Censuses of graphs and maps of certain symmetry types

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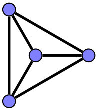
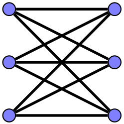
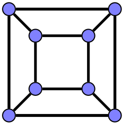


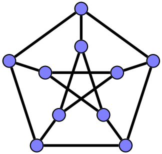
4, K_4



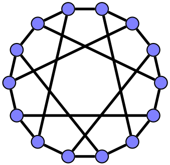
6, $K_{3,3}$



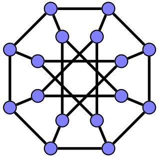
8, Q_3



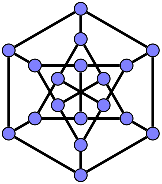
10, Petersen graph



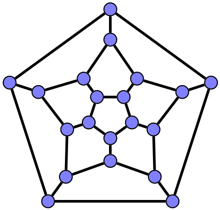
14, Heawood



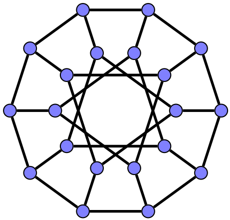
16, Möbius-Kantor



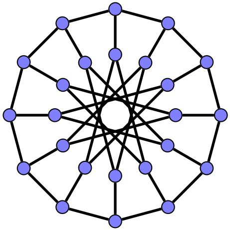
18, Pappus



20, Dodecahedron



20, Desargues

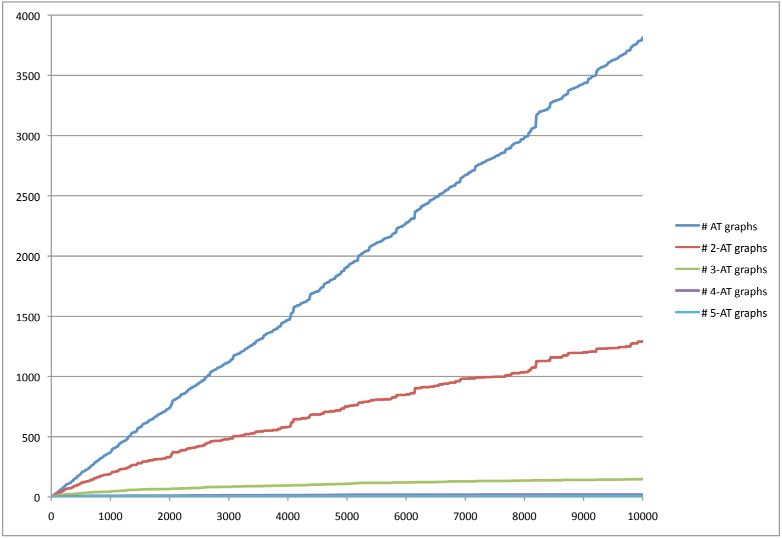


24, Nauru

Foster's census of cubic arc-transitive graphs

- These were the first 10 graphs from the Foster census of arc-transitive graphs of valence 3.
- A version of the census was first presented at the "Conference on Graph Theory and Combinatorial Analysis, Waterloo, 1966".
- In 1988, a book was published, containing graphs up to order 512 (some where missing).
- First complete version (up to 768 vertices) was obtained by Conder and Dobcsányi in 2001.
- The census is now extended up to 10 000 vertices.
- There are 3 815 such graphs (on up to 10 000), 1 293 of them are 2-arc-transitive, 149 of them are 3-arc-transitive, 20 of them are 4-arc-transitive, and 7 of them are 5-arc-transitive.

The number of cubic arc-transitive graphs



How did Marston do it?!

Marston's method relies on the following result of William Tutte.

Theorem

If Γ is a connected cubic arc-transitive graph, then $|\mathrm{Aut}(\Gamma)_v| \le 48$.

$$|\mathrm{Aut}(\Gamma)| \le 48|V(\Gamma)|$$

How does that help?

Observation

For any given integers k (e.g. $k = 3$) and m (e.g. $m = 48$), there exists a finite set $\mathcal{T}$ of triples (U, H, a) where U is a finitely presented group generated by a subgroup H and an element a, such that the following holds:
For any connected k-valent graph Γ and any arc-transitive group $G \le \mathrm{Aut}(\Gamma)$ satisfying $|G_v| \le m$, there exists a triple $(U, H, a) \in \mathcal{T}$ and an epimorphism $\wp \colon U \to G$, such that:

- *$\wp$ maps H isomorphically onto G_v;*
- *$\wp(a)$ maps v to a neighbour of v.*

Consequence: Every k-valent graph on at most M vertices admitting a G-arc-transitive group satisfying $|G_v| \le m$ can be obtained as a coset graph $\mathrm{Cos}(U/N, HN/N, aN/N)$ for some normal subgroup $N \le U$ of index at most mM.

Demonstration – cubic case

TASK: Find all connected cubic arc-transitive graphs with at most 30 vertices.

For $k = 3$ (cubic graphs) and $m = 48$ (Tutte's bound) , we need the set of triples (U, H, a) from the theorem: There are 7 such triples, explicitly determined by Djoković and Miller.

SWITCH TO MAGMA DEMONSTRATION

There are some practical issues:

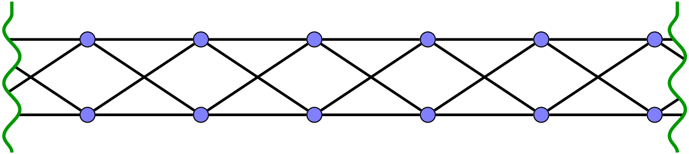
- Current version of magma computes LINS only up to index $5 \cdot 10^5$;
- For some types of FP groups, computing LINS is VERY hard even for much smaller indices;
- Examples of such groups are $G_1 = C_3 * C_2$ and $G_2^1 = S_3 *_{C_2} C_2^2$.
- G_1 up to index $3 \cdot 10^4$ is harder than G_4^1 up to index $24 \cdot 10^4$.

Tetravalent arc-transitive graphs

- Can we use the same approach for arc-transitive graphs of valence 4? Suppose we want to find them all up to M vertices.
- Problem: In the 4-valent case, $|\mathrm{Aut}(\Gamma)_v|$ is not bounded by a constant.
- But we don't need that! We just need a bound on $|\mathrm{Aut}(\Gamma)_v|$ for graphs that have at most M vertices. There certainly is one (say $M!$).

Bad news

- Consider $\mathrm{W}_t = C_t[2K_1]$:



- $|V(\mathrm{W}_t)| = 2t$;
- $|\mathrm{Aut}(\mathrm{W}_t)_v| = 2^t$ and so $|\mathrm{Aut}(\mathrm{W}_t)| = 2^t \cdot 2t$;
- In order to have $|\mathrm{Aut}(\mathrm{W}_t)| \leq 5 \cdot 10^5$, we need $t \leq 15$.
- Not very impressive!

Good idea

- Is this a hopeless project?

- Let's try to "isolate" the problem.

Theorem (Spiga, Verret, PP; "The lost paper")

Let Γ be a connected 4-valent G-arc-transitive graph. Then one of the following occurs:

- *Γ is a Praeger-Xu graph or one of a small number of graphs;*
- *$G_v^{\Gamma(v)}$ is doubly-transitive and $|G_v| \leq 2^4 3^6$;*
- *$|V(\Gamma)| \geq 2|G_v| \log_2(|G_v|/2)$.*

Example: $M = 640$

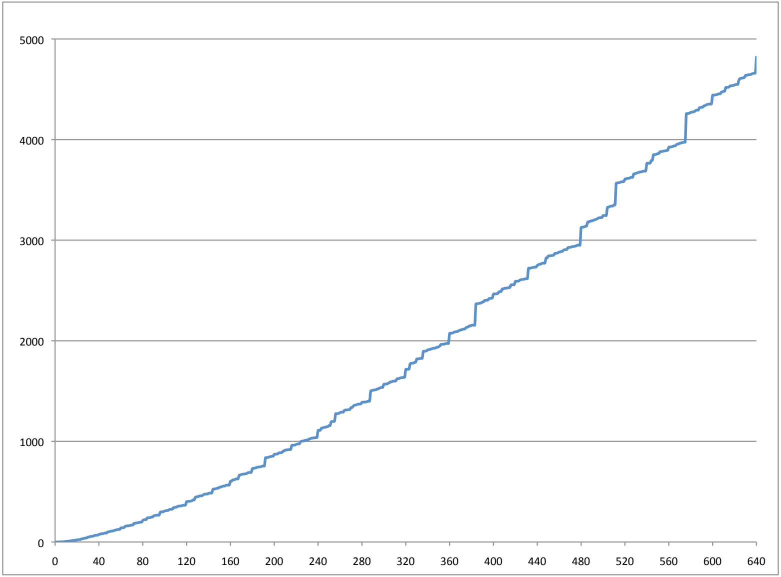
Let Γ be a 4-valent G-arc-transitive graph. Suppose Γ is not a PX-graph or a sporadic exception. Then one of the following holds

- G_v is 2-transitive on $\Gamma(v)$ and $|G_v| \leq 2^4 3^6$;
  - there are 9 universal triples (U, H, a), "largest" two determined by Weiss (1987) and the "smaller" ones by PP (2009).
  - a census of 2-arc-transitive 4-valent graphs up to 768 vertices was computed in 2009.
- $|G_v| \leq 32$;
  - there are 11 universal triples (U, H, a), determined by Djoković (1980).

A census of 4-valent AT graphs (Spiga, Verret, PP)

- All 4-valent arc-transitive graphs on at most 640 vertices are known.
- There is $4\,820$ of such graphs.
- This is more than the number of cubic-arc-transitive graphs on $10\,000$ vertices!

The number of 4-valent arc-transitive graphs



A census of 2-out-valent AT digraphs (Spiga, Verret, PP)

- A similar approach works for arc-transitive digraphs of out-valence 2. (equivalently, 4-valent graphs admitting a half-arc-transitive group action).
- All such digraphs on at most $1\,000$ vertices are known.
- There is $26\,457$ such digraphs, giving rise to $11\,941$ underlying graphs. Out of the latter, $8\,695$ are arc-transitive, and $3\,246$ are half-arc-transitive.

Cubic vertex-transitive graphs

We extended the Foster census from valence 3 to valence 4. Can we extend it to vertex-transitive graphs of valence 3?

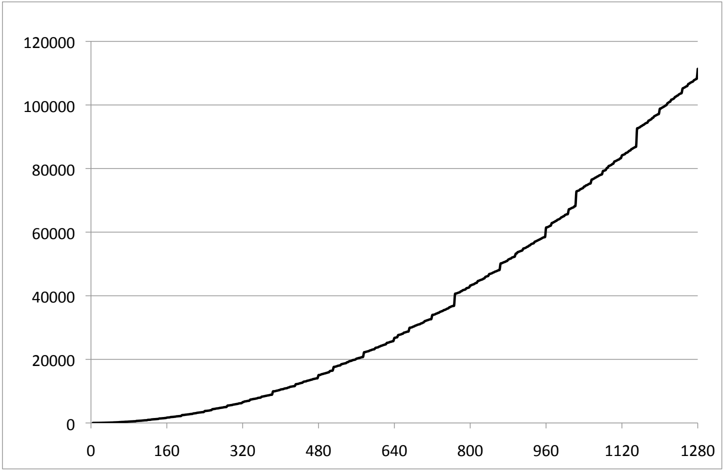
Let Γ be cubic G-vertex-transitive, let $v \in V(\Gamma)$, and let $G_v^{\Gamma(v)}$ be the permutation group induced by G_v on $\Gamma(v)$.

- If $G_v^{\Gamma(v)}$ is transitive, then G is arc-transitive. DONE
- If $G_v^{\Gamma(v)}$ is trivial, then G acts regularly on $V(\Gamma)$ and $\Gamma = \mathrm{Cay}(G, S)$.
  Here $G = |V(\Gamma)|$ ... small .. EASY
- If $G_v^{\Gamma(v)} \cong C_2$, then we're in troubles ($|G_v|$ can be as big as $2^{n/4}$). Fortunately, there is a correspondence between suchs graphs on $2n$ vertices and tetravalent G-arc-transitive graphs with $G_v^{\Gamma(v)} \cong D_4$ on n vertices.

A census of cubic VT graphs (Spiga, Verret, PP)

- All cubic vertex-transitive graphs up to order 1280 are known.
- There is $111\,360$ of them.
- Only $1\,434$ of them are non-Cayley.
- Only 482 of them are arc-transitive.

The number of cubic vertex-transitive graphs



Back to 4-valent AT graphs

Recall : in order to get all 4-valent AT graphs of order up to 640, we had to compute normal subgroups of "small" index of several infinite, finitely presented groups. Among others:

$$U = \langle x, y, a \mid x^2, y^2, a^2, [x, y] \rangle$$

of index at most $4 \cdot 640$. (This corresponds to $G_v \cong V_4$.)

Even though the index is small, applying finding them is computationally difficult!

Is there an alternative way ?

Regular maps

- The group $U = \langle x, a, y \mid x^2, y^2, a^2, [x, y] \rangle$ is related to the notion of regular maps.
- Let $\wp \colon U \to G$ be an epimorphism, and $x = \wp(x)$, $a = \wp(a)$, $y = \wp(y)$.
- Then $(G; x, y, a)$ determines a regular map :
  - vertices: cosets of $\langle a, y \rangle$;
  - edges: cosets of $\langle x, y \rangle$;
  - faces: cosets of $\langle x, a \rangle$;
  - incidence: non-empty intersection;
- To avoid degeneracy, we may require:
  - $|\langle x, y \rangle| = 4$;
  - $a \notin \langle x, y \rangle$

- Two regular maps (G_i, x_i, y_i, a_i), $i = 1, 2$, are isomorphic iff there is an isomorphism $f\colon G_1 \to G_2$ s.t. $f(x_1) = x_2$, $f(y_1) = y_2$, and $f(a_1) = a_2$.
- Note: if there is such an f, then it is unique.

TASK : Find all regular maps with up to M edges, up to isomorphism.

- Marston (2013) produced a census up to $1\,000$ edges, using "the standard method". Computations took several months.
- However, there seems to be a quicker method.

Finding regular maps with few edges

- First trick: rather then considering the regular map $G = \langle x, a, y \mid x^2, y^2, a^2, [x, y], \ldots \rangle$, consider the "rotation" subgroup G_0 generated by the $R = xa$ (face rotation) and $S = ay$ (rotation about a vertex).
- Note that $(RS)^2 = 1$ and $R^a = R^{-1}$, $S^a = S^{-1}$. Hence

$$G_0 = \langle R, S \mid (RS)^2, \ldots \rangle.$$

- Note: $|G : G_0| \leq 2$.
  - If $|G : G_0| = 2$, then (G, x, y, a) is orientable ;
  - If $|G : G_0| = 1$, then (G, x, y, a) is non-orientable ;

Reconstructing the regular map

- If (G, x, y, a) is orientable, then it can be uniquely constructed from (G_0, R, S):
  - Consider $G_0 \rtimes \langle a \mid a^2 \rangle$ with $R^a = R^{-1}$ and $S^a = S^{-1}$,
  - let $x = Ra$ and $y = aS$.
  - Then $(G_0 \rtimes \langle a \rangle, x, y, a)$ is the original regular map.
- If (G, x, y, a) is non-orientable, then (G_0, R, S) does not determine (G, x, y, a) uniquely (even though $G = G_0$).
- Namely, given (G_0, R, S), there might be several involutions $a \in G_0$, s.t. $R^a = R^{-1}$ and $S^a = S^{-1}$.
- But since we want to find all regular maps, we don't care.

Strategy:

- Find all triples (G_0, R, S) with $G_0 = \langle R, S\rangle$, $(RS)^2 = 1$, and $|G_0| \leq 4M$;
- Determine whether $\iota\colon R \mapsto R^{-1}$, $\iota\colon S \mapsto S^{-1}$ extends to an automorphism of G_0. For those that it does, do the following:
  - Let $G = G_0 \rtimes \langle\iota\rangle$, $a = \iota$, $x = Ra$, $y = aS$, and construct the regular map (G, x, y, a). This gives us all orientable regular maps with at most $2M$ edges.
  - Find all involutions $a \in G_0$, such that $R^a = R^{-1}$, $S^a = S^{-1}$. For each such a, construct the regular map (G_0, Ra, aS, a). This will give us all non-orientable regular maps with at most M edges.

Chiral maps

- If there is no automorphism ι of G_0 mapping (R, S) to (R^{-1}, S^{-1}), then the triple (G_0, R, S) is a chiral map .
    - Vertices: cosets of $\langle S \rangle$;
    - Faces: cosets of $\langle R \rangle$;
    - Edges: cosets of $\langle RS \rangle$;
    - Incidence: non-trivial intersection.
- This gives us all chiral maps on at most $2M$ edges.

To summarise: If we could find all triples (G, R, S) with $G = \langle R, S \rangle$, $(RS)^2 = 1$, and $|G| \leq 4M$, then it would be easy to get all:

- orientable maps with at most $2M$ edges (both chiral and regular);
- non-orientable maps with at most M edges.

Observe: This task is equivalent to finding all triples (G, R, t) with

$$G = \langle R, t \mid t^2, \dots \rangle$$

We call such a group G a $(2, *)$-group.

Construction a census of $(2,*)$-groups

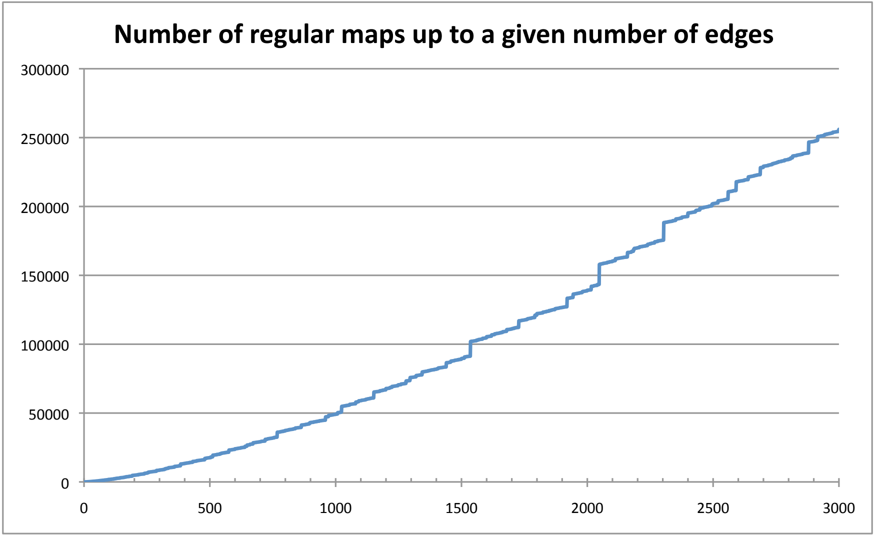
The $(2,*)$-groups up to order $4M$ can be constructed inductively:

- First find all such $(2,*)$-groups that contain no proper non-trivial abelian normal subgroups. These are:
  - cyclic of prime order;
  - $(2,*)$-groups G satisfying $\mathrm{soc}(G) \leq G \leq \mathrm{Aut}(\mathrm{soc}(G))$ with $\mathrm{soc}(G)$ being a product of non-abelian simple groups.
- Then inductively compute extensions of these by elementary abelian groups, at each step throwing away the extensions that are not $(2,*)$-groups.
- At each step, one can also find all generating pairs (R,t) of the group G, $t^2 = 1$, up to conjugacy in $\mathrm{Aut}(G)$.

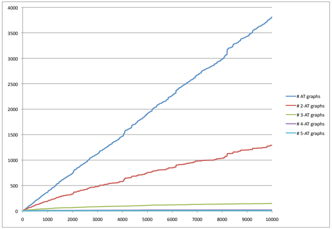
...

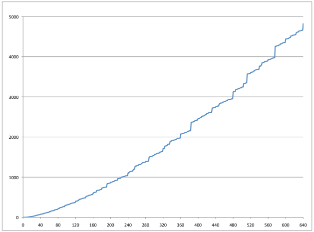
- After a few weeks of computations, MAGMA spit out all $(2,*)$-groups of order up to $6\,000$ (together with all the generating pairs (R,t)).
- There are $129\,340$ $(2,*)$-groups of order up to 6000, giving rise to $345\,070$ generating pairs.
- As a result, we obtained a complete list of all:
  - orientable maps with at most $3\,000$ edges (both chiral and regular);
  - non-orientable regular maps with at most $1\,500$ edges.
- We also computed all non-orientable regular maps with up to $3\,000$ edges that have at least one orientable Wilson's mate.

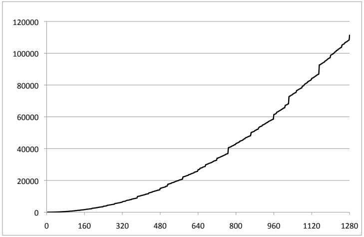
The number of regular maps

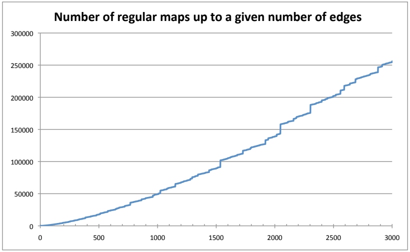


Asymptotic enumeration









Asymptotic enumeration

$$n^{C \log n}$$

Upper bound

Theorem (Lubotzky)

There exists a positive constant a such that the number of isomorphism classes of groups which are d-generated and of order at most n is at most $n^{ad \log n}$.

$$+$$

For these structures, Aut is bounded by a polynomial function of n (except for a "small" number of exceptions).

Lower bound

For the lower bound, further results are needed:

- (Jaikin-Zapirain) The number $f_d^{[p]}(m)$ of d-generated p-groups of order p^m is:

$$p^{\frac{1}{4}(d-1)m^2+o(m^2)} \leq f_d[p](m) \leq p^{\frac{1}{2}(d-1)m^2+o(m^2)}.$$

- (PSV) An analogue result for 2-groups generated by d involutions:

$$2^{\frac{(d-2)^2}{8d}m^2+o(m^2)} \leq g_d[2](m) \leq 2^{\frac{1}{2}(d-2)m^2+o(m^2)}$$

- A paper T. W. Müller and J.-C. Schlage-Puchta, Normal growth of large groups, II, *Arch. Math.* **84** (2005), 289–291 , which implies that for any G-arc-transitive graph (of valence at least 3), there are at least $n^{C\log n} - D$ non-equivalent G-admissible regular covering projections of order at most n.