SKEW-MORPHISM: A concept between Group Theory and Topological Graph Theory

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Skew-Morphisms

Definition (RJ,Širáň)

A *skew-morphism* of a group G is a permutation φ of G preserving the identity and satisfying the property

$$\varphi(gh) = \varphi(g)\varphi^{\pi(g)}(h)$$

for all $g, h \in G$ and a function $\pi : G \to \mathbb{Z}_{|\varphi|}$, called the *power function* of G.

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- skew-morphisms were originally introduced for the study of regular Cayley maps
- they have since proved central in the theory of cyclic group extensions
- the focus of this talk is on the interplay between their original use in the topological graph theory and their group-theoretical properties

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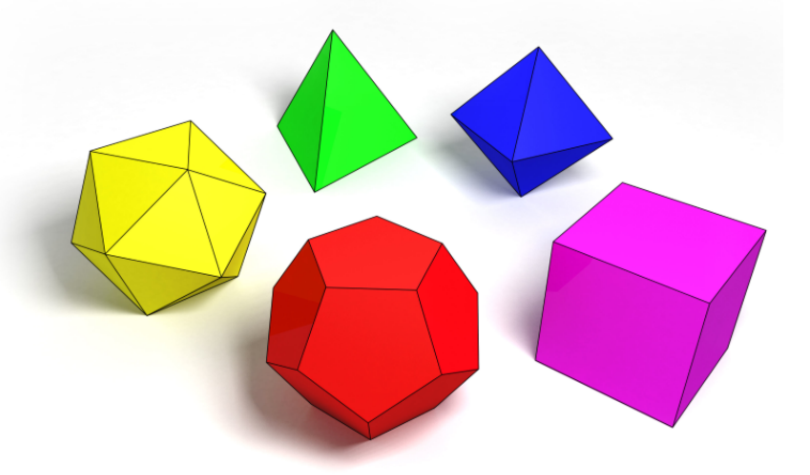
- a **map automorphism** of a map $\mathcal{M}$ is a permutation of its darts that preserves the adjacency and the faces

Classical Examples - The Five Platonic Solids



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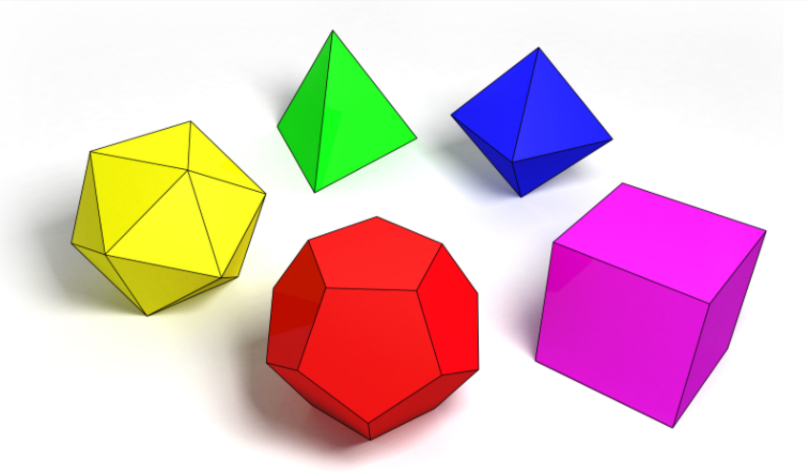

All five platonic solids possess automorphism groups that act regularly on their darts

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A map $\mathcal{M}$ is regular if and only if

$$|Aut\mathcal{M}| = |D(\mathcal{M})|$$

Cayley Maps

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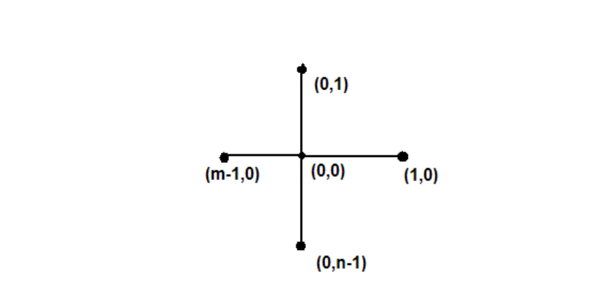

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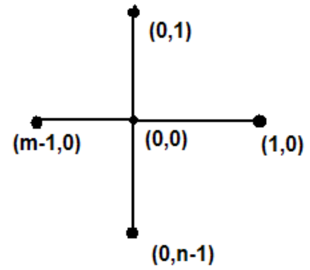
- in order for a Cayley map to be regular, the stabilizer of any vertex in $Aut(CM(G,X,p))$ must be of size $|X|$
- since the stabilizers of orientable maps are cyclic, in order for a Cayley map to be regular, there must exist an automorphism Φ that maps $(1,x)$ to $(1,p(x))$

$$\Phi(1_G) = 1_G \text{ and } \Phi((1_G, x)) = (1_G, p(x))$$



Skew-Morphisms

Theorem (RJ,Širáň)

Let $\mathcal{M} = CM(G, X, p)$ *be any Cayley map. Then* $\mathcal{M}$ *is regular iff there exists a skew-morphism* φ *of* G *satisfying the property* $\varphi(x) = p(x)$ *for all* $x \in X$*.*

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Proof.

- If $\Phi(1_G) = 1_G$ and $\Phi((1_G, x)) = (1_G, p(x))$ is the map automorphism generating the stabilizer of 1_G, then the mapping $\varphi : G \to G$ induced by Φ on G is a skew-morphism satisfying the properties $\varphi(1_G) = 1_G$ and $\varphi(x) = p(x)$.

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- If $\varphi : G \to G$ satisfying the properties $\varphi(1_G) = 1_G$ and $\varphi(x) = p(x)$ is a skew-morphism of G, then Φ defined by $\Phi(g, x) = (\varphi(g), \varphi(g)^{-1}\varphi(gx))$ is the required map automorphism.

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- every regular Cayley map on G is of the form

$$CM(G, \{x, \varphi(x), \ldots, \varphi^{n-1}(x)\}, (x, \varphi(x), \ldots, \varphi^{n-1}(x))),$$

  where φ is a skew-morphisms with a generating orbit $\{x, \varphi(x), \ldots, \varphi^{n-1}(x)\}$ that is closed under inverses

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- the order of φ in this case equals the length of its generating orbit that is closed under inverses

Algebraic Properties of Skew-Morphisms

Lemma (RJ, Širáň)

Let φ be a skew-morphism of a group G and let π be the power function of φ. Then the following holds :

1. *the set* $Ker\varphi = \{g \in G \mid \pi(g) = 1\}$ *is a subgroup of* G*;*
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Lemma (Conder, RJ, Tucker)

If A is a finite abelian group and φ is a skew-morphism of A, then

1. φ *preserves* $Ker\,\pi$ *setwise;*
2. *the restriction of* φ *to* $Ker\,\pi$ *is a group automorphism.*

Special Classes of Skew-Morphisms

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- **skew-morphisms whose orbits are contained within the cosets** of the kernel *(c.o.p.f.)*

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- regular, non-balanced Cayley maps over a dihedral group D_{2n}, n odd (Kovács, Marušič, Muzychuk)
- recent work of Jun-Yang Zhang suggests possibilities for classifying regular Cayley maps on cyclic groups with kernel of index 3

Lemma (RJ, Nedela)

Let φ be a skew-morphism of a finite group G, and π be its associated power function. The orbit $\mathcal{O}_a$ of any element a in G under the action of φ, $\mathcal{O}_a = \{a, \varphi(a), \varphi^2(a), \ldots\}$, satisfies the following two properties:

(i) The restriction of φ to the subgroup $\langle \mathcal{O}_a \rangle$ of G generated by $\mathcal{O}_a$ is a skew-morphism of $\langle \mathcal{O}_a \rangle$.

(ii) $\mathcal{O}_a$ is either closed under inverses or it contains no involutions and no inverses of the elements included in the orbit and the inverses of the elements included in the orbit constitute another orbit of φ of the same size.

Half-Regular Cayley Maps

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Corollary

Each skew-morphism of G gives rise to a regular or a half-regular Cayley map on a non-trivial subgroup of G.

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for some unique $a' \in A$ and some unique nonnegative integer i less than the order of ρ.

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Already observed in the 1930's (e.g., Oystein Ore, 1938).

Cyclic Extensions from Skew-Morphisms

Let H be a group, and φ be a skew-morphism of H with power function π,

$$s(i,b) = \sum_{j=0}^{i-1} \pi(\varphi^j(b)).$$

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Define a multiplication $*$ on $H \times \langle \varphi \rangle$ as follows:

$$(a, \varphi^i) * (b, \varphi^j) = (a\varphi^i(b), \varphi^{s(i,b)+j}),$$

for all $a, b \in H$ and all $i, j \in \mathbb{Z}_{|\varphi|}$.

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Theorem (Conder,RJ,Tucker; Kovács and Nedela)

Let H be a group and φ be a skew-morphism of H of finite order m and power function π. Then $A = (H \times \langle\varphi\rangle, *)$ is a group, $H \times \langle\varphi\rangle$ is a complementary factorization of A, and the skew-morphism of H associated with this factorization is equal to φ.

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Theorem

Let K be a subgroup of the kernel of a skew-morphism φ of a group A that is preserved by φ and is normal in A. Then φ induces a 'factor' skew-morphism φ^* on A/K defined by

$$\varphi^*(aK) = \varphi(a)K.$$

Cyclic Extensions from Skew-Morphisms

Theorem (Lucchini)

If P is a transitive permutation group of degree $n > 1$ with cyclic point-stabilizers, then $|P| \leq n(n-1)$.

Theorem (Herzog and Kaplan)

Let A be a non-trivial finite group of order n with a cyclic subgroup $\langle x \rangle$ satisfying the property $|x| \geq \sqrt{n}$. Then $\langle x \rangle$ contains a non-trivial normal subgroup of A.

Cyclic Extensions from Skew-Morphisms

Theorem (Conder, RJ, Tucker)

If G is any finite group with a complementary subgroup factorisation $G = AY$ with Y cyclic, then for any generator y of Y, the order of the skew morphism φ of A is the index in Y of its core in G, or equivalently, the smallest index in Y of a normal subgroup of G.

Moreover, in this case the quotient $\overline{G} = G/\mathrm{Core}_G(Y)$ is the skew product group associated with the skew morphism φ, with complementary subgroup factorisation $\overline{G} = \overline{A}\,\overline{Y}$ where $\overline{A} = AY/Y \cong A/(A \cap Y) \cong A$ and $\overline{Y} = Y/\mathrm{Core}_G(Y)$.

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Corollary (Conder, RJ, Tucker)

Every skew morphism of a non-trivial finite group has non-trivial kernel.

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Corollary (Conder, RJ, Tucker)

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Corollary (Conder, RJ, Tucker)

Every skew morphism of a cyclic group of prime order is an automorphism.

Kernels of Skew-Morphisms

Theorem (Conder, RJ, Tucker)

If A is a finite abelian group of order greater than 2, then the kernel of every skew morphism of A has order greater than 2.

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Let A be a finite abelian group of order greater than 2. If K is the kernel of any skew morphism of A, then every prime divisor of $|K|$ is larger than every prime that divides $|A|$ but not $|K|$.
In particular if q is the largest prime divisor of $|A|$, then the order of the kernel of every skew morphism of A is divisible by q when q is odd, or by 4 when $q = 2$.

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Corollary (Conder, RJ, Tucker)

Every skew morphism of an elementary abelian 2-group is an automorphism.

Further Results

Theorem (Conder, RJ, Tucker)

Let φ be a skew morphism of C_n. Then the order m of φ divides $n\phi(n)$. Moreover, if $gcd(m, n) = 1$ or $gcd(\phi(n), n) = 1$, then φ is an automorphism of C_n.

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Theorem

Let A be any finite abelian group. Then every skew morphism of A is an automorphism of A if and only if A is is cyclic of order n where $n = 4$ or $\gcd(n, \phi(n)) = 1$, or A is an elementary abelian 2-group.

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Classification and enumeration of the skew-morphisms of the cyclic groups C_{p^2} and C_{pq} and of $C_p \times C_p$.

Open Problem:

The set of all skew-morphisms of a finite group A is a subgroup of

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