Nilpotent regular maps and dessins

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Original results on the nilpotent maps were done in collaboration with **M. Conder, S.F. Du, A. Malnič and M. Škoviera.**

Original results on the nilpotent dessins come from the thesis of my former PhD. student **Naer Wang** defended on May, 29, 2014 and from subsequent papers in progress by **Kan Hu, Naer Wang** and N.

Maps

Topological map A 2-cell decomposition of a surface = 2-cell embedding of a connected graph into a surface,

Surfaces preferably compact connected and orientable and their universal covers,

Combinatorial Map Equivalence classes of topological maps up to isotopy can be described by triples $(D; R, L)$

D a set of darts, L is an (dart-reversing) involution, R rotation system,

Category of Ormaps Combinatorial maps and their homomorphisms,

Vertices, Edges and Faces Orbits of R, L, and RL,

Regular maps

Monodromy group the group $\langle R, L\rangle$ is by definition transitive on the set D of darts,

Automorphism group permutations in $Sym(D)$ centralising the monodromy group form the automorphism group $Aut(M)$ of a map M,

semiregularity $Aut(M)$ is semiregular on D;

regularity if $Aut(M)$ is regular on D, then M is called regular;

in a regular map $Aut(M) \cong Mon(M)$ and $|D| = |Aut(M)| = |Mon(M)|$;

Nilpotent maps

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A nilpotent map can be view as a triple $(G; x, y)$, where $G = \langle x, y \rangle$ is a (finite) nilpotent group, $y^2 = 1$,

Two such maps (triples) are **equivalent** if there the assignment $x_1 \mapsto x_2$ and $y_1 \mapsto y_2$ extends to a group isomorphism,

Problem(s) considered

General problem: Characterisation (classification) of nilpotent maps.

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Since **non-abelian nilpotent maps are bipartite,** some of the ideas naturally generalise to **nilpotent regular hypermaps and dessins.**

Basic facts on the nilpotent maps

Theorem (Malnic, N., Škoviera)

A nilpotent map $(G; x, y)$ decomposes into S. Wilson's product: $G_2 \times Ss_m$, where $G_2 = \langle x_2, y_2 \rangle$ is the Sylow 2-group and Ss_m is an odd semistar,

Corollary

The automorphism group of a nilpotent map with a simple underlying graph is a 2-group.

Abelian regular maps

Theorem

A folklore result: Abelian regular maps are the semistars, balanced regular embeddings of bouquets of loops and embeddings of n-dipoles with exponent 1.

An argument: the group is either cyclic, or $Z_2 \times Z_n$, check how one can generate them by an involution and a non-involution.

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Conclusion: **Nilpotent maps form a broad class of maps**

Nilpotent maps of class two

Theorem (Malnic, N., Škoviera EJC 2012)

Every nilpotent map of class two belong to one of the two families defined by presentations:

$$G_1(n) = \langle x, y | x^{2^n} = y^2 = 1, [x, y] = x^{2^{n-1}} \rangle, n \geq 2$$

and

$$G_2(n) = \langle x, y, z | x^{2^n} = y^2 = z^2 = [x, z] = [y, z] = 1, z = [x, y] \rangle.$$

Nilpotent maps of class three I

joint work: Ban, Du, Liu, N., Škoviera (2014+):

$$
\begin{aligned}
H_1(k) &= \langle a, b \mid a^{2^k} = b^2 = 1,\ [a,b] = c,\ [c,a] = [c,b] = e, \\
&\quad [e,a] = [e,b] = 1\rangle, k \geq 1; \\
H_2(k) &= \langle a, b \mid a^{2^k} = e, b^2 = 1,\ [a,b] = c,\ [c,a] = [c,b] = e, \\
&\quad [e,a] = [e,b] = 1\rangle, k \geq 1; \\
H_3(k) &= \langle a, b \mid a^{2^k} = b^2 = 1,\ [a,b] = c,\ [c,a] = d,\ [c,b] = 1, \\
&\quad [d,a] = [d,b] = 1\rangle, k \geq 2; \\
H_4(k) &= \langle a, b \mid a^{2^k} = d, b^2 = 1,\ [a,b] = c,\ [c,a] = d,\ [c,b] = 1, \\
&\quad [d,a] = [d,b] = 1\rangle, k \geq 2;
\end{aligned}
$$

Nilpotent maps of class three II

$$
\begin{aligned}
H_5(k) &= \langle a, b \mid a^{2^k} = b^2 = 1,\ [a,b] = c,\ [c,a] = d,\ [c,b] = e,\\
&\quad [d,a] = [d,b] = [e,a] = [e,b] = 1\rangle, k \geq 2;\\
H_6(k) &= \langle a, b \mid a^{2^k} = d, b^2 = 1,\ [a,b] = c,\ [c,a] = d,\ [c,b] = e,\\
&\quad [d,a] = [d,b] = [e,a] = [e,b] = 1\rangle, k \geq 2;\\
H_7(k) &= \langle a, b \mid a^{2^k} = e, b^2 = 1,\ [a,b] = c,\ [c,a] = d,\ [c,b] = e,\\
&\quad [d,a] = [d,b] = [e,a] = [e,b] = 1\rangle, k \geq 3;\\
H_8(2) &= \langle a, b \mid a^4 = de, b^2 = 1,\ [a,b] = c,\ [c,a] = d,\ [c,b] = e,\\
&\quad [d,a] = [d,b] = [e,a] = [e,b] = 1\rangle.
\end{aligned}
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Projection and lifting of nilpotent maps

Given a nilpotent map one can form a chain of quotients by factoring out central subgroups, and one can do it in the way that for each class $c' < c$ a map of class c' is included!

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Example: The underlying graphs of maps of class 2 are 4-cycles with multiple edges, or dipoles. Consequence: Non-abelian nilpotent maps are **bipartite**!

Summary of basic properties

Theorem

The family of nilpotent maps of bounded class is:

(a) *closed under Wilson's products,*

(b) *closed under taking quotients,*

(c) *admits lower central series intersecting all the families of smaller classes.*

Universal map of class c

$\Gamma = C * C_2 \cong \langle x, y | y^2 = 1 \rangle$, (the universal map)

Set $\Gamma_1 = [\Gamma, \Gamma]$ and in general $\Gamma_{n+1} = [\Gamma, \Gamma_n]$.

Then the universal map of class c is

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Conjecture: there are just finitely many nilpotent maps of bounded class without parallel edges.

The number of vertices of the universal map?

Group theoretical formulation:
Determine the size of $U(c) = H(c)/core(\bar{x})$.

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Determine the size of $U(c) = H(c)/core(\bar{x})$.

Surprise: $U(c)$ is finite!!!

A result by Du, N., Škoviera

later, with a significant help of M. Conder we were able to determine precise formula for the size of $|U(c)|$, its number of vertices, its type and genus.

Three integer sequences

Set $h_1 = 0$, and then for $n \geq 2$ take

$$h_n = \frac{1}{n} \sum_{d|n} \mu(n/d) \Big(\sum_{0 \leq i \leq d/3} (-1)^i \frac{d}{d-2i} \binom{d-2i}{i} 2^{d-3i} \Big),$$

$$g_n = h_1 + h_2 + \cdots + h_n = \sum_{j=1}^{n} h_n,$$

$$f_n = g_1 + g_2 + \cdots + g_n = \sum_{i=1}^{n} g_n.$$

As usual, $\mu(m)$ denotes the Möbius function.

The number of vertices of the universal map of class c

Theorem (Labute77)

The rank of the factor group Γ_{n-1}/Γ_n is g_n, for all $n \geq 2$.

Theorem (Conder, Du, N.,Škoviera, 2014...)

For every integer $c \geq 1$ there exists a unique nilpotent regular map $\mathcal{U}_c$ of class c with simple underlying graph, having 2^{1+f_c} vertices, and type $\{2^c, 2^{c-1}\}_{2^c}$.
Furthermore, this map is 'universal', in the sense that every nilpotent regular map of class at most c with simple underlying graph is a quotient of $\mathcal{U}_c$.

Generation of simple universal maps of given class

Just add relation $x^{c-1} = 1$ to the presentation of $H(c)$, since $core(x) = \langle x^{c-1} \rangle$ and identify all the normal subgroups.
Example: There are 21 nilpotent maps of class c, $2 \leq c \leq 4$, with a simple underlying graph.

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M2.8.1	$[x^2, y^2, (xy)^4]$,
M3.64.1	$[x^4, y^2, x^{-1}yx^{-2}yxyx^2y, (x^{-1}y)^4(xy)^4]$,
M3.32.1	$[x^4, y^2, (x^{-1}y)^4, (xyx^{-1}y)^2]$
M3.32.2	$[x^4, y^2, x^{-1}yx^{-1}yxyxy]$
M3.16.1	$[x^2, y^2, (xy)^8]$

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Conclusion: **Nilpotent maps of given class can be sorted into families characterised by the smallest members which have no parallel edges!**

Note on the sequences h_n and g_n

First values of h_n are

$$0, 1, 1, 1, 2, 2, 4, 5, 8, 11, 18, 25, 40, 58, 90, 135, 210, 316, \ldots$$

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Sloane on-line Encyclopedy of integer sequences:
"A006206 Number of aperiodic binary necklaces of length n with no subsequence 00, excluding the necklace "0". (Formerly M0317)

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The partial sums (sequence $g_n + 1$ appears there as well as A001461 in connection with some results in knot theory and field theory)

Two-step classification

1. Finding a presentation of the smallest member in a particular family,
2. Description of particular cyclic extensions adding multiple edges.

Theorem (Hu, N.,Škoviera, Wang, 2013...)

Let $\mathcal{M} = (G; x, y)$ be a regular map of valency d with underlying graph $X^{(m)}$. Set $A = \langle x^{d/m} \rangle$ and $B = \langle x^{d/m}, y \rangle$. Then:

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(i) *The group A is a normal subgroup of G, the map $\mathcal{M}/A = (G/A; xA, yA)$ is a regular embedding of X, and the natural projection $\mathcal{M} \to \mathcal{M}/A$ is a map homomorphism bijective on the vertices.*

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(ii) *$\mathcal{M}' = (B; x^{d/m}, y)$ is a dipole map isomorphic to $\mathcal{D}(m, e)$ for some integer e such that $e^2 \equiv 1 \pmod{m}$.*

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(iii) *$G = \langle x, y \mid x^{km} = y^2 = 1, yx^k y = x^{ek}, \dots \rangle$, where $e^2 \equiv 1 \pmod{m}$ and $k = d/m$ is the valency of X.*

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(iv) *If $\mathcal{M}$ is not bipartite, then $e \equiv 1 \pmod{m}$.*

Parameter e in a nilpotent map coming from a 2-group

We have a relation $yx^{2^t}y = x^{e2^t}$, where $k = 2^t$ and $m = 2^s$.

Since $e^2 = 1 \bmod m$, only four values $e = \pm 1$ and $e = 2^{s-1} \pm 1$ are admissible.

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Example: Consider a famility of regular embeddings of $C_n^{(m)}$, where both n, m are powers of two. By the classification (Hu, N., Škoviera and Wang) for each fixed $n \geq 4$ and m there are exactly 8 such maps:

$$G(n, m; e, f) = \langle x, y \mid x^{2m} = y^2 = 1, y^{-1}x^2y = (x^2)^e, (xy)^n = (x^2)^f \rangle,$$

where $e^2 = 1 \mod m$ and $f = \frac{1}{4}(e+1)n$ or $f = \frac{1}{4}((e+1)n + 2m)$.

Maximal discrete actions of genus g

Let $\lambda(g)$ denote the order of a largest group of conformal automorphisms of a compact Riemann surface of genus g. Accola (1968) and MacLachlan (1969) independently proved that

$$8(g+1) \leq \lambda(g) \leq 84(g-1)$$

for $g \geq 2$ and there are infinitely many integers $g \geq 2$ for which the equality $\lambda(g) = 8(g+1)$ holds.

Maximal nilpotent actions of genus g

If we take $n = 4$, $e = -1$, and $f = 0$, we get that the genus of $\mathcal{C}(4, m; -1, 0)$ is $g = m - 1$ with the automorphism group G of order $|G| = 8(g + 1)$, the lower bound of $\lambda(g)$.
The same problem for nilpotent groups has a solution

$$8(g + 1) \leq \lambda_{nil}(g) \leq 16(g - 1),$$

see Zamarrodian 1985, both extremal maps come from 2-groups.

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A dessin is abelian, nilpotent, if it is regular and $Aut_0(M)$ is abelian, nilpotent (respectively).

Representation of regular dessins

If D is regular, then we can label the edges of D by the elements of $G = Aut(D)$. Let $\langle x \rangle$ and $\langle y \rangle$ be generators of the stabilisers of two adjacent vertices (one is black the other is white). Hence we have an associated triple $(G; x, y)$, where $G = \langle x, y \rangle$.

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Vice-versa, given $(G; x, y)$ we set $E = G$, define the black vertices to be the (right) cosets of $\langle x \rangle$ and white vertices to be the cosets of $\langle y \rangle$. The edge-vertex incidence is given by the containment relation.

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If $\langle x \rangle g$ is a vertex, then the incident edges are cyclically ordered $(g, xg, x^2g, x^3g, \dots)$ by the left multiplication by x. Similarly, there is a natural cyclic order of edges based at white vertices. Then the right action of G preserves the incidence relation and the cyclic order. Thus we have a regular dessin with the automorphism group G and stabilisers $G_b \cong \langle x \rangle$, $G_w \cong \langle y \rangle$.

Nilpotent maps versus nilpotent dessins

Theorem

A nilpotent non-abelian map determines a nilpotent dessin.

Vice-versa it is not true. So, nilpotent dessins form a broader family of structures.

Decomposition theorem for nilpotent dessins

Theorem (Decomposition Theorem)

Let $G \cong \prod_{i=1}^{k} G_i$ be the direct product decomposition of a finite two-generated nilpotent group G where G_i are the Sylow p_i-subgroups of G.
Then a regular dessin $D = (G, x, y)$ is the parallel product of regular dessins $D_i = (G_i, x_i, y_i)$ where x_i and y_i are the images of x and y under the natural projections $\pi_i : G \to G_i$.

Conclusion: It is sufficient to restrict investigation to p-dessins.

Shadow dessins

Theorem

Let $D = (G, x, y)$ be a nilpotent regular dessin, and let $\bar{D} = (\bar{G}, \bar{x}, \bar{y})$ be its shadow dessin. If $c(G) = c \geq 2$, then $c - 1 \leq c(\bar{G}) \leq c$. Conversely, if $c(\bar{G}) = c$, then $c \leq c(G) \leq c + 1$.

Conclusion: Classification can be done in the two steps, first considering dessins without multiple edges.

The beggining: Abelian dessins

Theorem

Up to duality swapping the black and white vertices the abelian dessins with $Aut(D) \cong Z_{p^a} \times Z_{p^b}$, $0 \le a \le b$ *are in 1-1 correspondence with the pairs* (c, e) *of integers such that*

$$0 \le c \le b - a, \quad \textit{and } e \in Z^*_{p^c},$$

in which case $Aut(D)$ *has presentation*

$$\langle x, y | x^{p^b} = y^{p^{a+c}} = [x, y] = 1, y^{p^a} = x^{ep^{b-c}} \rangle.$$

How many abelian dessins there are?

Theorem

Up to isomorphism, the number of regular dessins A with $Aut(A) \cong Z_n \oplus Z_m$ where $0 \leq n \leq m$ is equal to $\psi(m/n)$ where ψ is the Dedekind's totient function.

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Remark. The Dedekind totient function $\psi(m) = m\prod_{p|m}(1 + \frac{1}{p})$ was introduced by Dedekind in connection with his investigation of modular forms. It appears in many other contexts, for instance in the formula describing the generating function of the Riemann zeta function.

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4. the shadow map is the standard embedding of $K_{n,n}$ (Hildalgo (2013)

Symmetric dessins of class 2

Theorem

Simple symmetric p-dessins of class two are in correspondence with quadruples (a, b, c, u) *such that*

$$max(0, 1 + c - a) \leq b \leq c \leq a \leq 2c - b, u \in Z^*_{p^{a-c}}$$

in which case it has presentation

$$\langle x, y | x^{p^a} = y^{p^a} = [x, z] = [y, z] = 1, z^{up^b} = x^{-p^c} u^{p^c}, z = [x, y] \rangle$$

.

Enumeration of simple dessins of class 2

Theorem

The number of symmetric simple dessins is given by the function

$$F(p,a) = (p+1)(p^{a/2}-1)/(p-1),$$

if a is even, and

$$F(p,a) = (2p^{\frac{a+1}{2}} - p - 1)/(p-1),$$

if a is odd.

Further results

1. complete classification of symmetric dessins of class 2,
2. regular embeddings of $K_{n,n}^{(m)}$ over standard embedding of $K_{n,n}$,
3. regular embeddings of $K_{p^e,p^e}^{(m)}$, for a prime $p > 2$,
4. regular embeddings of $K_{2^e,2^e}^{(m)}$ incomplete.

Open problems

1. Labute proved a more general results on the size of Γ_n/Γ_{n+1} for the one-relator group $\langle x, y | y^p = 1 \rangle$. We have employed it just in case $p = 2$. Is there a room for generalisation?
2. It seems regular dessins with the underlying graphs $K_{n,n}^{(m)}$ play important role. Can we complete the classification?
3. A project: Groups or graphs giving rise to a unique regular dessin (map).

A sample of the uniqueness phenomenon

An integer n is called singular, if $\gcd(n, \varphi(n)) = 1$.

Theorem

The graph $K_{n,n}^{(m)}$ has a unique regular embedding iff $m = 1$ and n is singular, or $m = 2$ and n is odd and singular.