Arc-transitive graphs and their vertex stabilizers

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Joint work with
Michael Giudici

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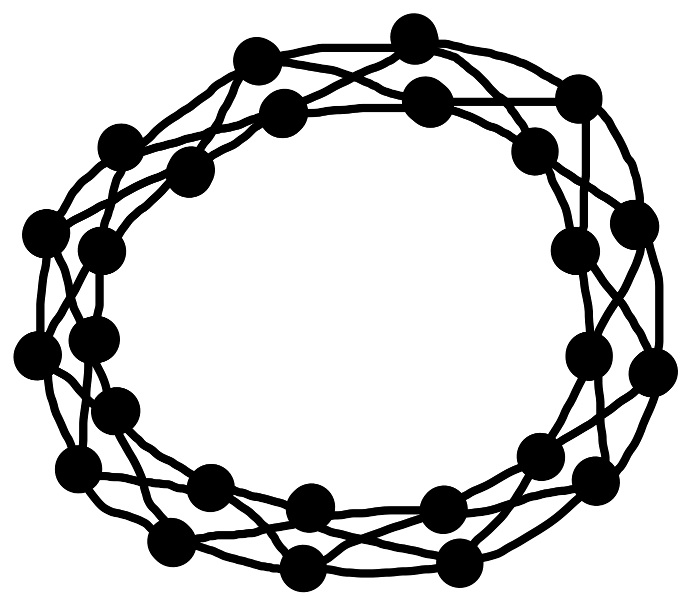

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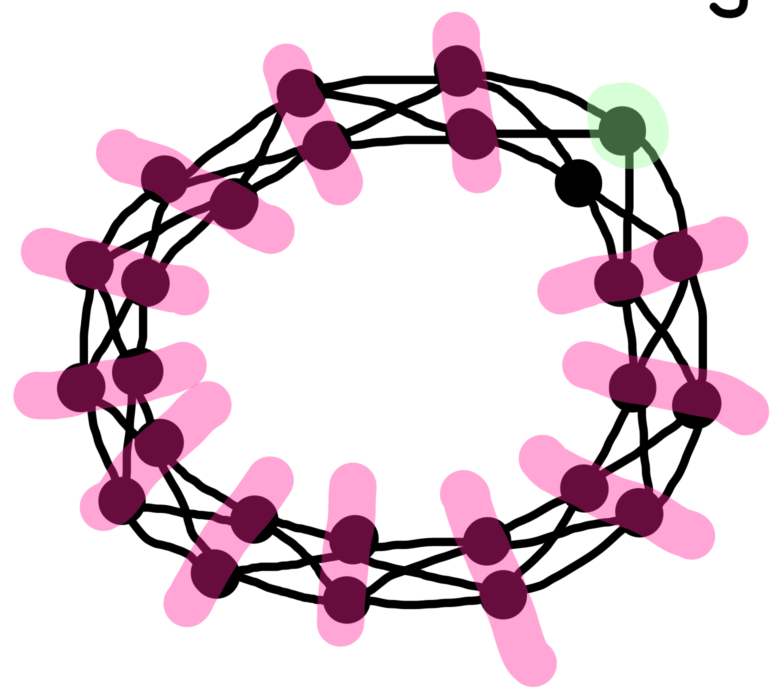

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<u>Theorem</u> (Tutte): Γ has valency three, a stabilizer has order at most 48.

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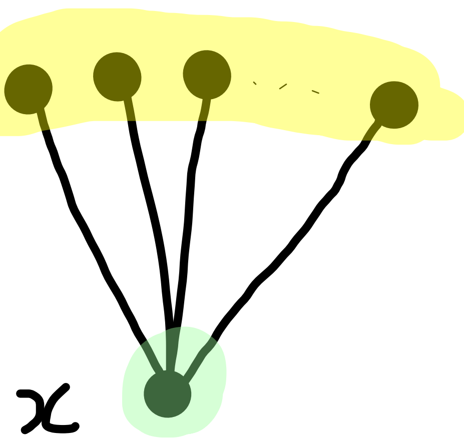
Local Approach:



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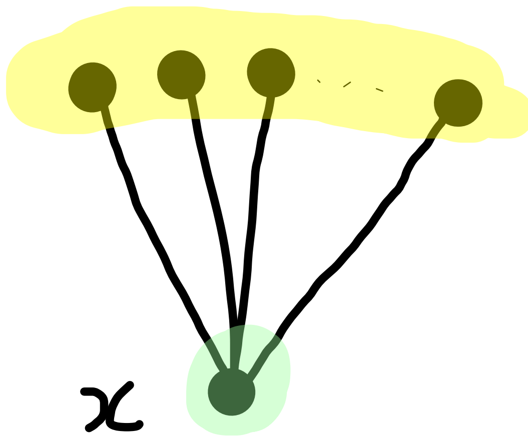

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Local action: $\quad G_x^{\Gamma(x)}$

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Definition: Let R be a permutation group.

Say (Γ, G) is locally R, if for some (all) $x \in \Gamma$

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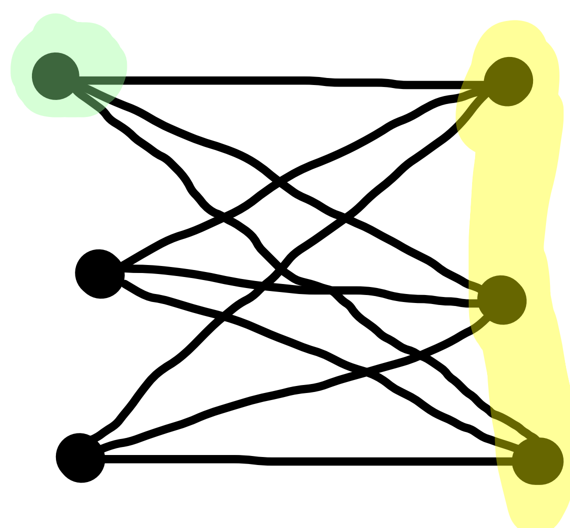

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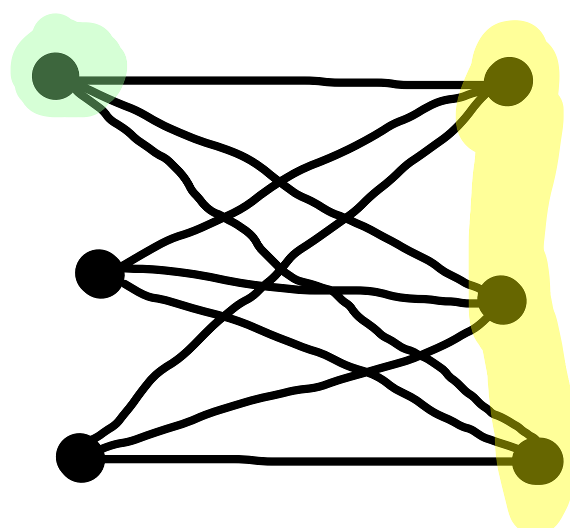
e.g. $\Gamma =$ 

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Definition: R is primitive on Ω if R preserves no nontrivial partition of Ω.

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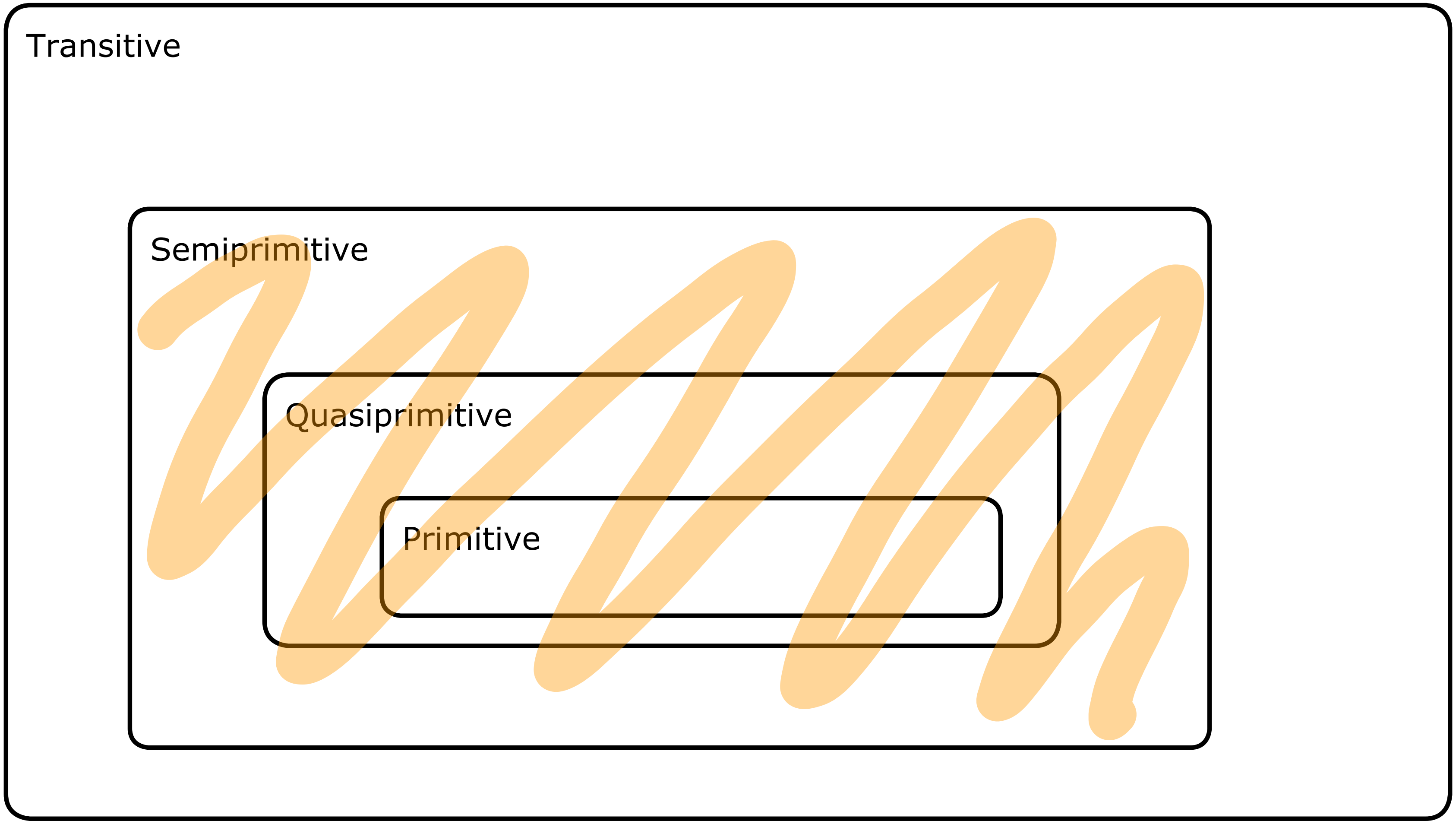
PSV Conjecture: R is graph-restrictive $\Leftrightarrow$ R is semiprimitive.



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- Strategy to prove L is graph-restrictive:

$$G_x^{[i]} = \{ g \in G \mid y^g = y \text{ if } d(x,y) \leq i \}.$$

$$G_x^{[0]} = G_x, \quad G_x^{[1]} = \mathrm{Stab}_G(\Gamma(x)).$$

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<u>Lemma</u>: $\exists K \in \mathbb{N}$ s.t. $G_x^{[K]} = 1$.

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Lemma: $\exists K \in \mathbb{N}$ s.t. $G_x^{[K]} = 1$.

Idea: Find a K which works for all locally L pairs.

PSV Conjecture

Known for: regular groups ($G_x^{[1]}=1$).

2-transitive groups (Trofimov, Weiss)

($G_x^{[6]}=1$).

Dihedral groups in odd degree (Sami).

S_4, A_4 (Gardiner).

$GL_2(p)$ on $(\mathbb{F}_p^2)^\#$ (PSV).

Smallest open imprimitive case: degree 9.

$3^2 : C_2$.

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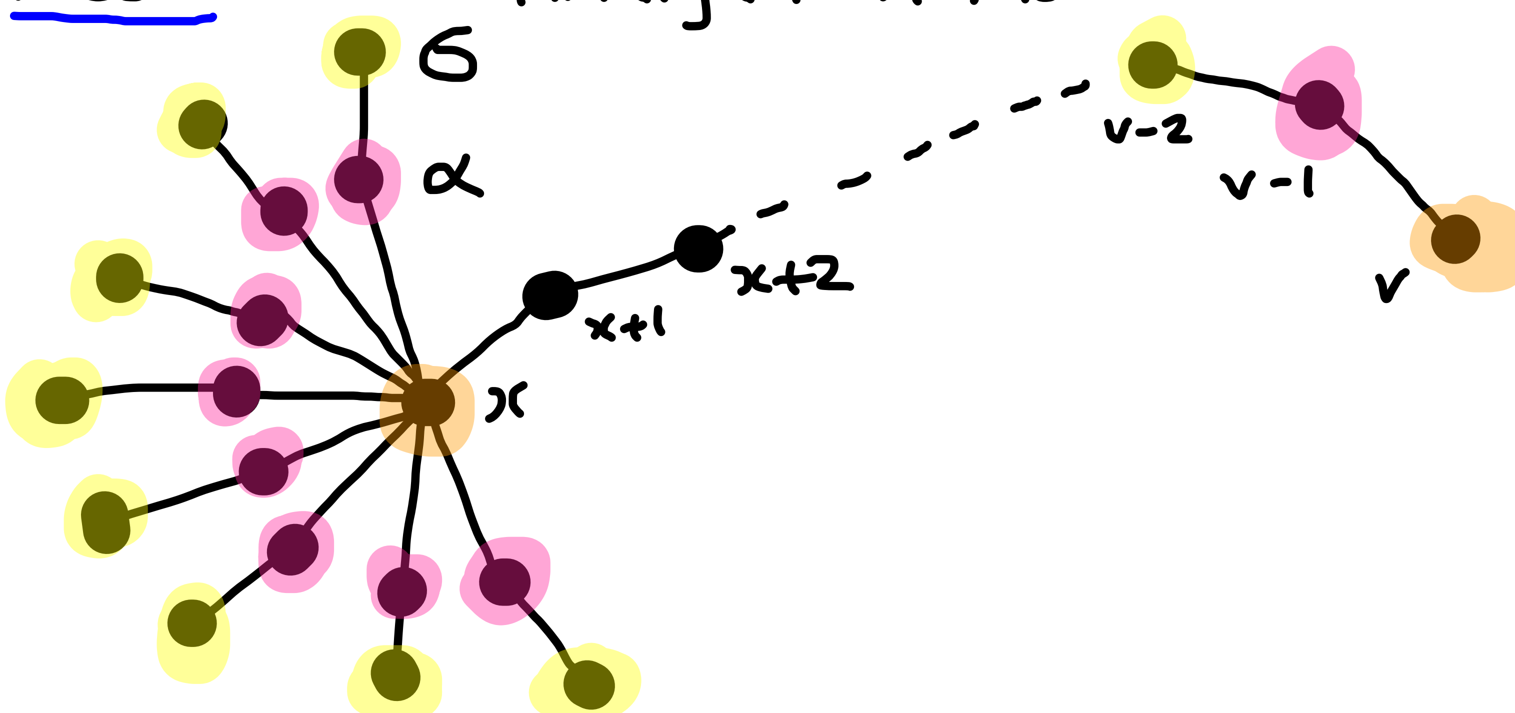
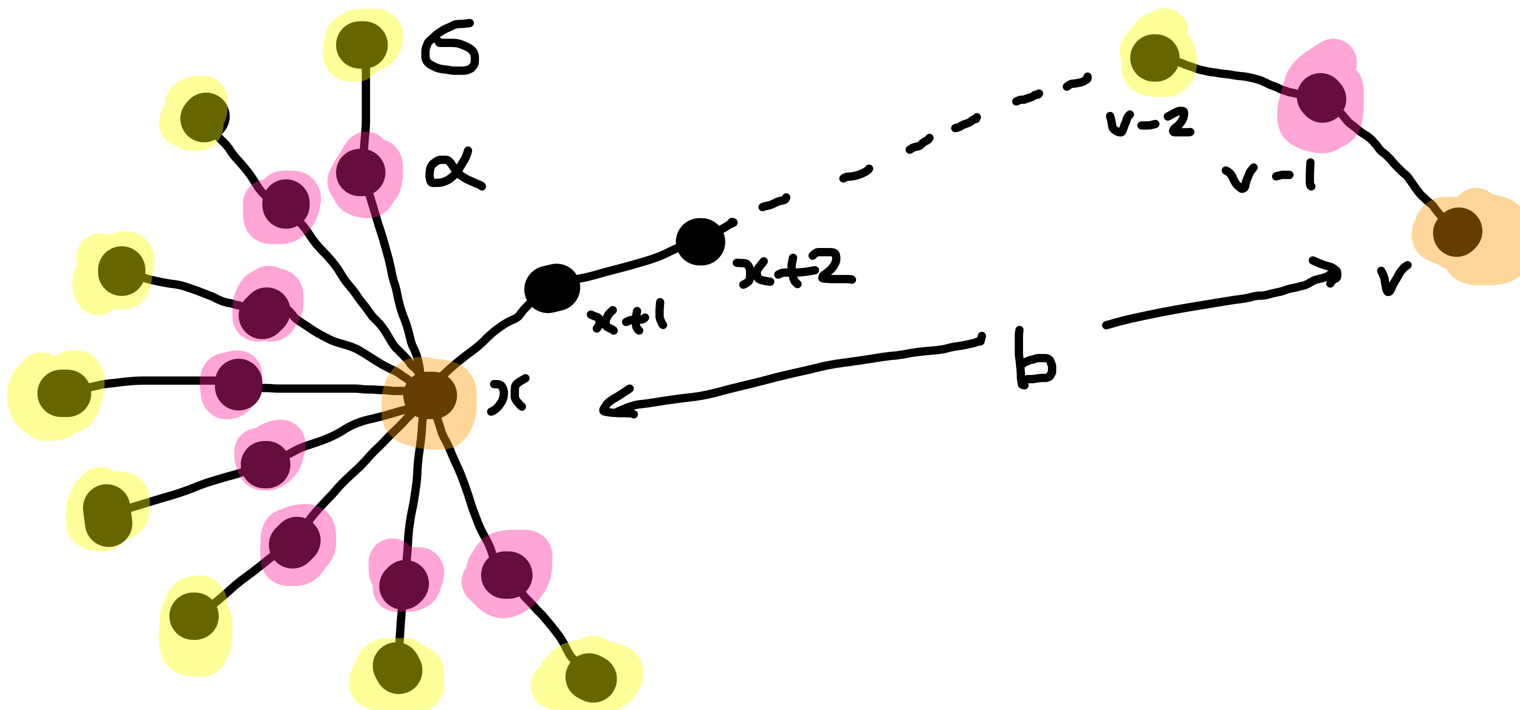
Proof: Amalgam Method.



Amalgam Method: Bound "Critical distance" = b.



- $b \le 2$.
- $G_x^{[b+2]} = 1$.

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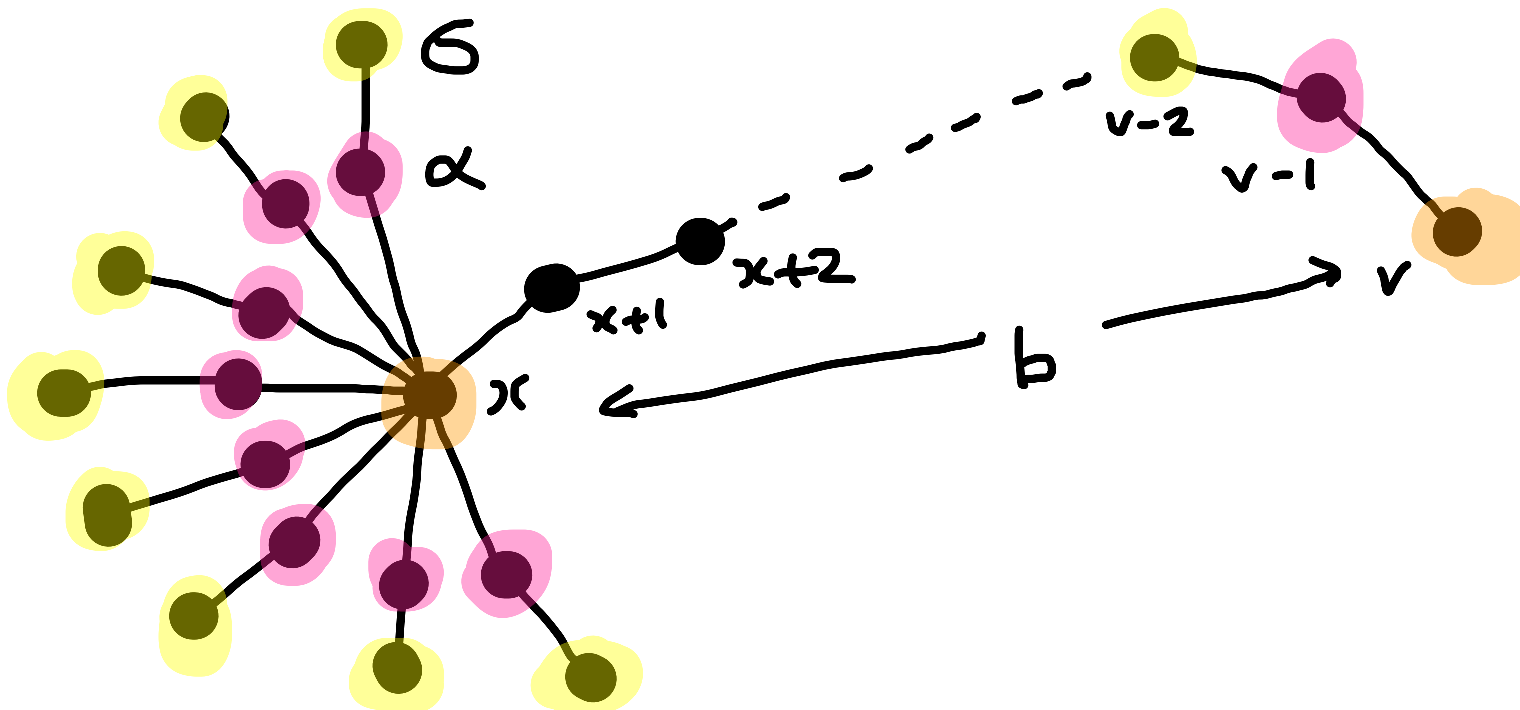

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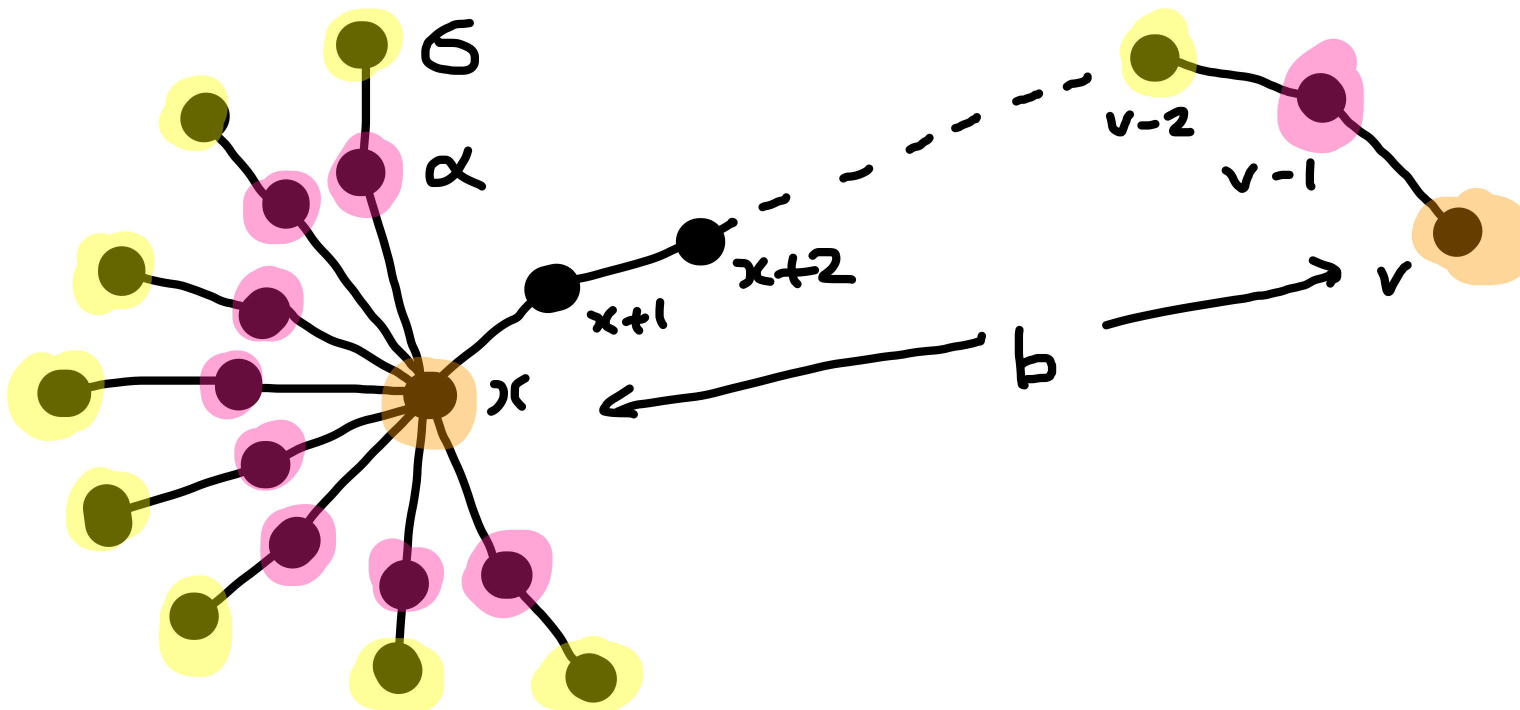

- $b \leq 2$.
- $G_x^{[4]} = 1$.

Remark: No examples known with $G_x^{[3]} \neq 1$.

O'Nan-Scott type :	HA	TW	HS	HC	else
Weiss Conjecture :	✓	✓	✓	✓	✗

Weiss

Spiga

HA type – Regular Abelian normal subgroup.

TW types – Regular non-Abelian
HS, HC

normal subgroup(s).

Theorem (Weiss): Let R be a primitive group of type HA.

Let A be the regular abelian normal subgroup.

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Suppose $(|A|, 6) = 1$.

Then R is graph-restrictive.

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<u>Definition</u>: A semiprimitive group is of type HN if it has a regular normal nilpotent subgroup.

Theorem: Let R be a semiprimitive group of type HN with regular normal nilpotent subgroup K.

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Proof: · Strengthen results of Bereczky and Maróti on semiprimitive groups.

· Employ p-local results of Stellmacher.

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Unknown if graph-restrictive.

Thank you

for listening!