flags with the adjacent flags always being in distinct orbits.

GEOMETRIC CHIRALITY:

RANK 2

There are no chiral polygons – all are geometrically regular.

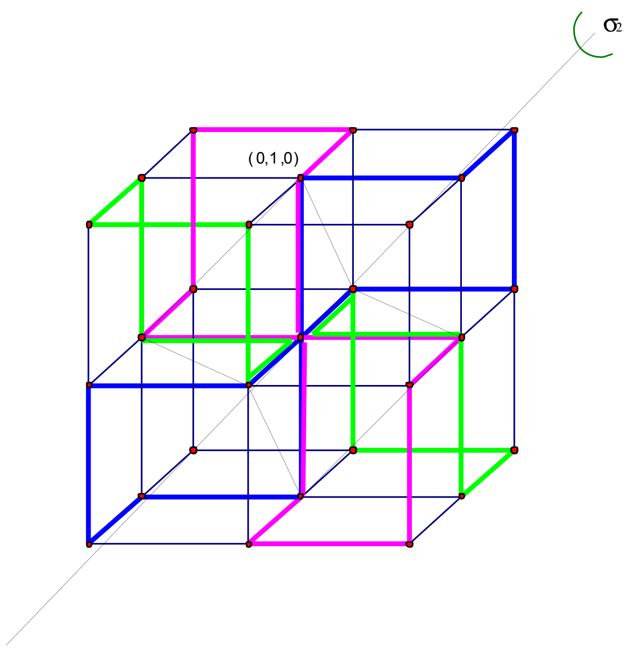
RANK 3

McMullen (1967) There are no convex chiral polyhedra.

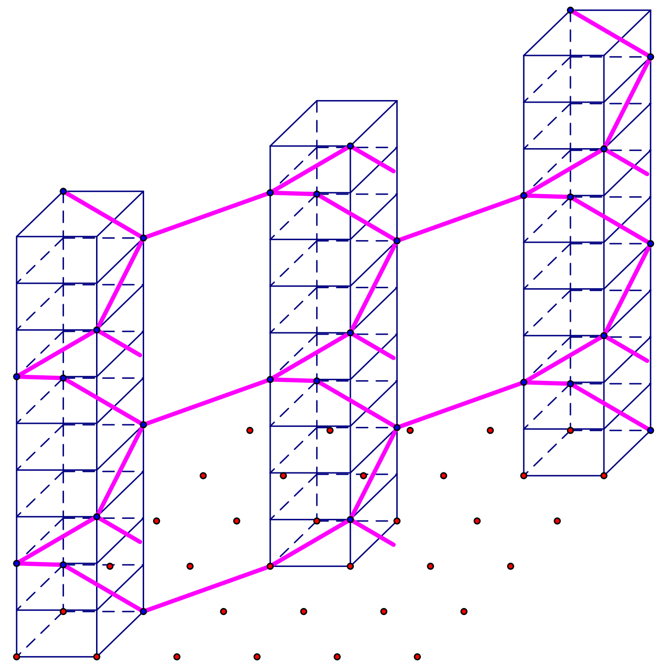
Schulte (2004-2005) There are no finite chiral polyhedra in Euclidean space. Infinite chiral polyhedra can be classified in six families; three families have finite faces and three have infinite faces.

Pellicer & Weiss (2010) Chiral polyhedra with finite faces are combinatorially chiral and the others are only geometrically chiral.

Abstractly (and geometrically) chiral:

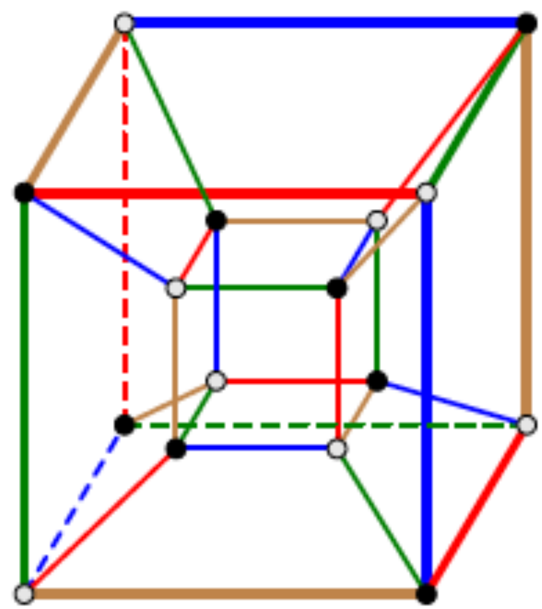


Abstractly regular, geometrically chiral:



RANK 4

Bracho, Hubard & Pellicer (2013) Found an abstractly regular but geometrically chiral rank 4 polytope of type $\{8,3,3\}$ in Euclidean 4-space, disproving McMullen's conjecture that such do not exist.



The 1-skeleton of the polytope is the 1-skeleton of the hemi 4-cube in the projective space. The 2-faces are 4-gonal Petrie polygons, which lift to helical 8-gons when taking the double cover to the 3-sphere; the 3-faces are chiral realizations of the cube which lift to the 3-sphere as chiral realizations of double covers of the cube with type $\{8,3\}$.

ABSTRACT CHIRALITY:

RANK 3

Coxeter (1948)
Classified regular and chiral maps on torus.

Garbe (1969)
There are no chiral maps on surfaces of genus 2, 3, 4, 5 or 6.

Heffter (1898)
Family of chiral maps of type $\{2^k-1, 2^k-1\}$ for $k>2$.

Edmonds (196?)
Rediscovers Heffter's map of type $\{7,7\}$ (on a surface of genus 8).

Sherk (1962)
Family chiral maps of type $\{6,6\}$ (smallest member on a surface of genus 7).

….

Conder (2001 -)
Lists of chiral maps by type, by genus and "size"…

RANK 4

Coxeter (1970)
Twisted Honeycombs.

Weber & Seifert (1933)
Two examples in rank 4; both with one polyhedral cell!

Colbourne & Weiss (1984)
Census of locally toroidal rank 4 polystroma.

Monson, Schulte & Weiss (1994 - 2005)
Construct number of families in rank 4.

ARBITRARY RANK

Schulte & Weiss (1991)
Basic structure theory of chiral polytopes.

Schulte & Weiss (1995)
Universal extension theorem for chiral polytopes with regular facets, leading to first examples in rank 5.

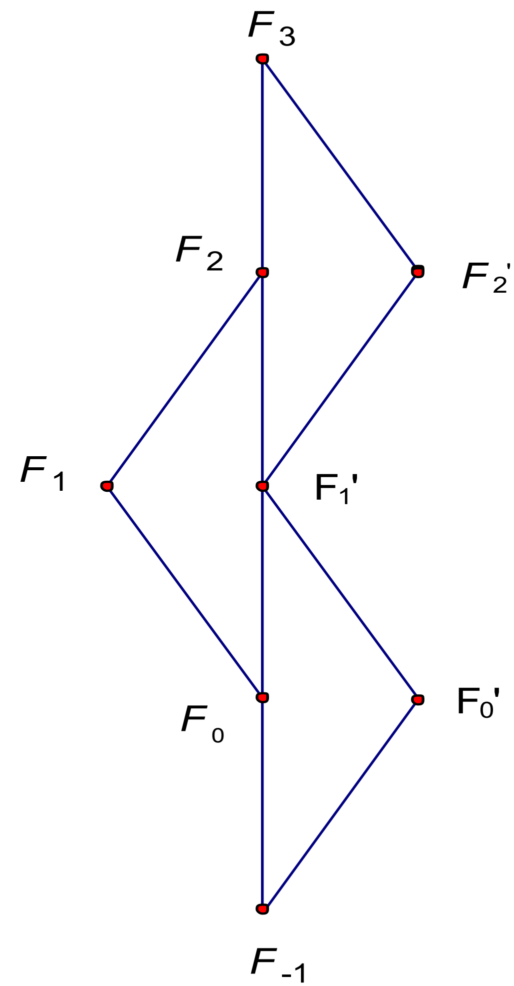
Conder, Hubard & Pisanski (2008)
First examples of finite chiral polytopes of rank > 4.

Pellicer (2010)
Constructs chiral polytopes of arbitrary rank.

Hartley, Hubard & Leemans (2011)
Two atlases of abstract chiral polytopes for small groups.

PS My apologies for omissions!

An (ABSTRACT) polytope P of rank *n*, or an *n*-polytope, is a poset, whose elements are called faces, with strictly monotone rank function with range $\{-1,0,1,\ldots,n\}$ satisfying the following properties:

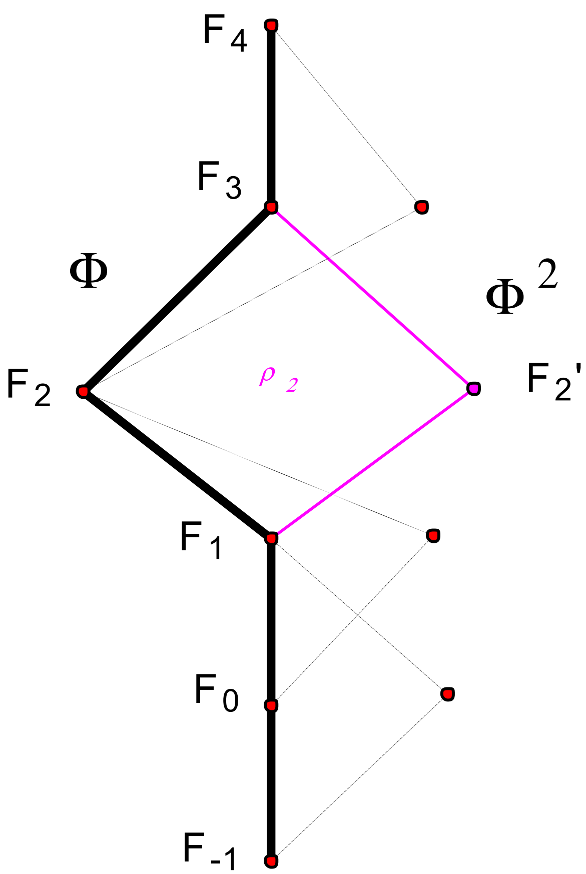


- P has a unique minimal face F_{-1} and a unique maximal face F_n.
- The maximal chains called flags of P contain exactly $n+2$ faces.
- P is strongly flag connected.
- P satisfies a homogeneity property (diamond condition).

Sections $F/G := \{H \in P \mid G \le H \le F\}$ of polytope P are polytopes and $rank(F/G) = rank(F) - rank(G) + 1$.
Section F_n/F_0 is called a vertex figure of *P* at F_0.

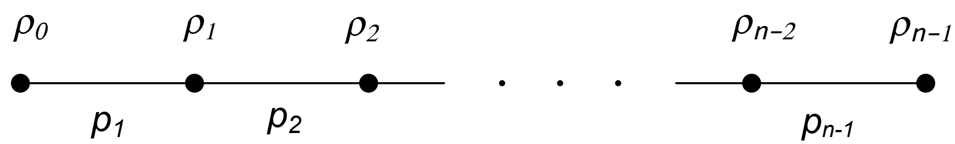
Aut(P) = group of all automorphisms (order preserving bijections)

A polytope P is said to be regular if its group of automorphisms Aut(P) is transitive on the flags.



⇒ Aut(P) is generated by involutions.

If P is of rank *n*, the generators $\rho_0,\cdots,\rho_{n-1}$ of Aut(P) satisfy the relations implicit in the $C-$ diagram



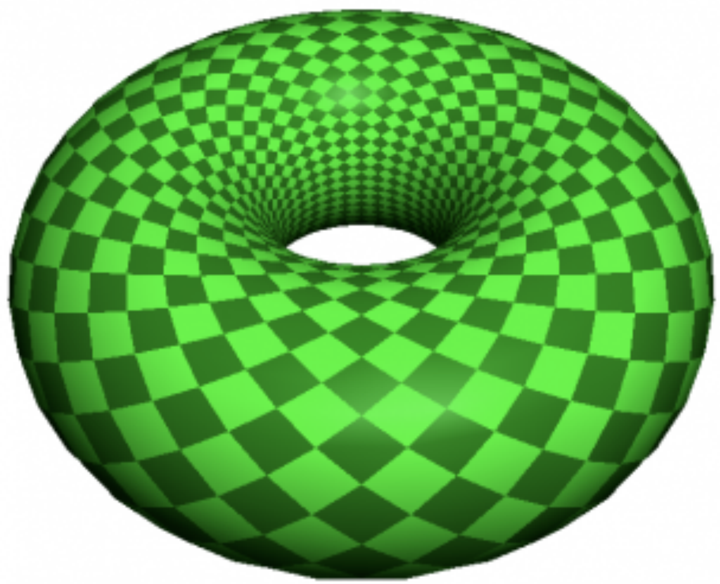
and P is said to have (Schläfli) type $\{p_1, p_2, \cdots, p_{n-1}\}$.

Furthermore, Aut(P) satisfies an intersection condition (IP):

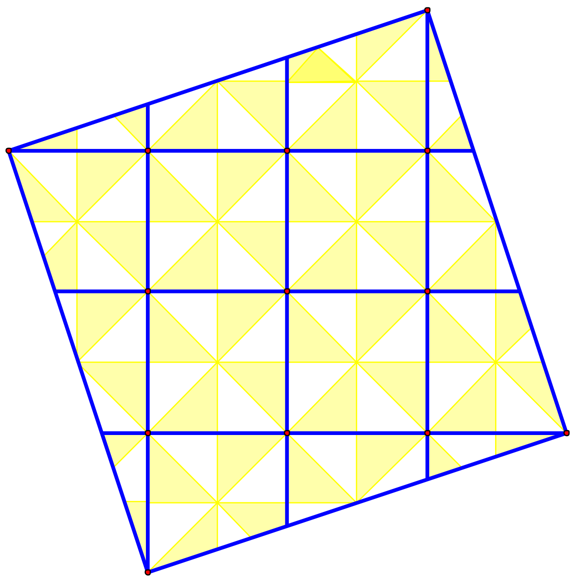
$$\langle \rho_i | i \in I \rangle \cap \langle \rho_i | i \in J \rangle = \langle \rho_i | i \in I \cap J \rangle, \quad I, J \subseteq \{0, \cdots, n-1\}.$$

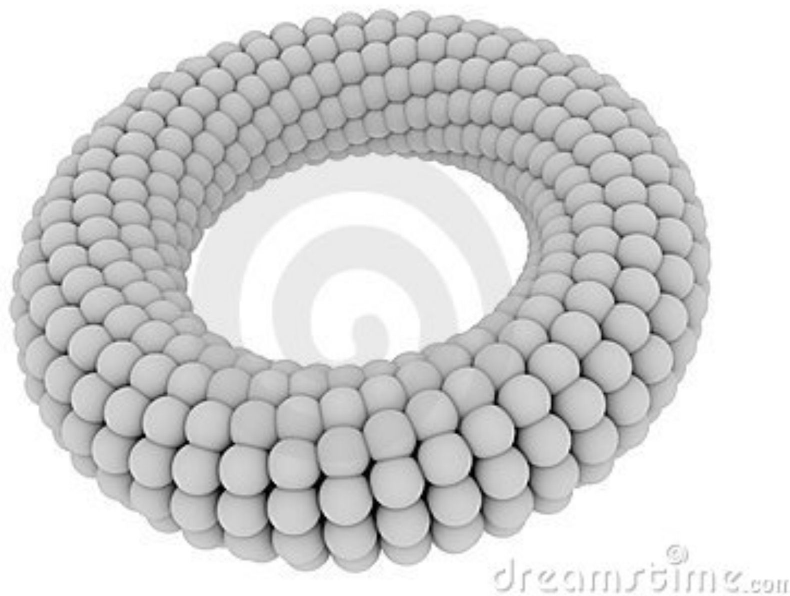
Schulte (1982): Given such group, called string $C-$group, one can construct a regular abstract polytope having this group as its automorphism group.

An example of a rank 3 regular abstract polytope of Schläfli type {4,4}



A polytope P is said to be **chiral** if its group Aut(P) has exactly two orbits on the flags, with adjacent flags in distinct orbits.



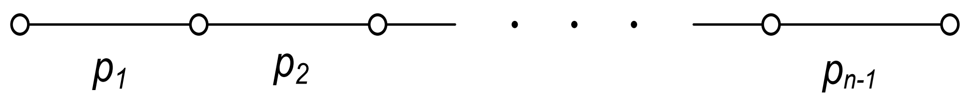


The group of automorphisms of a chiral *n*-polytope is generated by “rotations” $\sigma_1,\cdots,\sigma_{n-1}$ of periods $p_1,\cdots,p_{n-1}$ respectively, with the property that

$$(\sigma_i\sigma_{i+1}\cdots\sigma_j)^2=1 \text{ for } 1\le i<j\le n-1.$$

$\Rightarrow$ Chiral polytope can also be assigned a (Schläfli) type $\{p_1,\ldots,p_{n-1}\}$.

We can represent such groups by the following diagram



P also satisfies an intersection condition (IP^+), which for rank 3 can be written as

$$\langle\sigma_1\rangle \cap \langle\sigma_2\rangle = \{1\} = \langle\sigma_2\rangle \cap \langle\sigma_3\rangle,$$
$$\langle\sigma_1,\sigma_2\rangle \cap \langle\sigma_2,\sigma_3\rangle = \langle\sigma_2\rangle$$

and for higher rank can be stated inductively as follows. The group $\langle\sigma_1,\cdots,\sigma_{n-1}\rangle$ has the intersection property if $\langle\sigma_1,\cdots,\sigma_{n-2}\rangle$ has the intersection property and if

$$\langle\sigma_1,\cdots,\sigma_{n-2}\rangle \bigcap \langle\sigma_i,\cdots,\sigma_{n-1}\rangle = \langle\sigma_i,\cdots,\sigma_{n-2}\rangle \text{ for } i = 2,\cdots,n-1.$$

Schulte & Weiss (1991): Given such group, which we shall call string C^+ –group, one can construct an abstract polytope, which is regular whenever there exists an automorphism ρ such that

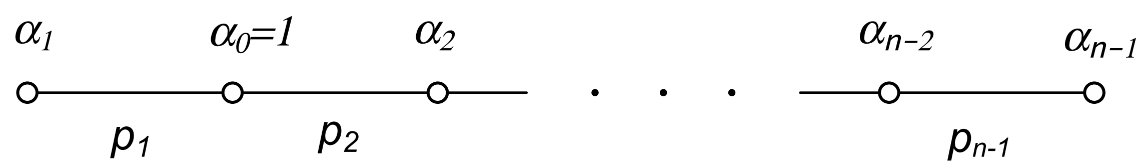
$$\sigma_1^\rho = \sigma_1^{-1}, \quad \sigma_2^\rho = \sigma_2^{-1}, \quad \sigma_3^\rho = \sigma_2^2\sigma_3, \quad \sigma_i^\rho = \sigma_i \text{ for } i = 4,\cdots,n-1,$$

and chiral otherwise.

In extending the definition of chirality to thin geometries it is more convenient to, instead of the above generators, use the following set of (independent) generators:

$$\alpha_1 = \sigma_1^{-1}, \quad \alpha_2 = \sigma_2, \quad \text{and} \quad \alpha_i = \sigma_2 \sigma_3 \cdots \sigma_i \quad \text{for } 2 \le i \le n-1.$$

To these groups we can then conveniently associate $B-$diagram: the complete graph on set of vertices labeled by $\alpha_0 = 1, \alpha_1, \cdots, \alpha_{n-1}$ and set of edges labeled by $o\left(\alpha_i^{-1}\alpha_j\right) = o\left(\alpha_j^{-1}\alpha_i\right) = o\left(\alpha_i\alpha_j^{-1}\right)$ with the convention of dropping an edge if its label is 2 and dropping the label if it is 3.



The condition on regularity with these generators requires the existence of an automorphism ρ such that

$$\alpha_i^{\rho} = \alpha_i^{-1} \quad \text{for } i = 1, \cdots, n-1.$$

An incidence system $\Gamma := (X, *, t, I)$ is a 4-tuple such that

- X is a set whose elements are called elements of Γ;
- I is a set whose elements are called the types of Γ;
- $t : X \rightarrow I$ is a type function, associating to each element its type $t(x)$;
- * is an incidence relation.

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A flag F of Γ is a set of mutually incident elements of Γ and its type is $\{t(x) | x \in F\}$
A chamber is a flag of type I. An incidence system is a geometry, or incidence geometry, if every flag of Γ is contained in a chamber. The rank of Γ is the cardinality of I.

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Polytopes and non-degenerate hypermaps are examples of thin geometries.

A hypertope is defined to be a thin incidence geometry that is strongly chamber-connected (SCC) (or, residually connected as commonly used in the terminology of incidence geometries).

An automorphism of $\Gamma := (X, *, t, I)$ is a mapping $\alpha : X \to X$ such that for all $x, y \in X$

- α is a bijection on X (inducing a bijection on I);
- $x * y$ if and only if $\alpha(x) * \alpha(y)$;
- $t(x) = t(y)$ if and only if $t(\alpha(x)) = t(\alpha(y))$.

The set of all automorphism of Γ is denoted by Aut (Γ). An automorphism is type preserving when for each $x \in X$, $t(\alpha(x)) = t(x)$. The set of all type preserving automorphism of Γ is denoted by $\mathrm{Aut}_I(\Gamma)$.

A thin geometry Γ is flag-transitive if $\mathrm{Aut}_I(\Gamma)$ is transitive on all flags of a given type J for each $J \subseteq I$; Γ is chamber-transitive if $\mathrm{Aut}_I(\Gamma)$ is transitive on all chambers of Γ. In fact, these two conditions are equivalent for incidence geometries.

A hypertope (that is a thin, SCC incidence geometry) Γ is said to be

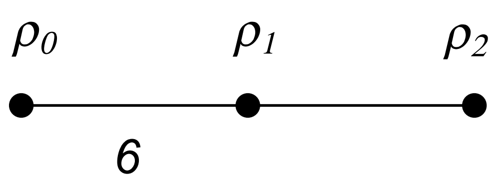
- regular if $\mathrm{Aut}_I(\Gamma)$ has one orbit on the chambers of Γ;
- chiral if $\mathrm{Aut}_I(\Gamma)$ has two orbits on the chambers of Γ such that any two adjacent chambers lie in distinct orbits.

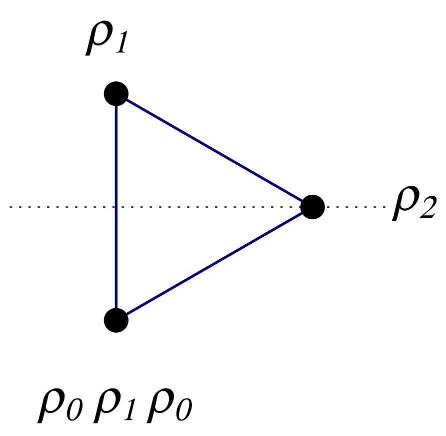
Let Γ be a regular hypertope and ϕ one of its chambers. Then for each $i \in I$ there exists an involutory automorphism ρ_i that interchanges ϕ with its $i-$adjacent chamber ϕ^i. The group of automorphisms $\text{Aut}_I(\Gamma)$ is then generated by distinguished generators $\{\rho_0, \cdots, \rho_{r-1}\}$, where $r = |I|$, which satisfy the intersection condition IP

$$\langle \rho_i | i \in I \rangle \cap \langle \rho_i | i \in J \rangle = \langle \rho_i | i \in I \cap J \rangle, \quad I, J \subseteq \{0, \cdots, n-1\}.$$

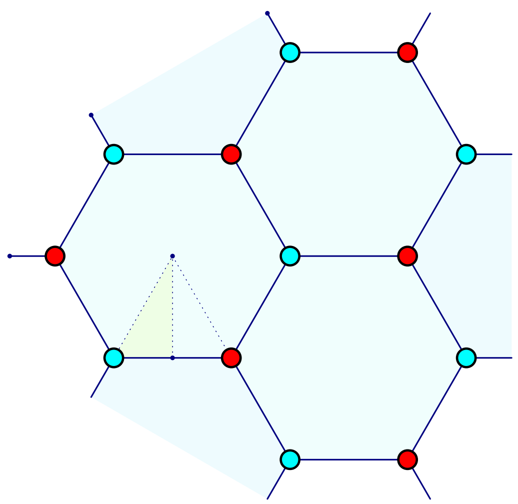
Furthermore, $\text{Aut}_I(\Gamma)$ satisfies the relations implicit in the $C-$diagram, the complete graph on r vertices whose vertices labeled by the $\rho_{0,\cdots,}\rho_{r-1}$ and edges $\rho_i \rho_j$ with $o(\rho_i \rho_j)$.

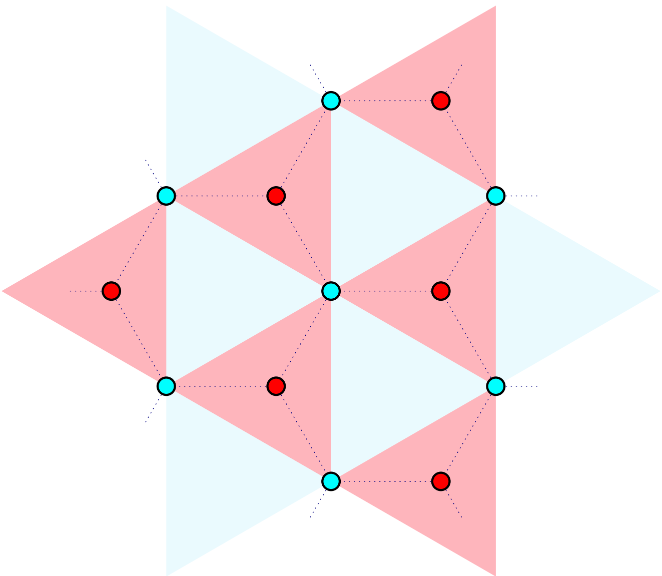
Example:





(3,3,3)





Similarly, if Γ is a chiral hypertope and ϕ one of its chambers, then for each pair $i, j \in I, i \neq j$, there exists an automorphism σ_{ij} mapping the flag ϕ to $\left(\phi^i\right)^j$.
We define the distinguished generators:

$$\alpha_0 = 1,\ \alpha_1 = \sigma_{10},\ \alpha_2 = \sigma_{12},\ \alpha_i = \sigma_{12} \cdots \sigma_{i-1,i} \quad \text{for } i = 3,\cdots,r-1.$$

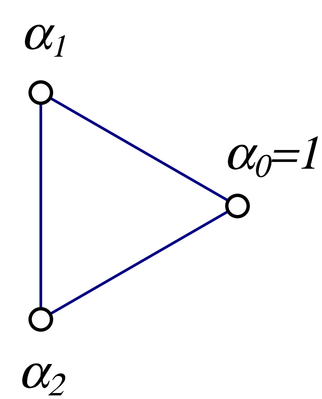
With so generated $\mathrm{Aut}_I(\Gamma)$ we can associate a $B-$diagram, the complete graph on r vertices labeled by $\alpha_0,\cdots,\alpha_{r-1}$ and edges labeled by $o(\alpha_i^{-1}\alpha_j)$ which must satisfy the intersection condition IP^+

$$\left\langle \alpha_i^{-1}\alpha_j \middle| i,j \in I \right\rangle \cap \left\langle \alpha_i^{-1}\alpha_j \middle| i,j \in J \right\rangle = \left\langle \alpha_i^{-1}\alpha_j \middle| i,j \in I \cap J \right\rangle$$

for all $I, J \subseteq \{0,\cdots,r-1\}$ with $|I|,|J| \geq 2$.

Example:

$B-$diagram of a chiral hypermap $(3,3,3)$



REVERSING THE CONSTRUCTION

Starting from a group and some of its subgroups construct an incidence system:

Tits (1961) Let n be a positive integer, $I=\{1,\cdots,n\}$, G a group and $(G_i)_{i\in I}$ a family of subgroups of G. Define

- X to be the set of all cosets $G_ig,\ g\in G, i\in I$;
- $t:X\to I$ such that $t(G_ig)=i$
- $G_ig_1 * G_jg_2$ if and only iff $G_i\cap G_j\neq\varnothing$

Then

- $\Gamma:=(X,*,t,I)$ is an incidence system having a chamber;
- G acts by right multiplication as an automorphism group on Γ;
- G is transitive on flags of rank less than 3.

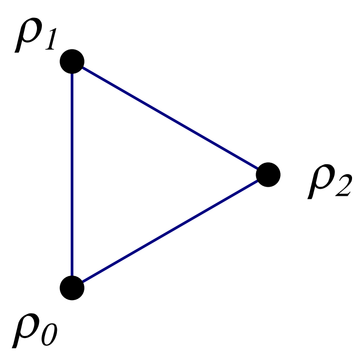
When the above construction gives us a geometry we denote it by $\Gamma\left(G,(G_i)_{i\in I}\right)$ and call it a coset geometry.

When the kernel K, under the action of G on Γ (the largest normal subgroup of G contained in every G_i) is the identity we say that G acts faithfully on Γ. If G acts faithfully on Γ and is transitive on chambers we say that Γ is regular.

A pair (G,S) such that G is a group and $S:=\{\rho_0,\cdots,\rho_{r-1}\}$ its generating set of involutions which satisfies the condition IP is called a $C-$ group. With each $C-$ group we associate a $C-$diagram which need not be linear and which we view as a complete graph on r vertices.

Theorem (Rank 3): Let $(G,\{\rho_0,\rho_1,\rho_2\})$ be a $C-$ group of rank tree. Then the coset geometry $\Gamma(G;(\langle\rho_1,\rho_2\rangle,\langle\rho_0,\rho_2\rangle,\langle\rho_0,\rho_1\rangle))$ is thin if and only if it is regular. Moreover, if it is thin (or regular) it is strongly chamber-connected.

Example: The $C-$group of order 18 given by the following relations

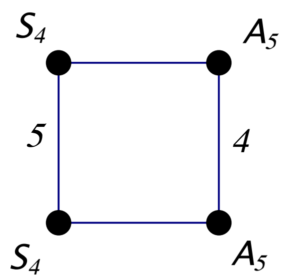


$$+ \quad (\rho_0\rho_1\rho_2)^2=1$$

is the automorphism group of the hypermap $\{6,3\}_2$. However, the implied coset geometry is a $K_{3,3,3}$ and hence G cannot be flag transitive on Γ which has 27 chambers. In this case Γ is not thin, but it is SCC.

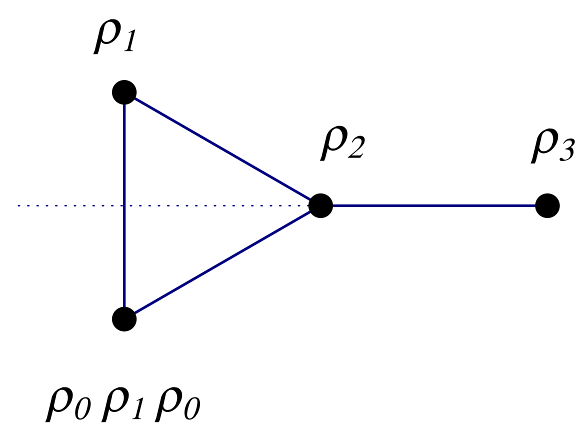
Remark: Unfortunately, in higher ranks even thinness need not suffice!

Example:



Is a $C-$ group, but the coset geometry it gives is not thin, it is not SCC, nor flag-transitive.

Theorem: Let $\left(G, S=\left\{\rho_0, \rho_1, \cdots, \rho_{r-1}\right\}\right)$ be a $C-$ group of rank r and let $\Gamma := \Gamma\left(G;\left(G_i\right)_{i \in I}\right)$ with $G_i := \left\langle \rho_j \middle| \rho_j \in S, j \in I \setminus \{i\} \right\rangle$ for all $i \in I := \{0, \cdots, r-1\}$. If Γ is flag-transitive, then Γ is a regular hypertope.



Similarly, starting with $\left(G^+, R\right)$, where G^+ is a group with a set of independent generators $R := \{\alpha_1, \cdots, \alpha_{r-1}\}$ satisfying the condition IP^+ we can construct the coset geometry

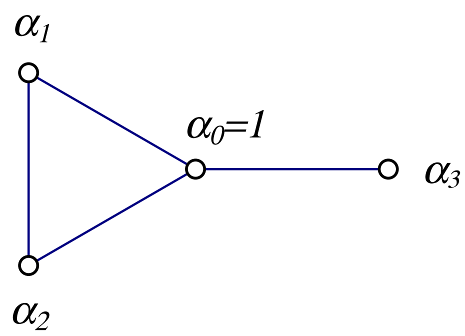
$$\Gamma\left(G^+, R\right) := \Gamma\left(G^+, (G_i)_{i \in \{0, \cdots, r-1\}}\right)$$

where $G_i := \left\langle \alpha_j \middle| j \neq i \right\rangle$ for $i = 1, \cdots, r-1$ and $G_0 := \left\langle \alpha_1^{-1}\alpha_j \middle| j \geq 2 \right\rangle$.

Theorem: Let $\Gamma = \Gamma\left(G^+, R\right) := \Gamma\left(G^+, (G_i)_{i \in \{0, \cdots, r-1\}}\right)$ be coset geometry constructed from G^+ and a set of independent relations $R := \{\alpha_1, \cdots, \alpha_{r-1}\}$. If Γ is a hypertope (that is, thin and SCC) it is chiral if and only if there is no automorphism of G^+ that inverts all elements of R.

Examples:

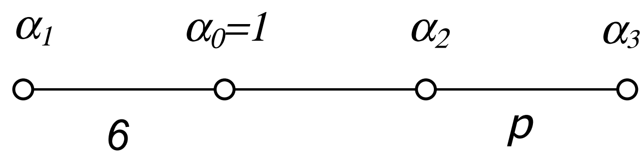
The group G^+, denoted by $(3,3,3;3)^+$, and given by the $B-$diagram



is an infinite group with the following defining relations:

$$\alpha_1^3 = \alpha_2^3 = \alpha_3^3 = 1,$$
$$\left(\alpha_2\alpha_1^{-1}\right)^3 = \left(\alpha_2\alpha_3^{-1}\right)^2 = \left(\alpha_3\alpha_1^{-1}\right)^2 = 1.$$

The addition of $(\alpha_2\alpha_1)^b = (\alpha_1\alpha_2)^c$ to the relations in the $B-$diagram



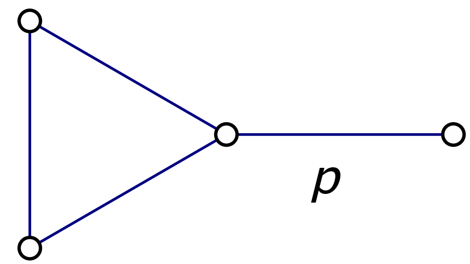
gives a finite group whenever $b+c \leq 5$. In each case it is a C^+-group and the induced coset geometry Γ is (thin, SCC) a regular or a chiral hypertope:

p	$\mathbf{s}$	v	f	g	Group	Chiral/Regular
3	$(2,0)$	10	5	240	$S_5 \times C_2$	regular
	$(3,0)$	54	12	1296	$[1\ 1\ 2]^3 \rtimes C_2$	regular
	$(4,0)$	640	80	15360	$[1\ 1\ 2]^4 \rtimes C_2$	regular
	$(1,2)$	28	8	336	$PGL_2(7)$	chiral
	$(1,3)$	182	28	2184	$PSL_2(13) \times C_2$	chiral
	$(1,4)$	672	64	8064	$SL_2(7) \rtimes A_4 \rtimes C_2$	chiral
	$(2,2)$	120	20	2880	$S_5 \times S_4$	regular
	$(2,3)$	570	60	6840	$PGL_2(19)$	chiral
4	$(1,1)$	6	8	288	$S_3 \rtimes [3,4]$	regular
	$(2,0)$	16	16	768	$[3,3,4] \times C_2$	regular
	$(1,2)$	84	48	2016	$PGL_2(7) \times S_3$	chiral
5	$(2,0)$	240	600	28800	$[3,3,5] \times C_2$	regular

TABLE 1. Finite polytopes of type $\{\{6,3\}_{\mathbf{s}},\{3,p\}\}$

Similarly, for each $p=3,4,5$ and 6 from the groups $(3,3,3;p)$ and $(3,3,3;p)^+$, which can be seen as the subgroups of the symmetry groups of $3-$dimensional hyperbolic honeycombs $[6,3,p]$, one can construct "locally toroidal" regular and chiral hypertopes by addition of appropriate relations.

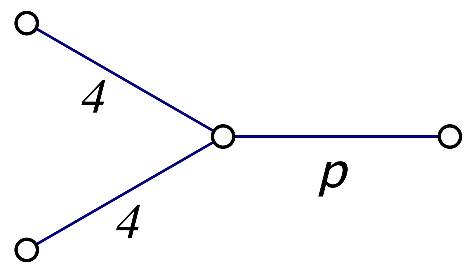
Here the $B-$diagram for $(3,3,3;p)^+$ is



p	s	v	f	g	Group	Chiral/Regular
3	$(2,0)$	5	5	120	S_5	regular
	$(3,0)$	27	12	648	$[1\ 1\ 2]^3$	regular
	$(4,0)$	320	80	7680	$[1\ 1\ 2]^4$	regular
	$(1,2)$	14	8	168	$PSL_2(7)$	chiral
	$(1,3)$	91	28	1092	$PSL_2(13)$	chiral
	$(1,4)$	336	64	4032	$SL_2(7) \rtimes A_4$	chiral
	$(2,2)$	60	20	1440	$A_5 \times S_4$	regular
	$(2,3)$	285	60	3420	$PSL_2(19)$	chiral
4	$(1,2)$	42	48	1008	$PSL_2(7) \times S_3$	chiral
	$(2,0)$	8	16	384	$[3,3,4]$	regular
5	$(2,0)$	120	600	14400	$[3,3,5]$	regular

TABLE 2. Finite thin geometries of type $[3,3,3;p]$

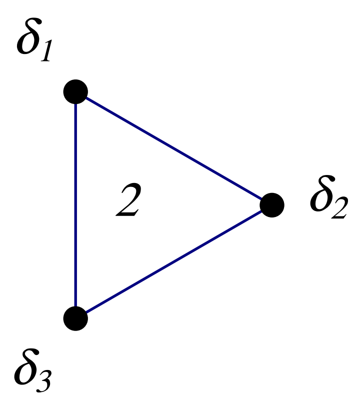
Furthermore, regular and chiral hypertopes with the $B-$diagram



for the group $(2,4,4;p)^{+}$ are derived with and $p=3$ and 4.

Regular hypertopes exist in each rank: Examples are obtained from the symmetric group S_{n+1} together with its generating transpositions $\delta_i=(i \;\; n+1)$ for $i=1,\ldots,n$. Its Coxeter diagram is the complete graph on n vertices and unlabeled edges, that is $(\delta_i\delta_j)^3=1$ whenever $i\neq j$, and additional relations $(\delta_i\delta_j\delta_k\delta_j)^2=1$ for all i,j,k such that $i\neq j\neq k\neq i$.

S_4:



Thank you !

Carbon (chiral) nanotube
An equivelar non-regular map on a torus