Dessins d'enfants: a historical perspective

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From Dessins to Riemann surfaces.

In 1974, I attended a seminar in Southampton given by Norman Biggs. The topic of this seminar was representation of map automorphism groups into the symplectic group. It is well known that a group of automorphisms of a Riemann surface of genus g has a representation as a group of $2g \times 2g$ symplectic matrices so I thought that if we could show that if there was a natural Riemann surface structure underlying a map then Biggs result would follow.

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An (abstract) Riemann surface is a two dimensional manifold (or a one dimensionsal complex manifold) (so we have charts and atlases) where the transition functions are holomorphic (=complex analytic.)

The Riemann surface associated to a map.

A *map* $\mathcal{M}$ is a two-cell embedding of a graph $\mathcal{G}$ into a surface $\mathcal{S}$. This means that we require the cells of $\mathcal{S} \setminus \mathcal{G}$ to be simply connected, so that they are topological polygons. The important features of a map are its vertices, edges and faces. The genus can then be computed using the Euler-Poincaré formula

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Now if an edge joins two vertices then we can form two darts, (ordered pairs (v, e) where v lies on the edge e.) Now on the set Ω of darts there is an obvious involution X which interchanges the two darts belonging to each edge. If we consider all the darts pointing to a vertex v, then anticlockwise orientation around v gives us an m-cycle, Y where m is the valency of v. We then see that $Y^{-1}X$ describes the darts around a face. (Note that $Y^{-1}X$ is conjugate to XY^{-1}.)

The Riemann surface associated to a map continued

Thus if m is the LCM of the vertex valencies and n is the LCM of the face valencies, then we have the relations

$$X^2 = Y^m = (XY)^n = 1.$$

Let $G = gp\langle X, Y\rangle$ denote the group generated by X and Y and let $\Gamma = \Gamma(2, m, n)$ denote the $(2, m, n)$ triangle group which has presentation $(x, y | x^2 = y^m = (xy)^n = 1)$. There is then an epimorphism $\theta : \Gamma \longrightarrow G$, defined by $\theta(x)$=X,$\theta(y) = Y$. Let $H = G_\alpha$, the stabiliser of the dart α.

Continuation

Then the index $|G : H| = N$ where N is the number of darts. Now let $M = \theta^{-1}(H)$. Then is a subgroup of Γ of index N and we can compute that the genus of $\mathbf{H}/M$ and we find it is equal to the genus of the map $\mathcal{M}$. We associate the Riemann surface $X = \mathbf{H}/M$ to the map $\mathcal{M}$. It follows that every automorphism of the map $\mathcal{M}$ is an automorphism of the Riemann surface X and then Biggs' result automatically follows.

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Thus given a map $\mathcal{M}$ we can construct a permutation group which is an epimorphic image of a triangle group. The stabilizer of a dart then gives a subgroup M of a triangle group which then gives a Riemann surface $X = X(\mathcal{M}) = \mathbf{H}/M$.

The fundamental group of a map

The importance of this subgroup M of $\Gamma(2, m, n)$ was realised in a paper of G.A. Jones and D.S. in 1978, "Theory of Maps on Orientable surfaces". There it was called a map subgroup, but it behaves just like a fundamental group. One map $\mathcal{M}_1$ covers another map $\mathcal{M}_2$ iff $M_1 < M_2$, and two maps of type $\{m, n\}$ are isomorphic iff their map subgroups are conjugate in $\Gamma(2, m, n)$. For this reason, I will now call M the fundamental group of $\mathcal{M}$.
Also, a map is regular (meaning its automorphism group acts transitively on the darts) iff $M \triangleleft \Gamma$.

Fundamental theorem 1. The uniformization theorem.

Every simply connected Riemann surface is isomorphic to
(i) The Riemann sphere $\mathbf{P}^1(\mathbf{C})$
(ii) The complex plane $\mathbf{C}$,
(iii) The upper half-plane $\mathbf{H}$
(The latter could be replaced by the open unit disc $\mathbf{D}$.)

The Geometry of these spaces

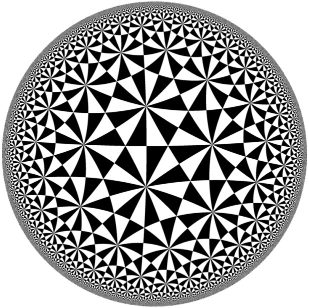
The Riemann sphere carries spherical geometry (positive curvature), The complex plane carries Euclidean geometry, (zero curvature) and the upper half-plane carries hyperbolic geometry (negative curvature).

Triangle groups

We can tessellate these spaces by triangles with angles $\pi/l, \pi/m, \pi/n$. If

$$\frac{1}{l} + \frac{1}{m} + \frac{1}{n} <, =, > 1$$

we get a tessellation of $\mathbf{H}$, $\mathbf{C}$, or $\mathbf{P}^1(\mathbf{C})$ respectively.



Triangle groups

Let a, b, c be the reflection in the sides of the above triangle. Then a, b, c generate a group with presentation

$$\langle a, b, c | a^2 = b^2 = c^2 = (ab)^l = (bc)^m = (ca)^n = 1. \rangle \tag{1}$$

This group is called an extended triangle group $\Gamma^*(l, m, n)$. To get an ordinary triangle group, we let $x = ab$, $y = bc$, $z = ca$ and then we get the group

$$\langle x, y, z | x^l = y^m = z^n = xyz = 1. \rangle \tag{2}$$

This is called the triangle group $\Gamma(l, m, n)$.

Hypermaps

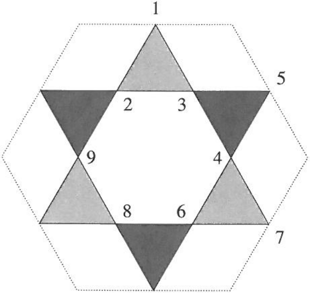
We have seen that maps can be studied via their fundamental groups, which are subgroups of the triangle group $\Gamma(2, m, n)$. The '2' here might seem a bit restrictive. What corresponds to subgroups of $\Gamma(l, m, n)$, where l might not be 2?
In 1975, Robert Cori wrote a paper "Un Code pour les graphs planaires et ses applications". Here he introduced hypermaps. (hypergraph embeddings) . (Notice that he restricted himself to planar hypermaps because he had applications to computer science in mind.) A hypergraph is like a graph except that "edges" can have more than two vertices. A simple example is the Fano plane, which has seven edges each containing three vertices.

Hypermaps continued

In Cori's theory we have a set S of hypervertices and a set A of hyperedges. The set $B = S \cap A$ are called the brins. (rather like vertices). Each component of the complement $X \setminus (S \cup A)$ is called a hyperface.

A genus 1 hypermap

D. Singerman, J. Wolfart: Cayley Graphs, Cori Hypermaps, and Dessi



The Fano plane in Klein's surface

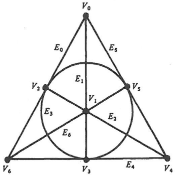
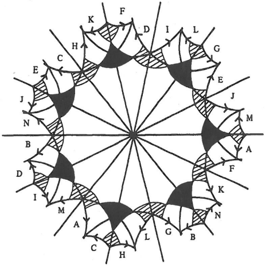


FIG. 2

There is a homomorphism $\theta:\Gamma(3, 3, 7) \rightarrow G$, given by $\theta(x) = \sigma$, $\theta(y) = \alpha$. If H is the kernel of θ then by Proposition 2, H is a hypermap subgroup for a regular hypermap. This is pictured in Fig. 3.



The associated permutation group

Notice that if we go around the hypervertices and hyperedges of the genus 1 hypermap we get the following permutations of the brins. $\sigma = (1, 3, 2)(4, 7, 6)(3, 9, 6)$
and
$\alpha = (5, 4, 3)(2, 8, 4)(3.9.6)$
and their product is $(1, 5, 7)(2, 8.4)(3, 9, 6)$ These permutations generate a group H isomorphic to $C_3 \times C_3$. There is an epimorphism from $\Gamma(3, 3, 3) \longrightarrow G$ whose kernel is a genus 1 surface group. This shows that we have a regular hypermap of type (3,3,3) on the torus.

Bipartite maps

From a hypermap we can easily get an associated bipartite map. Just put a white vertex in a hyper face and a black vertex in a hyper edge and join them if the hypervertex intersects the hyperedge. In the above example we just get a $K_{3,3}$ in the torus.

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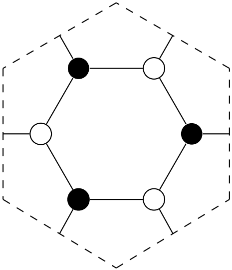
From a hypermap we can easily get an associated bipartite map. Just put a white vertex in a hyper face and a black vertex in a hyper edge and join them if the hypervertex intersects the hyperedge. In the above example we just get a $K_{3.3}$ in the torus. A bipartite map is what Grothendieck called a dessin d'enfant. Now given a map, we can convert it into a bipartite map by placing a white vertex in between two vertices of the map.

The standard embedding of $K_{3,3}$



Algebraic functions

An analytic function $w(z)$is called an *algebraic function* if it satisfies a functional equation

$$A(z,w) = a_0(z)w^n + a_1(z)w^{n-1} + \cdots + a_n(z) = 0, \quad a_0(z) \neq 0.$$

Here, the $a_i(z)$ are polynomials in z , with coefficients in some subfield F of the complex numbers. For example, F could be the field of complex numbers $\mathbf{C}$, the field of real numbers $\mathbf{R}$ the field $\overline{\mathbf{Q}}$ of algebraic numbers, or the field $\mathbf{Q}$ of rational numbers.

Branch points and critical values

In some senses the Riemann surface of $A(z, w)$ could be regarded as a graph of the equation $A(z, w) = 0$.
Now for most values of z there are n values of w, but for some isolated values of z there will be less than n values of w. In these cases the n sheets intersect at some points. These points are called branch-points and their images on the sphere are called critical values.

Riemann surfaces

The above equation $A(z, w) = 0$ defines w as an n-valued function of z, so w is not a function defined in $\mathbf{C}$ or the Riemann sphere $\mathbf{C} \cup \{\infty\}$. (The Riemann sphere is denoted by $\mathbf{P}^1(\mathbf{C})$.) So what we do is to is to construct an n-sheeted cover of the Riemann sphere where each sheet carries one of the solutions of the equation $A(z, w) = 0$.

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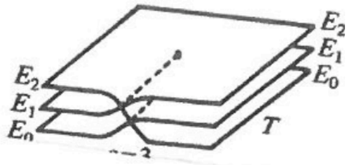
This n-sheeted cover is called the Riemann surface of the algebraic function (or curve w.)


E_2
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Fundamental theorem 2. Riemann's existence theorem

Every abstract compact Riemann surface is the Riemann surface of an algebraic function.

This usually takes a whole textbook of around 300 pages to prove, e.g. G.Springer, "Introduction to Riemann surfaces"

Fundamental theorem 3. Belyi's theorem

As we have seen, an algebraic curve and hence a Riemann surface can be defined over a subfield F of the complex numbers. Usually, there is no geometric connection between the field and the Riemann surface. A function $\beta; \mathcal{S} \longrightarrow \mathbf{C}$ is called a Belyi function if β has at most three critical values. These could be taken to be $0, 1, \infty$

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Belyi's theorem: A compact Riemann surface $\mathbf{X}$ is defined over the field $\overline{\mathbf{Q}}$ of algebraic numbers if and only if there exists a Belyi function $\beta : \mathbf{X} \longrightarrow \mathrm{P}^1(C)$. The Riemann surface of a dessin d'enfants gives such a surface as the vertices,edges and face-centres can be taken to be the inverse images of $0, 1, \infty$ under β.

Grothendieck's esquisse

The importance of Belyi's theorem was noted by Grothendieck in his "Esquisse d'un programme." There he wrote Never, without a doubt was such a deep and disconcerting result proved in so few lines. He pointed out the importance of dessins. He goes on to write of this theorem and goes on to write

To me, its essential message is that there is a profound identity between the combinatorics of finite maps on the one hand, and the geometry of algebraic curves defined over number fields on the other.

Personal history

Around 1990, Gareth and I were invited to be on the Jury in Paris, examining a student, Laurence Bessis, of Antonio Machi, who developed the theory of hypermaps with Robert Cori. Tony Machi showed us a paper of Shabat and Voevodsky from the Grothendieck Festschrift, which contained many ideas that we had been working on for around five years. We then discovered the Esquisse. Soon after, we found that there was a conference on dessins in Luminy, which we invited ourselves to. There we met many people, including Sasha Zvonkin, who gave some lectures on trees, (nice examples of dessins!) and Manfred Streit, a student of Juergen Wolfart.