HAMILTON CYCLES
in truncated triangulations of closed surfaces

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joint work with
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Graph symmetry and hamiltonicity

Question (Lovász, 1969)

Does every connected vertex-transitive graph have a Hamilton path, i. e., a simple path going through all vertices?

- Only 5 connected v-t graphs with no Hamilton cycle are known.
- None of them is a Cayley graph.

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Every Cayley graph (of order ≥ 3) has a Hamilton cycle.

Counter-Conjecture (Babai, 1995)

For some $c > 0$, there are infinitely many vertex-transitive graphs G, even Cayley graphs, without cycles of length $> (1 - c)|G|$.

Let $H = \langle r, l \rangle$ be a $(2, 3, s)$-presented finite group; i.e.,
$r^s = l^2 = (rl)^3 = 1$.

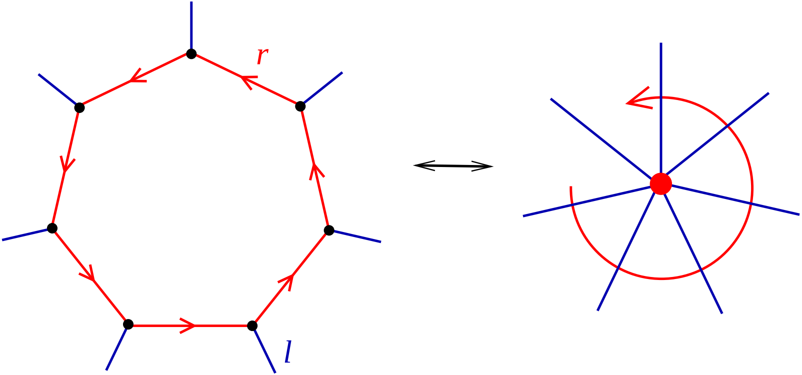
Then H is a finite quotient of the modular group $PSL(2, \mathbb{Z})$.

Theorem (Glover & Marušič, 2009)

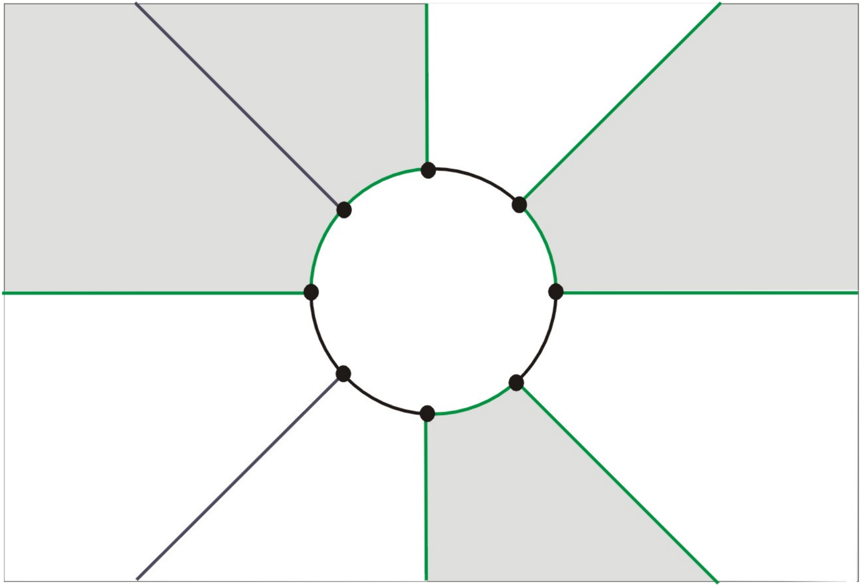
Let $K = \mathrm{Cay}(H;\ r, r^{-1}, l)$ be a cubic Cayley graph, where $H = \langle r, l \mid r^s = l^2 = (rl)^3 = 1, \dots \rangle$ is a finite quotient of the modular group $PSL(2, \mathbb{Z})$. Then K has a Hamilton path. Moreover,

- *if $|H| \equiv 2 \pmod 4$, then K has a Hamilton cycle*
- *if $|H| \equiv 0 \pmod 4$, then K has a cycle through all but two adjacent vertices.*

Proof I: Cayley map and the corresponding triangulation



Proof II: How to find a Hamilton cycle



Proof II: How to find a Hamilton cycle

We construct a Hamilton cycle in $\mathcal{CM}$ as $\partial(\bigcup \mathcal{F})$ of a set $\mathcal{F}$ of red-blue hexagonal faces of $\mathcal{CM}$.

- The boundary must be connected and must cover all vertices.
- To cover all the vertices, the complementary set of hexagons must be 'independent'.
- To get a connected boundary, $\bigcup \mathcal{F}$ must be connected and homologically trivial, i.e., a 'tree' of faces.

Proof III: Dual of the triangulation

hexagonal faces of $\mathcal{CM}$ $\longleftrightarrow$ faces of the triangulation $\mathcal{T}$
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In other words:

We need to find a partition of $V(G^*)$ into two sets A and J,
where A induces a tree and J is independent.

Proof IV: Symmetry and vertex-partitions in cubic graphs

Theorem 1 (Nedela & S., 1995)

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Theorem 2 (Payan & Sakarovitch, 1975)

Let G be a cyclically 4-edge-connected cubic graph with n vertices. Then the following hold:

(i) *If $n \equiv 2 \pmod 4$, then $V(G)$ has a partition $\{A, J\}$ where A induces a tree and J is independent.*

(ii) *If $n \equiv 0 \pmod 4$, then $V(G)$ has a partition $\{A, J\}$ where either A induces a tree and J induces a graph with a single edge, or A induces a forest with two components and J is independent.*

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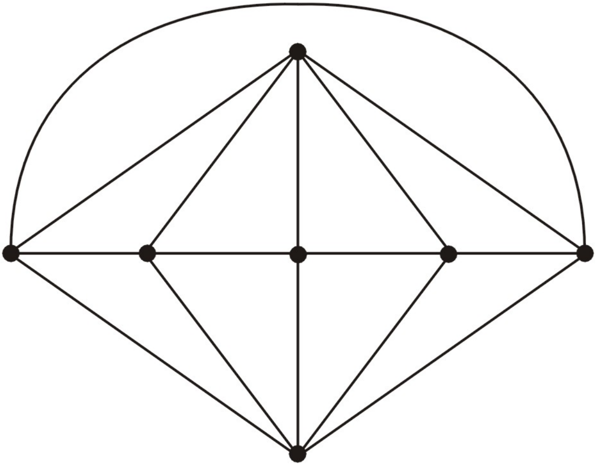
How far from symmetry can we go?

Example: Construction of a Hamilton cycle in $t(\mathcal{T})$



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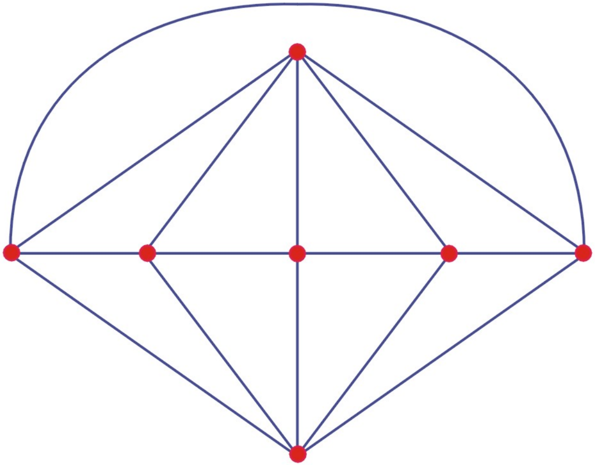

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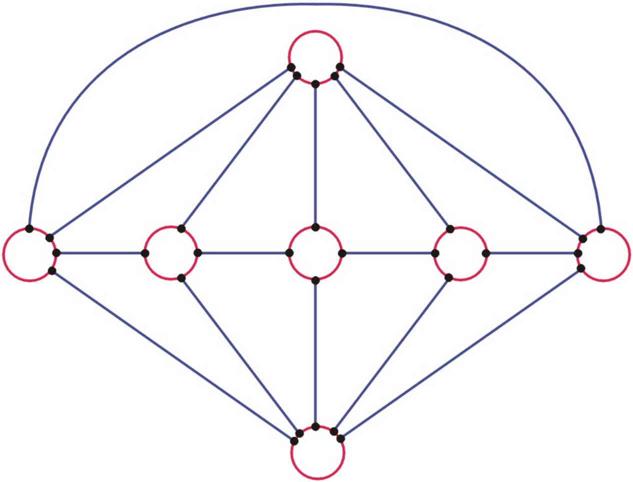

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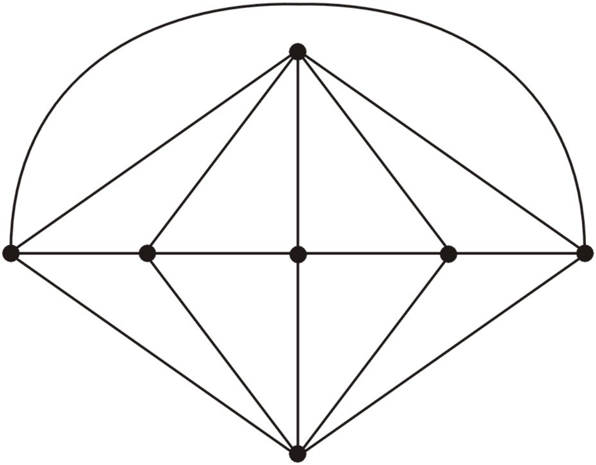

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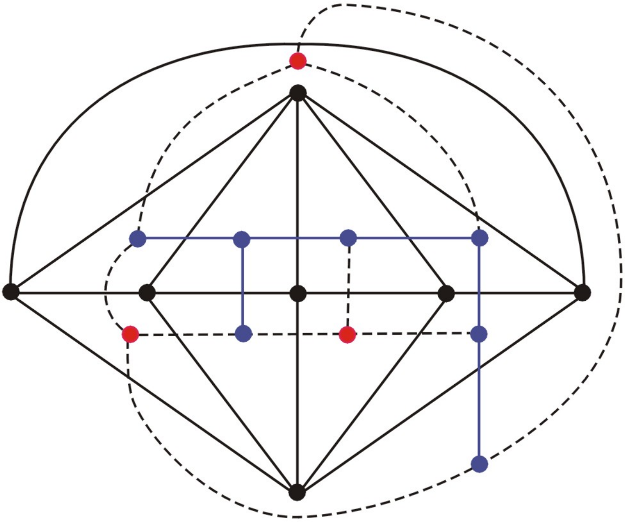

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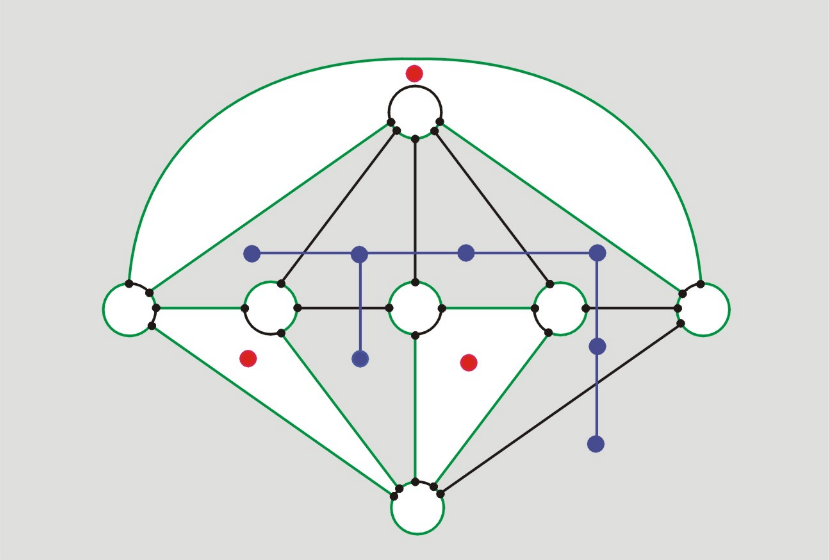
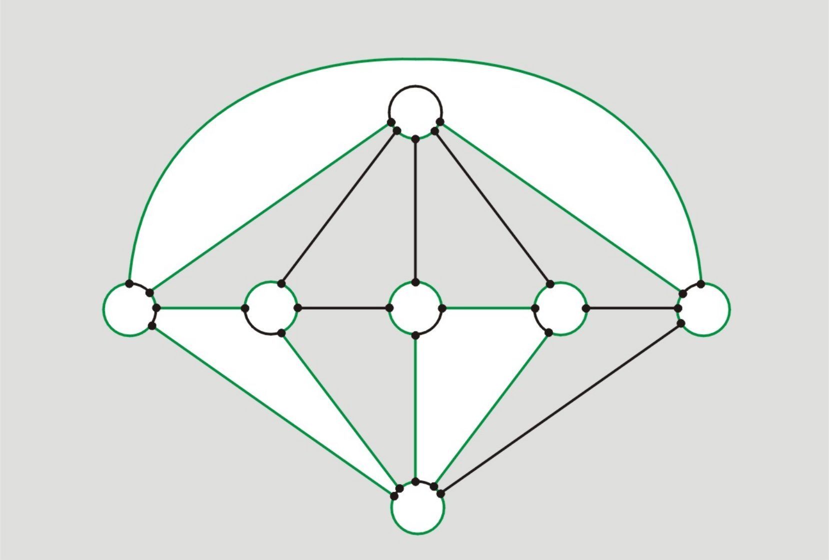
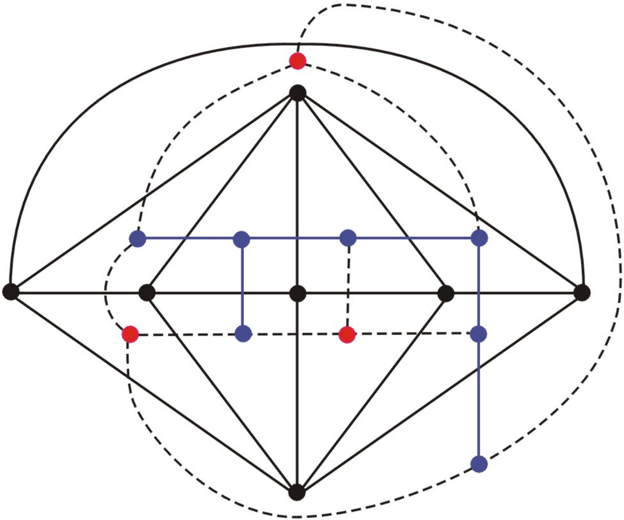


Example: The required Hamilton cycle



When does such a structure exist?



Maximum genus of a graph

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Definition. The maximum genus $\gamma_M(G)$ of a graph is the largest genus of an orientable surface in which G has a cellular embedding.

- By Euler-Poincaré Equation, $\gamma_M(G) \leq \lfloor \beta(G)/2 \rfloor$ where $\beta(G) = |E| - |V| + 1$ is the Betti number of G.

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Definition. A graph G is upper-embeddable if $\gamma_M(G) = \lfloor \beta(G)/2 \rfloor$; equivalently, if it has an embedding with one or two faces.

Upper-embeddable graphs

Theorem (Jungerman, 1978, Xuong, 1979; Nebeský, 1981)

The following statements are equivalent for every connected graph G:

(i) G *is upper-embeddable.*

(ii) G *has a spanning* T *such that* $G - E(T)$ *has at most one component of odd size.*

(iii) $ob(G - A) \leq |A| + 1$ *each* $A \subseteq E(G)$.

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Two types of upper-embeddable cubic graphs

- one-face embeddable $\iff n \equiv 2 \pmod 4$
  $\iff$ all Xuong cotree components are even
- two-face embeddable $\iff n \equiv 0 \pmod 4$
  $\iff$ one Xuong cotree component is odd

Upper-embeddable cubic graphs: even case

Theorem (K., N. & S., 2014+)

The following are equivalent for every connected cubic graph G.

(i) *G one-face-embeddable.*

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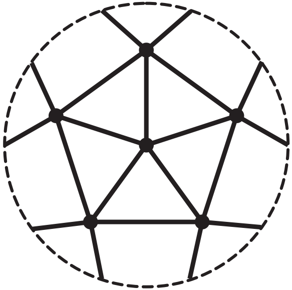
Corollary

Let $\mathcal{T}$ be a triangulation of a closed surface by f triangles. If the underlying graph of $\mathcal{T}^*$ is upper-embeddable and $f \equiv 2 \pmod 4$, then the truncation $t(\mathcal{T})$ has a Hamilton cycle.

Interesting example



Upper-embeddable cubic graphs: odd case

Theorem (K., N. & S., 2014+)

The following are equivalent for every connected cubic graph G.

(i) *G is two-face-embeddable.*

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- *A induces a tree and J is near-independent, or*
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Let $\mathcal{T}$ be a triangulation of a closed surface by f triangles. If the underlying graph of $\mathcal{T}^*$ is upper-embeddable and $f \equiv 0 \pmod 4$, then the truncation $t(\mathcal{T})$ has a Hamilton path.

Ample upper-embeddability

Definition.

1. A cubic graph G is amply upper-embeddable if

(1) G is upper-embeddable

(2) $G - \{x, y\}$ remains upper-embeddable for a suitable pair of adjacent vertices.

2. An upper-embeddable cubic graph G is called tightly upper-embeddable if it is not amply upper-embeddable.

Amply upper-embeddable cubic graphs: odd case

Theorem (K., N. & S., 2014+)

The following are equivalent for every connected cubic graph G.

(i) *G is amply two-face-embeddable.*

(ii) *G has a Xuong tree with a single odd cotree component, which is of size at least three.*

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Classes of amply upper-embeddable cubic graphs

Theorem (K., N. & S., 2014+)

Every cyclically 4-edge-connected cubic graph is amply upper-embeddable.

This strengthens [Payan & Sakarovitch, 1975]:

In the odd case we can always guarantee a partition $\{A, J\}$ where A induces a tree and J is almost independent.

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Every connected edge-transitive cubic graphs is amply upper-embeddable.

Applications

Theorem (K., N. & S., 2014+)

Let $\mathcal{T}$ be a triangulation of a closed surface by f triangles which is either edge-transitive or has no separating cycle of length ≤ 3. Then $t(\mathcal{T})$ has a Hamilton path. Moreover,

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Corollary (K., N. & S., 2014+)

Let $K = \mathrm{Cay}(H; x, y, z)$ be a cubic Cayley graph, where
$H = \langle x, y, z \mid x^2 = y^2 = z^2 = 1, (xy)^3 = (yz)^3 = 1, \dots \rangle$.
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Applications

Graphs admitting a 2-cell embedding with each face of size ≤ 7 are upper-embeddable [Huang & Liu, 2000].

Theorem (K., N. & S., 2014+)

Let $\mathcal{T}$ be a polyhedral triangulation of a closed surface by f triangles such that every vertex has valency ≤ 7. Then $t(\mathcal{T})$ has a Hamilton path, and if $f \equiv 2 \pmod 4$, then $t(\mathcal{T})$ has a Hamilton cycle.

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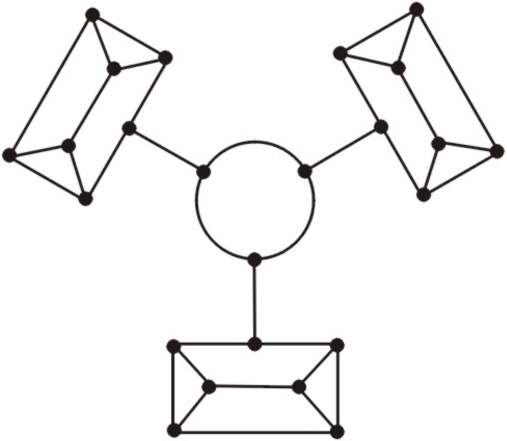

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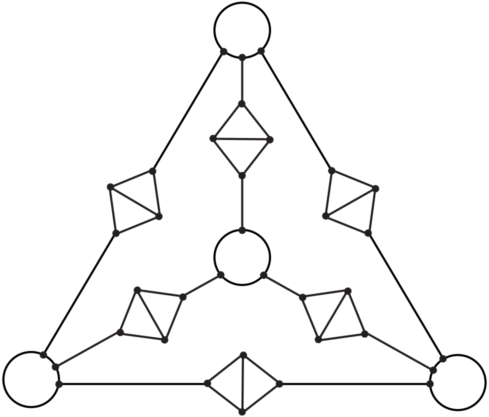

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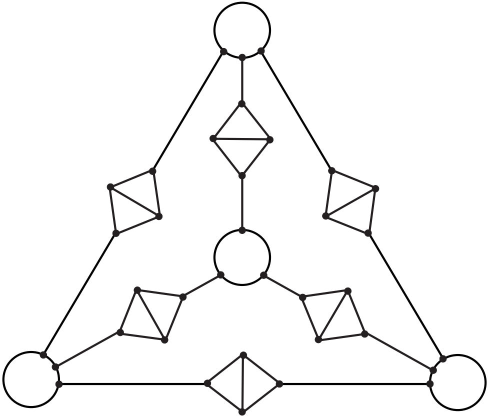

We believe that every 3-connected cubic upper-embeddable graph is amply upper-embeddable.

Final remarks and problems

Problem

What is the proportion of upper-embeddable cubic graphs in the class of all cubic graphs?

Thank you!