Generalised hemicuboctahedron

Daniel Pellicer

Regular polygons and polyhedra

Antiquity

Regular polygons and polyhedra

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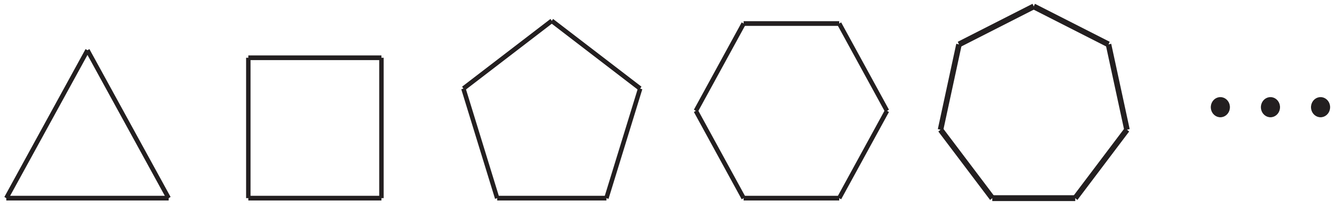
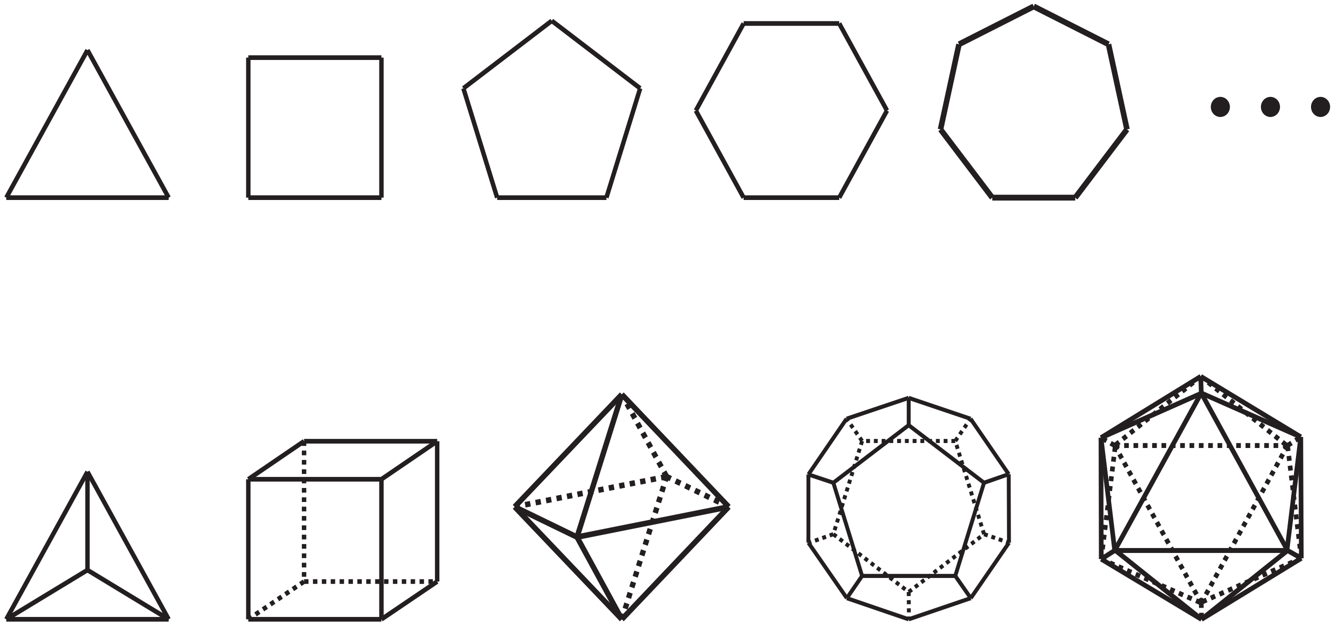


Regular polygons and polyhedra

Antiquity



Star polygons and polyhedra

Bradwardine

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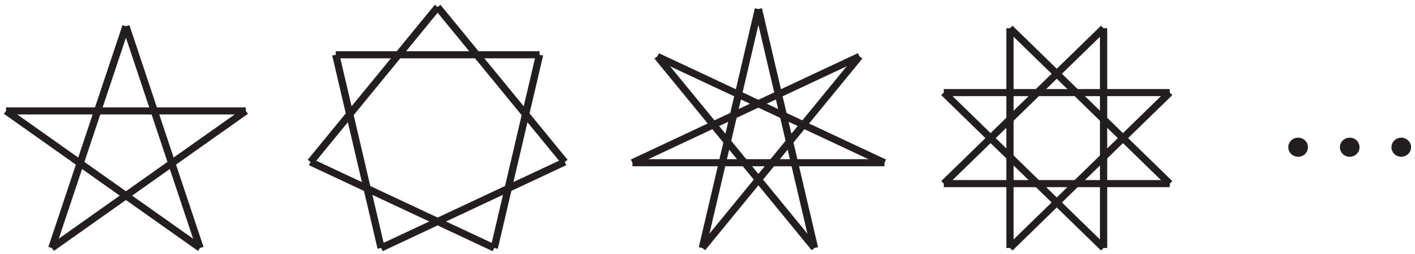

Star polygons and polyhedra

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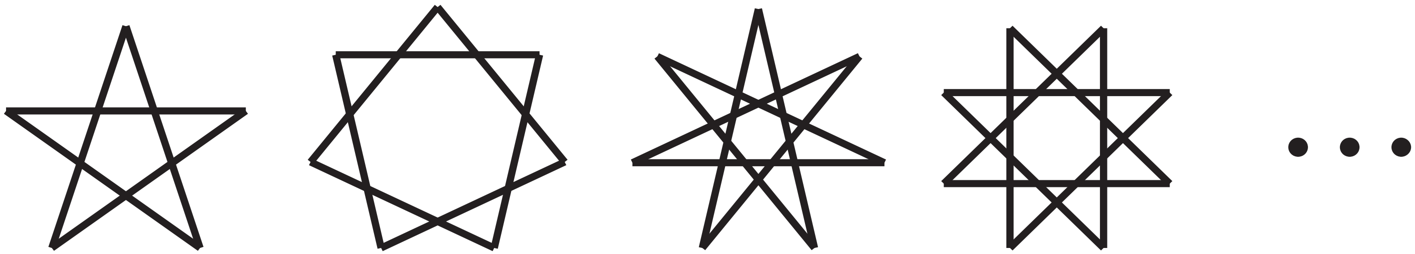

Kepler, Poinsot

Star polygons and polyhedra

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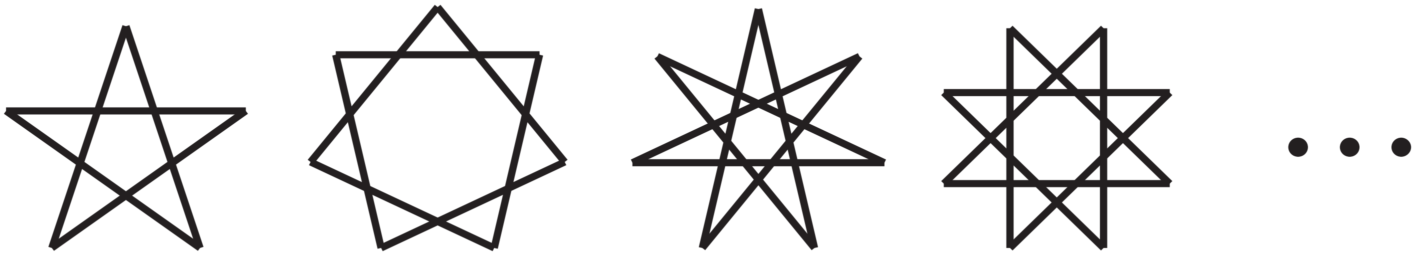

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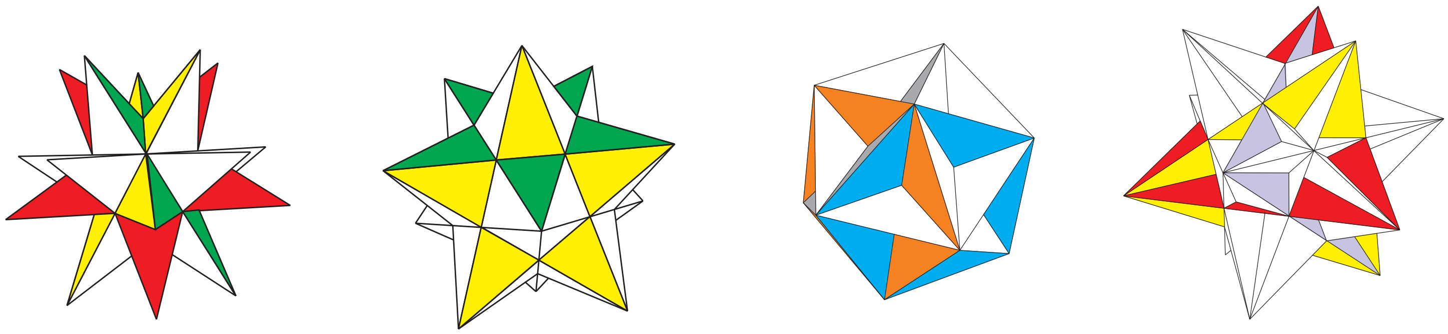

Finite regular polyhedra

Grünbaum

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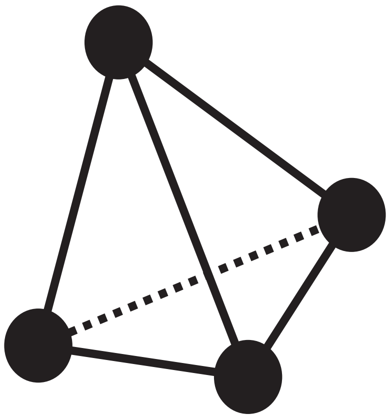

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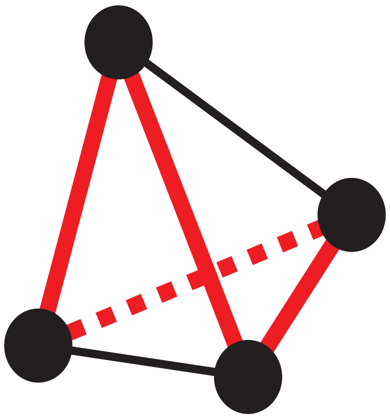

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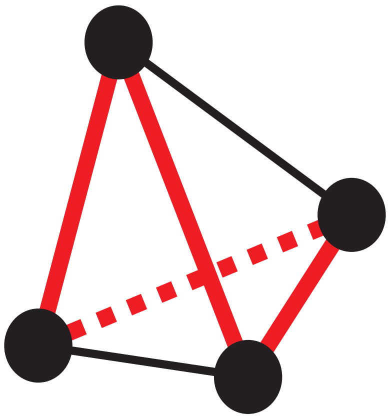

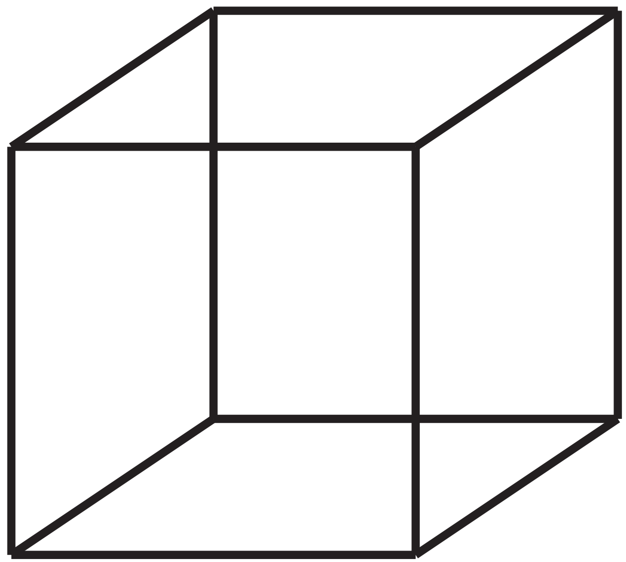

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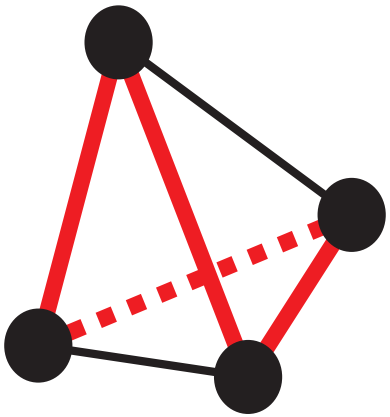

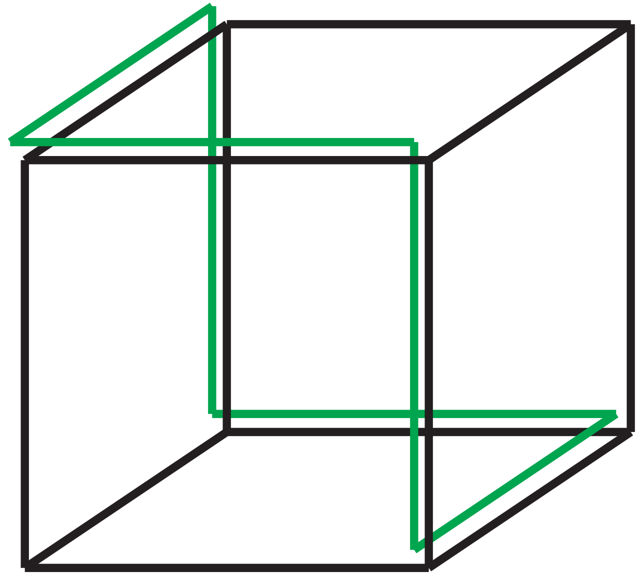

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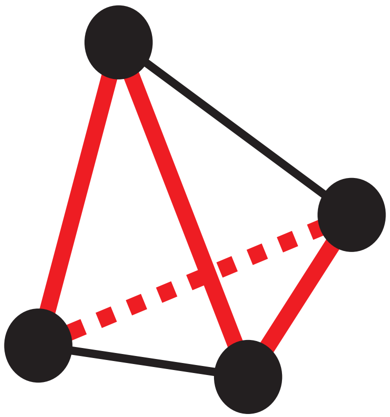

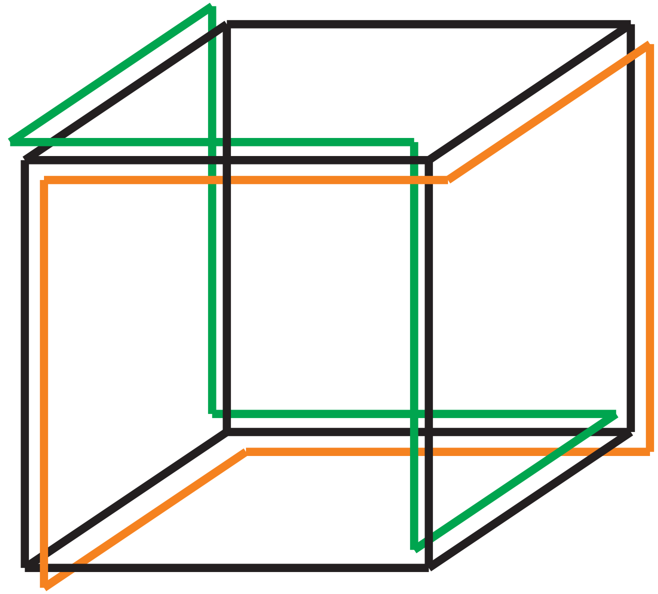

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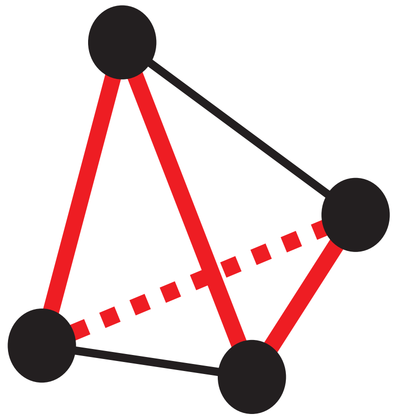

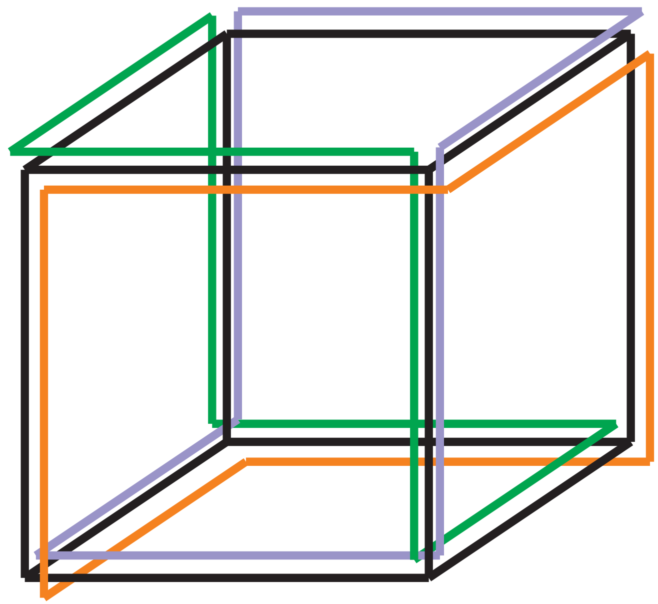

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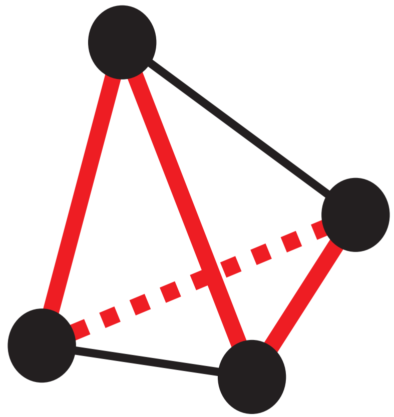

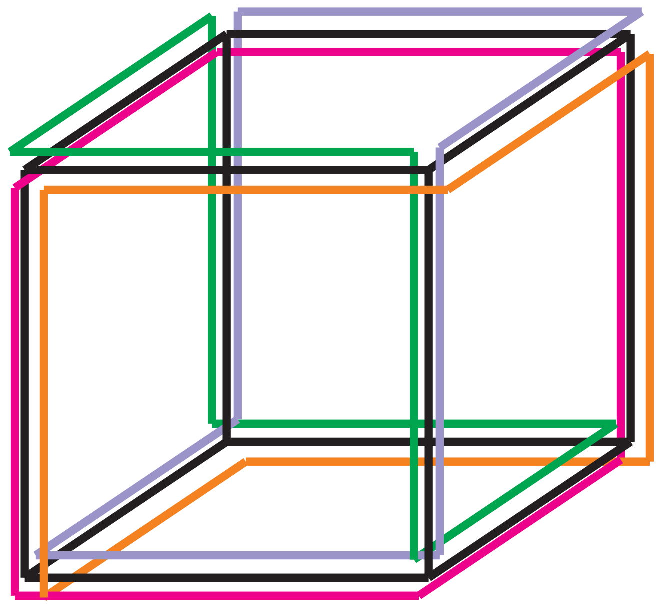

Finite regular polyhedra

18 finite regular polyhedra

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- 5 Platonic Solids

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- 9 Petrials

Finite regular polytopes

Finite convex regular n-polytopes

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- n-simplex

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Schläfli

Finite star polytopes

- Rank 2 $\longrightarrow$ infinitely many

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E. Hess, S. L. van Oss

Finite regular polytopes

Finite regular n-polytopes in $\mathbb{R}^n$

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P. McMullen

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Rank $n \geq 2$

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- k-faces are k-simplices ($k \leq n$)
- Geometric dual of the cube

Cross polytope

Automorphism group:

Cross polytope

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- $\mathbb{Z}_2^n \longrightarrow$ changing sign on each coordinate

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The Coxeter group B_n

2-orbit polytopes

Flag $\longrightarrow$ maximal totally ordered subset

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Combinatorial 2-orbit polytope $\longrightarrow$ Two flag orbits under the automorphism group

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rank 2: all polygons are combiantorially regular

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- Maybe others

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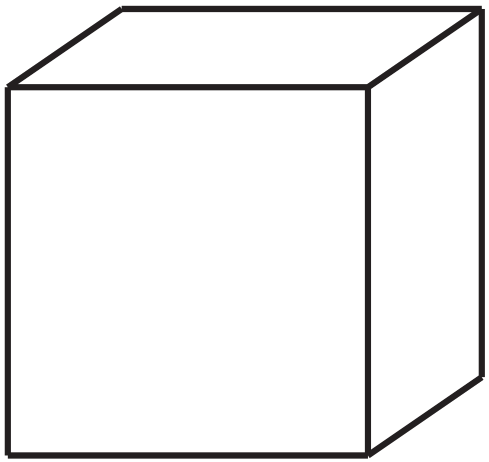
- No convex of dimension $n \geq 4$ (Matteo)
- No convex combinatorially 2-orbit n-polytope for $n \geq 4$ (Matteo)

Cuboctahedron



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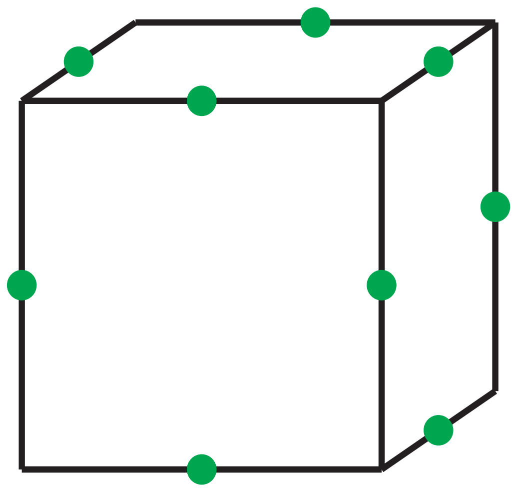

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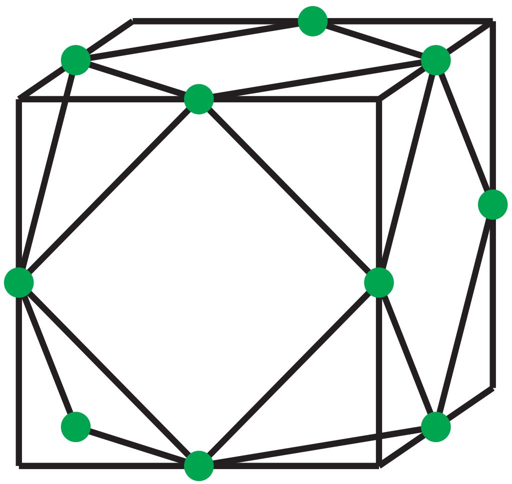

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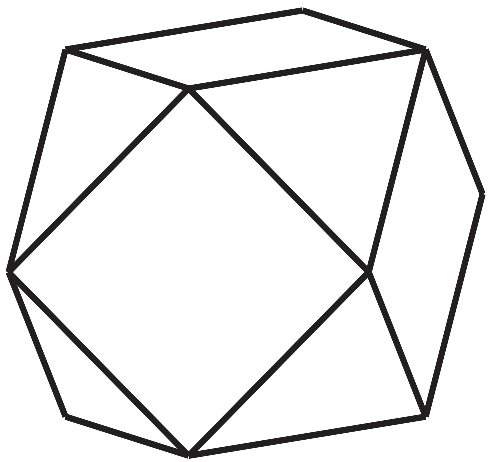

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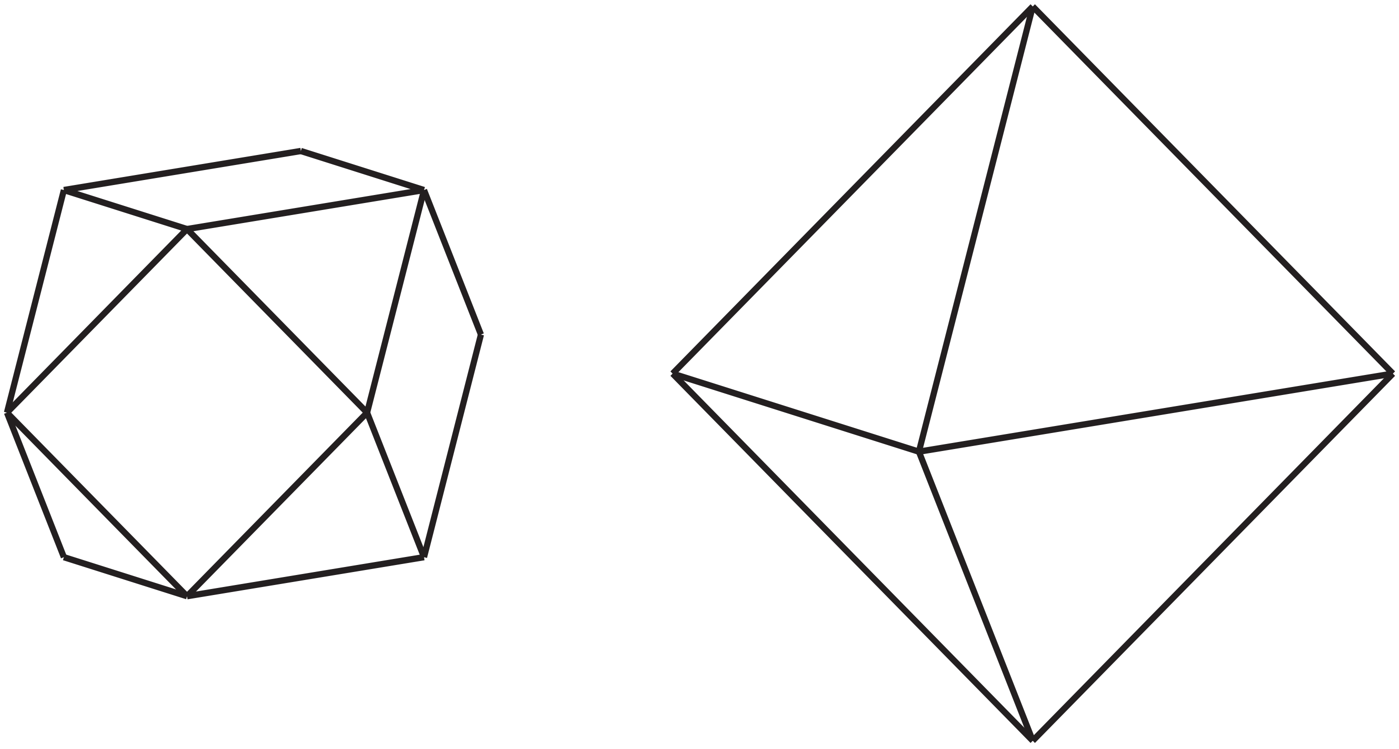

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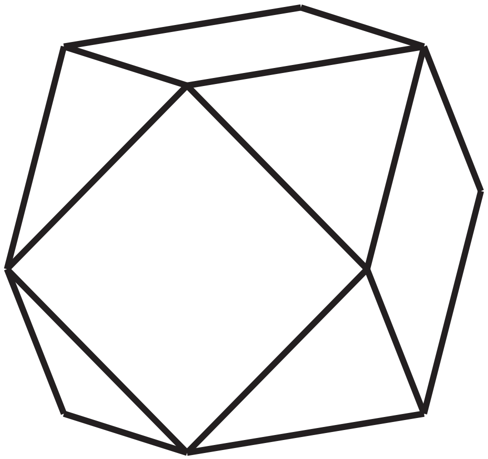

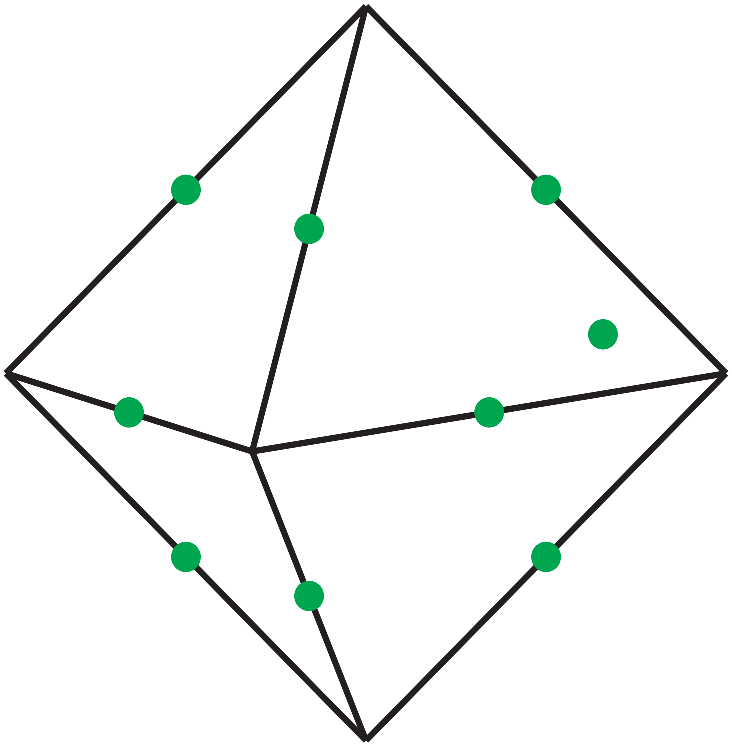

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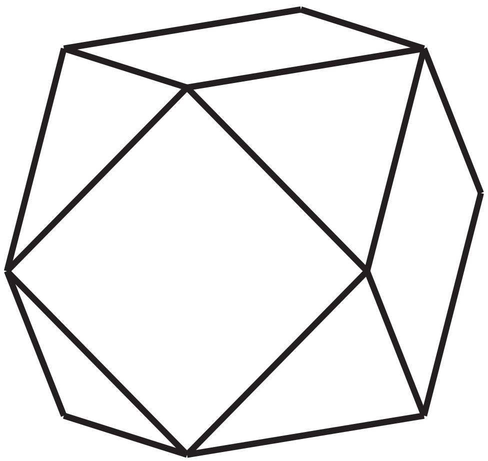

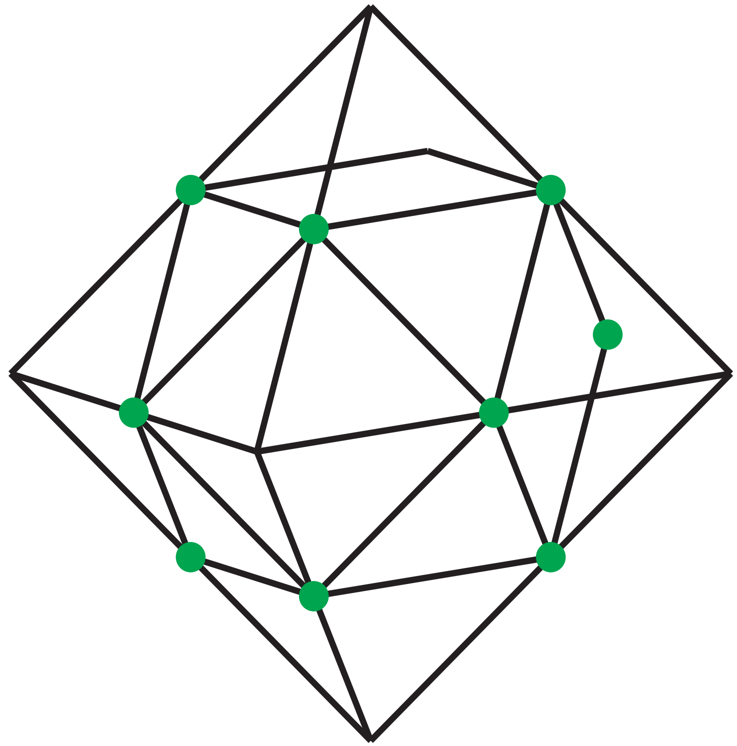

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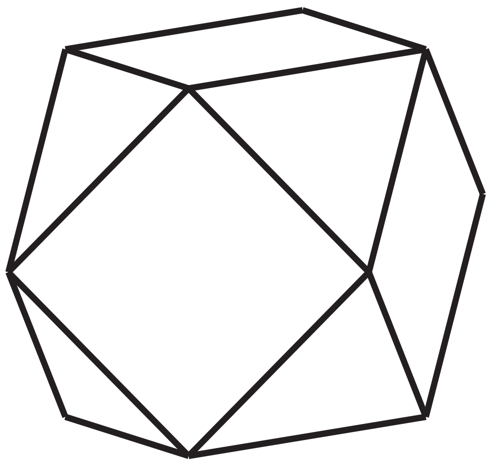
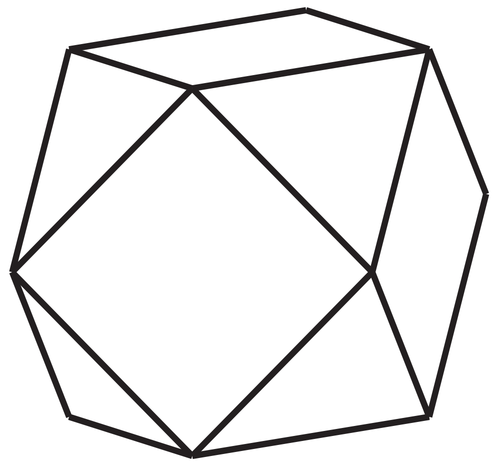
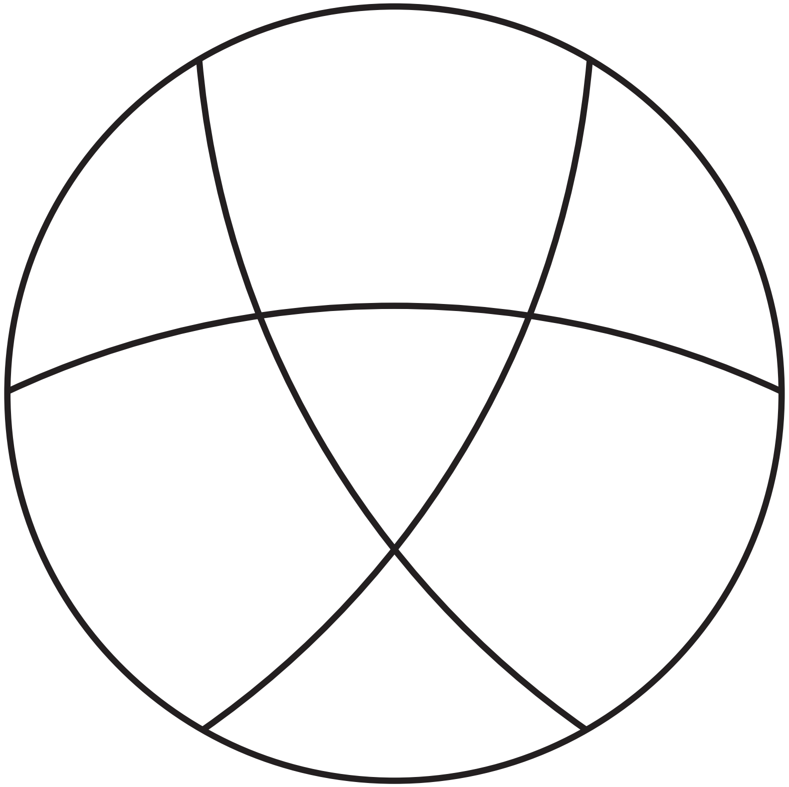




Hemicuboctahedron



Generalised hemicuboctahedron

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- The facets of the n-cross polytope are $(n-1)$-simplices
- The facets of the n-cross polytope admit a bipartition

Generalised hemicuboctahedron

n-hemicuboctahedron

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n-hemicuboctahedron

- Facets $\longrightarrow$

Generalised hemicuboctahedron

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Generalised hemicuboctahedron

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Generalised hemicuboctahedron

n-hemicuboctahedron

- Facets $\longrightarrow$
  - Half the simplicial facets of the n-Cross polytope
  - All $(n-1)$-cross polytopes from intersections with canonical hyperplanes
- The vertex-figures are isomorphic to the $(n-1)$-hemicuboctahedron

Generalised hemicuboctahedron

★ Every (simplicial) $(n-2)$-face belongs to precisely one $(n-1)$-simplex and to one $(n-1)$-cross polytope

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★ The quotient of the 4-hemicuboctahedron to the projective space is isomorphic to the Tomotope

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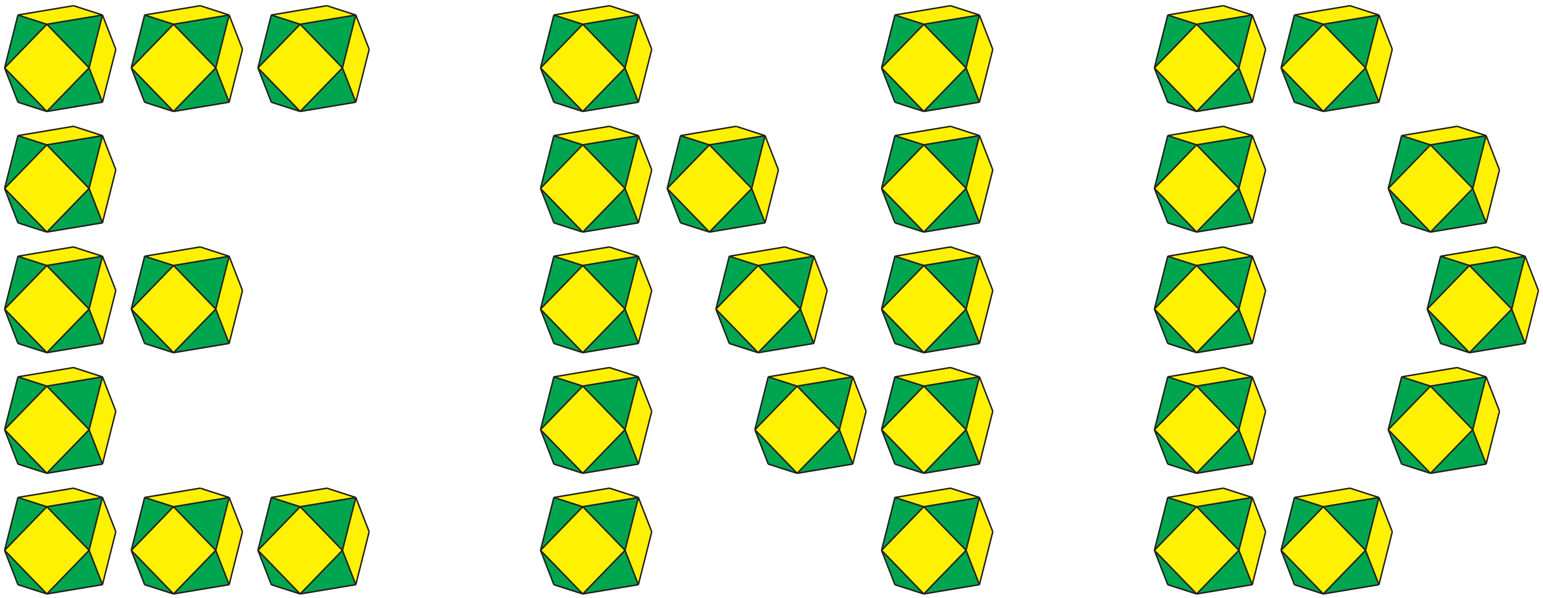
The Coxeter group D_n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