Classification of some Mobius regular maps

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Introductions of maps and regular maps

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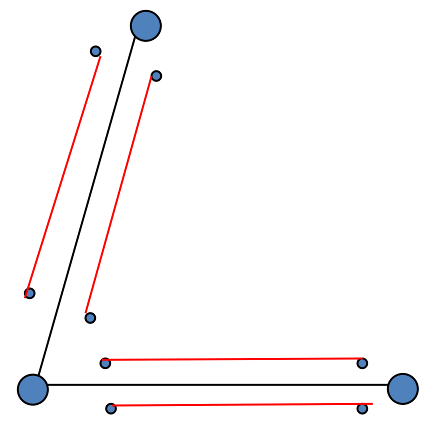

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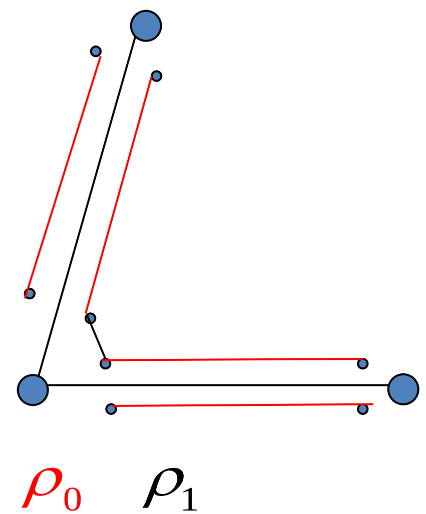

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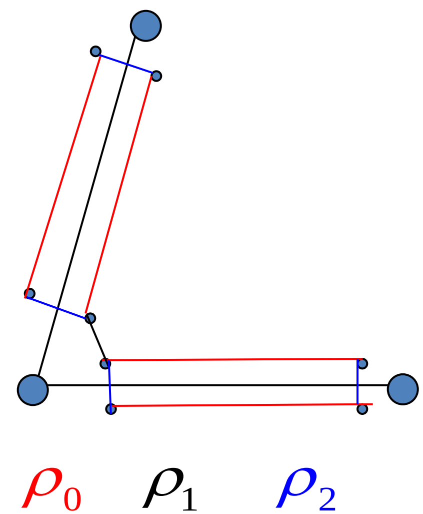

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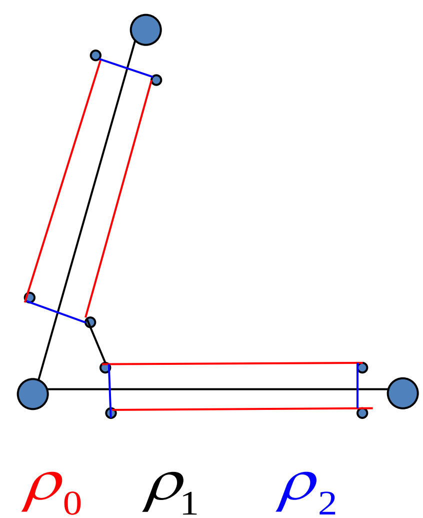
Any map $\mathfrak{M} = G \to S$ can be descibed by three involutions (ρ_0, ρ_1, ρ_2) acting on $\mathrm{F}(\mathfrak{M})$.

Note that $\langle \rho_0, \rho_1, \rho_2 \rangle$ acts transitively on $\mathrm{F}(\mathfrak{M})$ and $(\rho_0 \rho_2)^2 = 1$.



Conversely, any quadruple $(F;\ \rho_0, \rho_1, \rho_2)$ satisfying

(1) ρ_0, ρ_1, ρ_2 are fixed point free involutions of F.

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We call $\mathfrak{M} = (F;\ \rho_0, \rho_1, \rho_2)$ a *combinatorial map* and $\langle \rho_0, \rho_1, \rho_2 \rangle$ is called a *monodromy group* denoted by $\mathrm{Mon}(\mathfrak{M})$.

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[Definition]

1. For any map $\mathfrak{M} = G \to S$, a *map automorphism* is a graph automorphism of G which can be extended to self-homeomorphism of the surface S in the embedding.

An automorphism of $\mathfrak{M} \leftrightarrow$ a permutation of $F(\mathfrak{M})$ which commute ρ_0, ρ_1 and ρ_2.

$\mathrm{Aut}(\mathfrak{M}) \leftrightarrow$ the centralizer of $\langle \rho_0, \rho_1, \rho_2 \rangle$ in $\mathrm{Sym}(F(\mathfrak{M}))$.

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2. $|\mathrm{Aut}(\mathfrak{M})| \le |\mathrm{F}(\mathfrak{M})| \le |\langle \rho_0, \rho_1, \rho_2 \rangle|$

One equality holds $\Leftrightarrow$ both equalities hold

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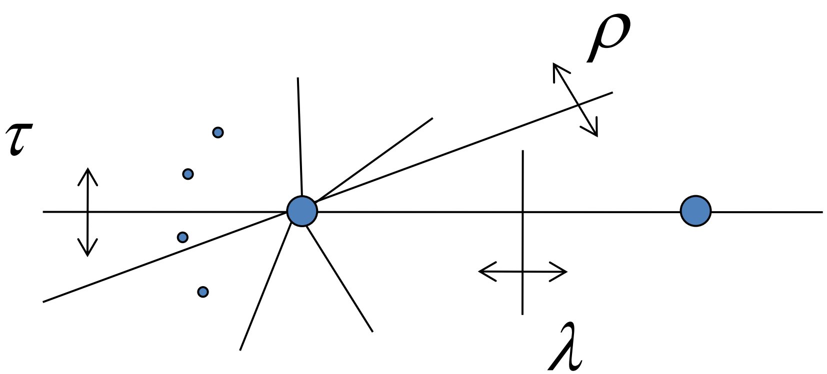
Classification of regular maps are pursued by fixed graphs, fixed surfaces, fixed automorphisms, etc.

$\mathfrak{M}$: regular embeddings of $G \Rightarrow \exists\, \lambda, \rho, \tau \in$ Aut(G) s.t.



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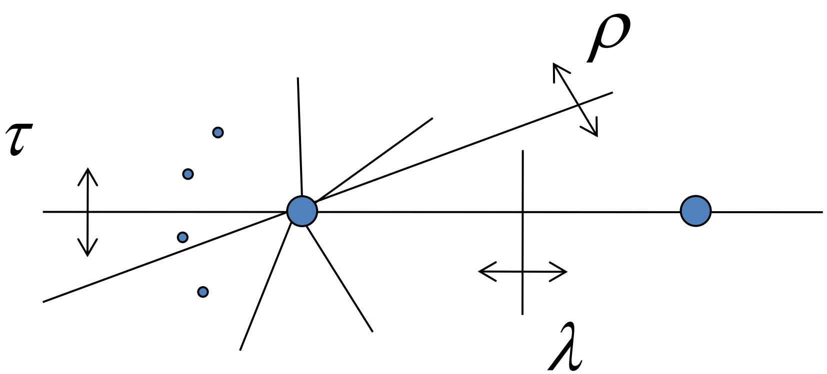

and $|\langle \lambda, \rho, \tau \rangle| = 4|E| = |F|$

We call such pair (λ, ρ, τ) an admissible triple for G.

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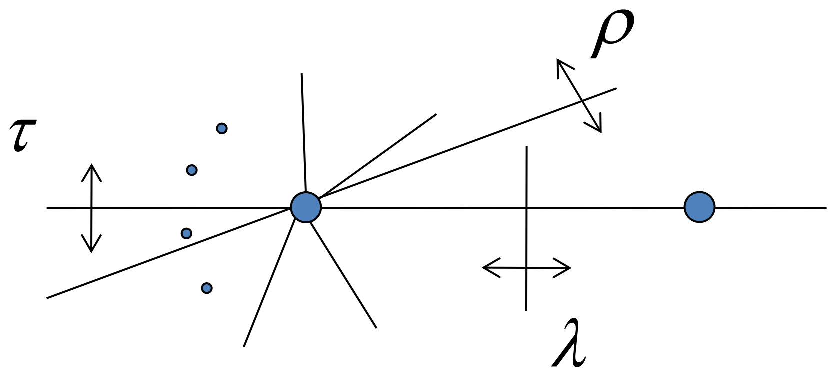

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 **[Theorem] ('99, Gardiner et. al)**

A graph G has a regular map iff there exists an admissible triple for G.

The number of regular embeddings of G up to isomorphism is the number of orbits of admissible triples (ρ, λ, τ) for G under the conjugate action by Aut(G).

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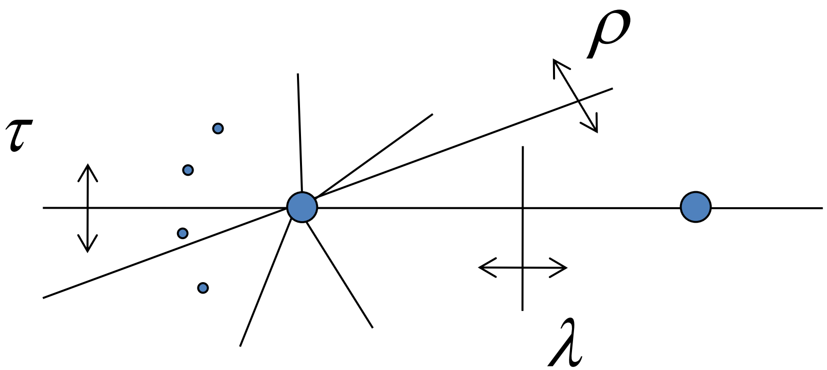

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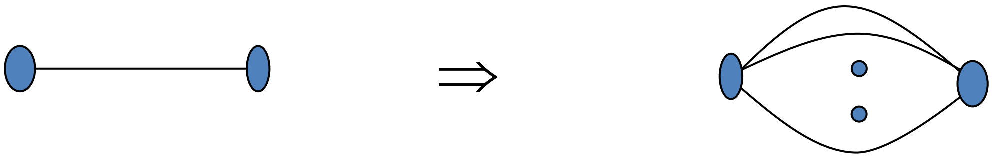
nonorientable $\Leftrightarrow$ $\langle \lambda, \rho, \tau \rangle = \langle \rho\tau, \lambda\tau \rangle$ $\Leftrightarrow$ $\tau \in \langle \rho\tau, \lambda\tau \rangle$.

Introduction of Mobius regular maps

G: graph $\Rightarrow$ $G^{(m)}$: m-multiple graph of G.



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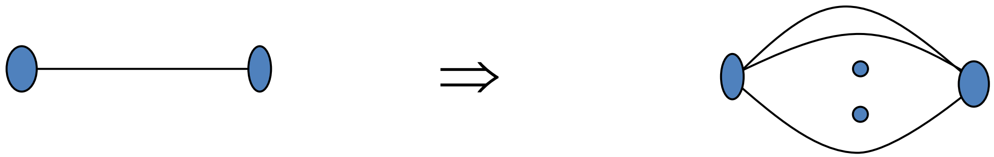

$\mathfrak{M}$: an embedding of $G^{(m)}$ $\Rightarrow$ $G^{(m)}$: underlying graph of $\mathfrak{M}$

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G: underlying simple graph of $\mathfrak{M}$

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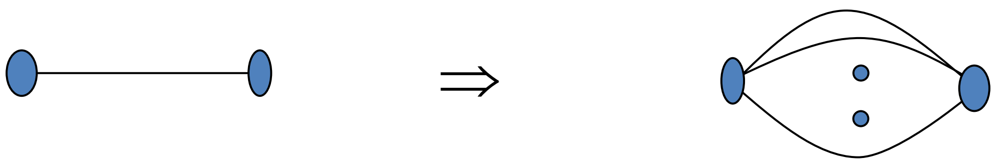

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G: underlying simple graph of $\mathfrak{M}$

For a regular m-multiple embedding $\mathfrak{M}$ of G, let

valency $= km$, $\quad$ $\text{Aut}(\mathfrak{M})=\langle \lambda, \rho, \tau \rangle$, $\quad$ $\text{Mon}(\mathfrak{M})=\langle \rho_0, \rho_1, \rho_2 \rangle$

$r = \rho\tau$ $\quad$ $\ell = \lambda\tau$ $\quad$ $R = \rho_1\rho_2$ $\quad$ $L = \rho_0\rho_2$

Two kinds of regular multiple embeddings

1. Any two parallel edges have neighborhood homeomorphic to a disc.

2. Any two parallel edges have neighborhood homeomorphic to a Möbius band.

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$m \geq 2$, $\mathfrak{M}$:orientable or nonorientable $\quad$ $\text{core}(\langle r \rangle) = \langle r^k \rangle$

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$m = 2$, $\mathfrak{M}$: nonorientable $\quad$ $\langle r \rangle$: core-free

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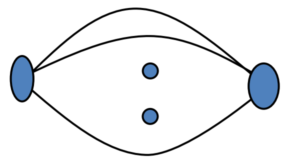
Restriction of embedding on 

orientable regular embedding of dipole $\mathrm{D}_m \quad \leftrightarrow \quad (\rho_0, \rho_1 R^k, \rho_2)$

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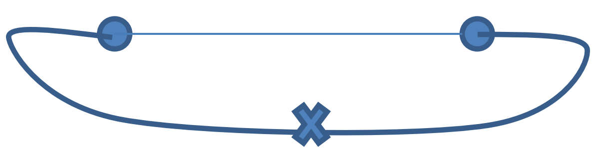

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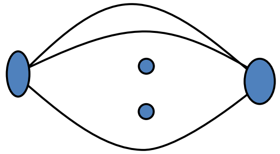
Restriction of embedding on 

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$\mathfrak{M}/C$: regular embedding of underlying simple graph G

$\mathfrak{M}$: orientable $\Leftrightarrow$ $\mathfrak{M}/C$: orientable

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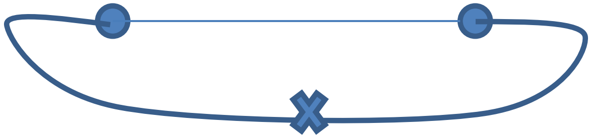
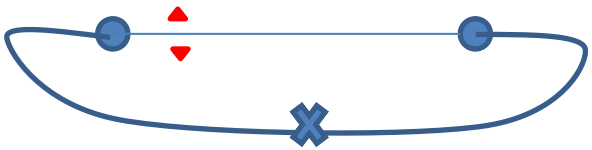


Möbius regular embedding of G

Some observation of Mobius regular maps

$\mathfrak{M}$: Möbius regular embedding of G with valency $2k$.

1. 

$\Rightarrow \quad \ell r^k \ell r^k = \tau \quad (\lambda r^k \lambda r^k = \tau)$

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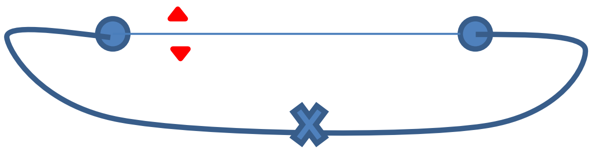
1. 

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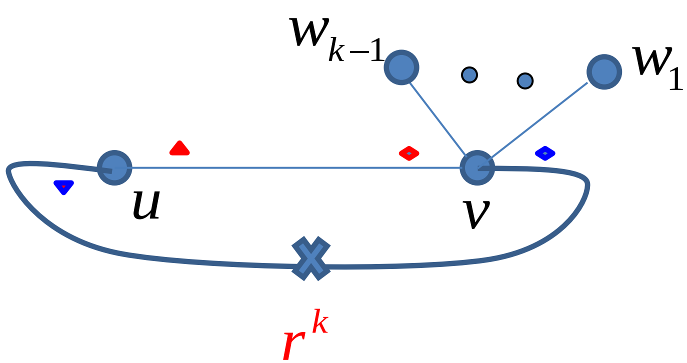
2. 

$\Rightarrow$ (1) r^k fixes all neighbors of u.

(2) r^k sends w_i to w_{k-i}.

k:odd $\Rightarrow \exists$ no element in $N(v)-\{u\}$ fixed by r^k

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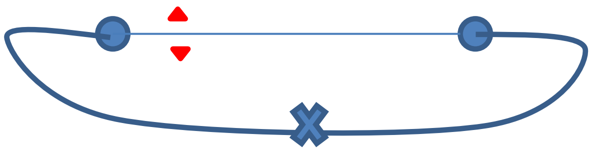

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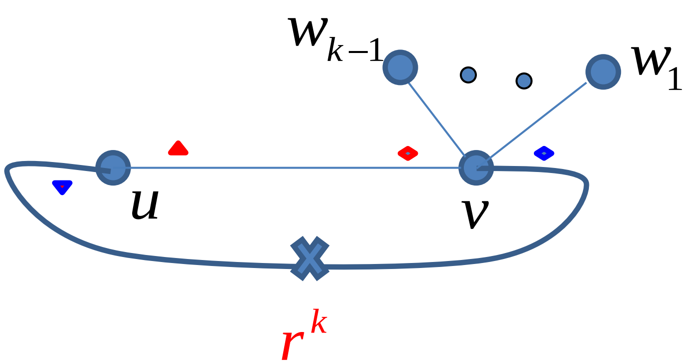

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3. $\mathrm{H} = \mathrm{Aut}(\mathfrak{M})_u = \langle \mathrm{r}, \tau \rangle \simeq \mathrm{D}_{2k}$. $\quad \mathrm{J} = \mathrm{Aut}(\mathfrak{M})_{\{u,v\}} = \langle \mathrm{r}^k, \tau, \lambda \rangle = \langle \mathrm{r}^k, \ell \rangle \simeq \mathrm{D}_4$.

$\mathrm{H} \cap \mathrm{J} = \langle \mathrm{r}^k, \tau \rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. $\quad \mathrm{Aut}(\mathfrak{M}) = \langle H, J \rangle$.

H and J do not have identical centers.

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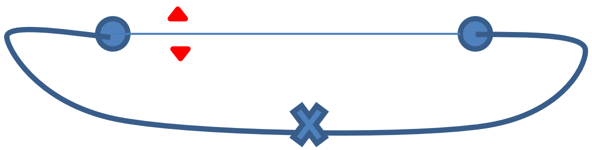
1. 

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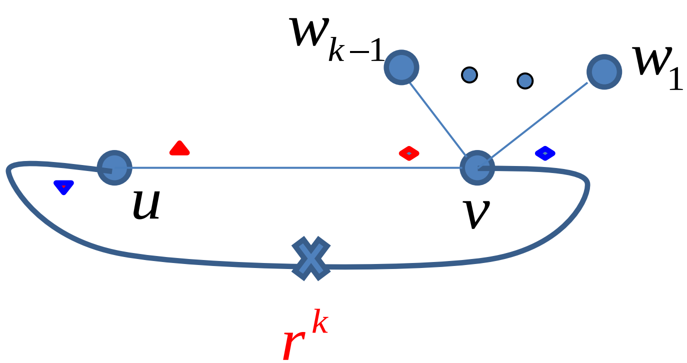
2. 

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3. $H=\mathrm{Aut}(\mathfrak{M})_u = \langle r, \tau \rangle \simeq D_{2k}$. $\quad J=\mathrm{Aut}(\mathfrak{M})_{\{u,v\}} = \langle r^k, \tau, \lambda \rangle = \langle r^k, \ell \rangle \simeq D_4$.

$H \cap J = \langle r^k, \tau \rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. $\quad \mathrm{Aut}(\mathfrak{M}) = \langle H, J \rangle$.

H and J do not have identical centers.

4. $\mathfrak{M}$ is self-Petrie dual.

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$\exists$ a Möbius regular emb. of $C_n \iff 3 \mid n$.

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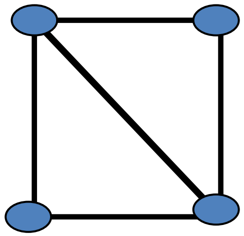
$\exists$ a Möbius regular emb. of C_n $\Leftrightarrow$ $3 \mid n$.

1.  $\Rightarrow$ $\exists$ no Möbius regular embedding of G

(1) For $K_n, K_{n,\ldots,n}, H(d,n)$ with $n \geq 3$

$\Rightarrow$ $\exists$ no Möbius regular embedding.

(2) $\exists$ no Möbius regular map of covalency 3.

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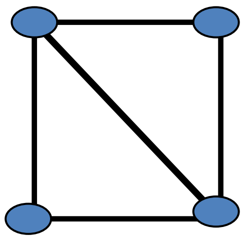
1. 

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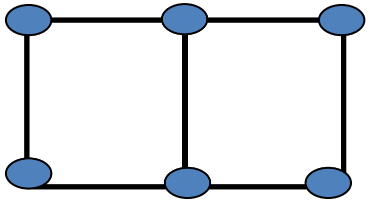
2. 

$\forall u, v \in V(G),\ |N(u) \cap N(v)| \leq 2$

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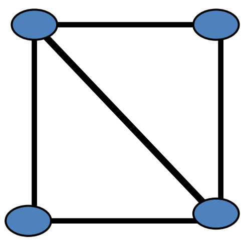
1. 

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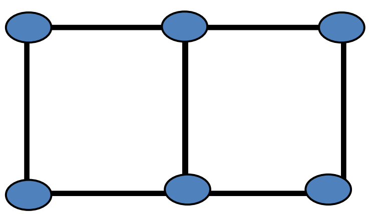
2. 

$\forall u,v \in V(G),\ |N(u) \cap N(v)| \leq 2$

$\Rightarrow \exists$ no Möbius regular embedding of G

For Q_n with $n \geq 2$

$\Rightarrow \exists$ no Möbius regular embedding.

3. k: even, $k\,|V(G)|$ is not a multiple of 3

$\Rightarrow \exists$ no Möbius regular embedding of G

$\because$ Each orbit of $\langle R^k, L, \tau \rangle$ has the same size and corresponds to Möbius regular embedding of a cycle $\Rightarrow 3 \mid k\,|V(G)|$

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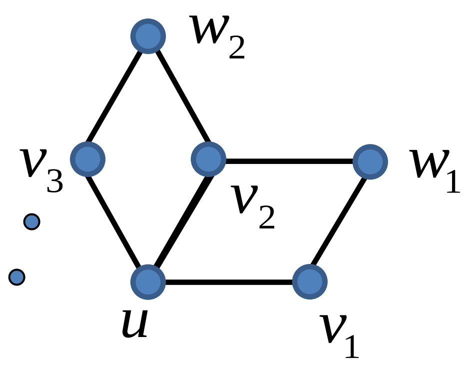
$\alpha = (0)(1,\ n\text{-}1)(2,\ n\text{-}2)\cdots$ and n is odd

$\Rightarrow$ For odd n, $\exists$ only one Möbius regular embedding of $K_{n,n}$.

For even n, $\exists$ no Möbius regular embedding of $K_{n,n}$.

For odd n, the order of $r_{\alpha}\ell_1$ is 4 $\Rightarrow$ covalency is 4.

Mobius regular embeddings of covalency 4



$w_1 = w_2 \Rightarrow w_1 = w_2 = w_3 = \cdots \Rightarrow r^k$ fixes $w_1 = w_2 = w_3 = \cdots$.

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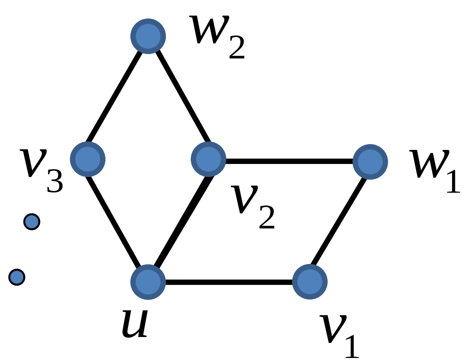

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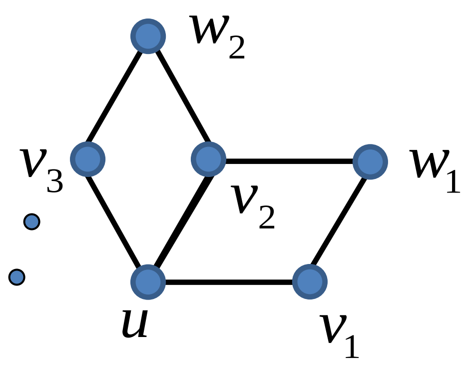

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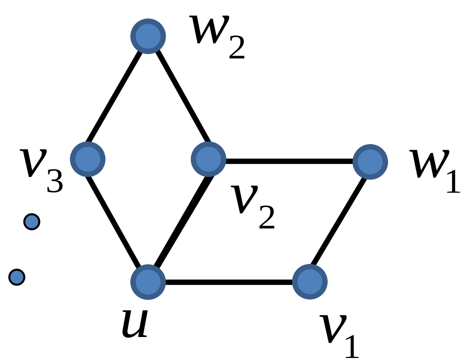

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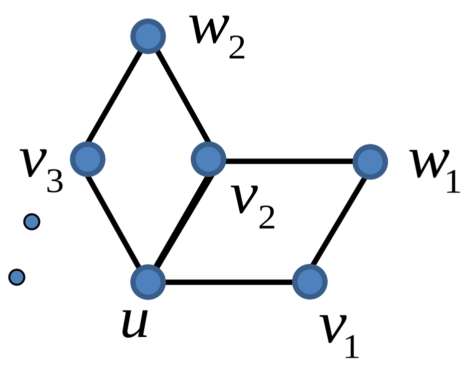

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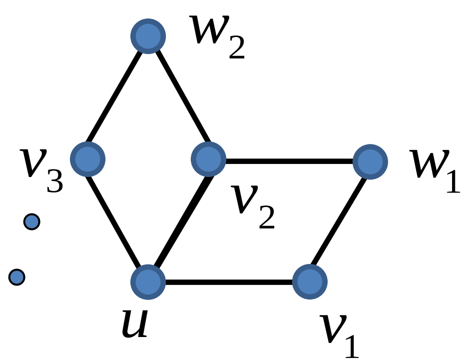

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$\Rightarrow$ For odd k(even k, resp), $\exists$ only one (no, resp) Möbius regular map whose valency and covalency are $2k$ and 4, respectively.

$k, t \in \mathbb{N},\ (k, t \geq 2)$, let

$$T^M(2, t, 2k) = \langle a, b, c \mid a^2 = b^2 = c^2 = (ac)^2 = (bc)^{2k} = (ab)^t = 1,\ a(bc)^k a(bc)^k = c \rangle$$

called Möbius triangle group of type $(t, 2k)$.

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For $t = 3,\ k \geq 3,\ T^M(2,t,2k)$ is not a smooth represented group.

For $k : even,\ t = 4,\ T^M(2,t,2k)$ is not a smooth represented group.

For $k : odd,\ t = 4,\ T^M(2,t,2k)$ is a smooth represented finite group

and all other relations are redundant. (property A)

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For $t=3,\ k \geq 3,\ T^M(2,t,2k)$ is not a faithfully defined group.

For $k: even,\ t=4,\ T^M(2,t,2k)$ is not a faithfully defined group.

For $k: odd,\ t=4,\ T^M(2,t,2k)$ is a faithfully defined finite group and all other relations are redundant. (property A)

Problem: 1. Classify k,t such that $T^M(2,t,2k)$ is a smooth represented group.

2. Classify k,t such that $T^M(2,t,2k)$ satisfy the property A.

3. Classify k,t such that $T^M(2,t,2k)$ is a smooth represented infinite group and check residual finiteness of it.

$k=2 \Rightarrow$ smooth represented only for $t=3t'$.

$t=3 \Rightarrow$ smooth represented only for $k=2$.

$t=4 \Rightarrow$ smooth represented only for odd k. a finite group.

$(t,k)=(5,3),(5,4)$: not smooth represented.

$(t,k)=(5,5),(5,6),(7,3)$: smooth represented finite.

$(t,k)=(5,10),(6,5),(6,6),\ldots$:smooth represented infinite.

Thank you!!!!