Discrete groups and surface automorphisms: Murray Macbeath's contributions.

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Abstract

This talk re-examines some of the evolving interaction between hyperbolic geometry and low-dimensional topology, focussing in particular on some contributions of Murray Macbeath to surface topology.

1 Introduction

The classical results of Klein, Hurwitz and others on automorphisms of Riemann surfaces were based on the theory of algebraic curves. This contrasts somewhat with the approach today, which makes essential use of the uniformisation theorem and covering spaces. A rigorous theory of uniformisation was worked out via Dirichlet's Principle by Hilbert and Courant and completed independently by Koebe and by Poincarë in the years before 1910, establishing a firm basis for a systematic account of surface topology. Group actions in the hyperbolic plane were analysed by Dehn and Nielsen, while the 2-volume book of Fricke and Klein (1898, 1912) explored at length the immense range of discrete hyperbolic plane groups involved in this theory, formulating a classification of Fuchsian groups into distinct parameter spaces associated with each *signature* (or geometric type). Afterwards, the formulation of an abstract notion of manifold, signalled by Weyl's ground-breaking book [8], now just over a hundred years old, heralded a surge of interest in topology generally and low dimensional manifolds in particular.

It is worth noting that a serious hiatus in the systematic development of discrete group actions in the plane lasted for over 30 years from the late 1920s. Thus, in spite of Fricke's construction of parameter spaces for Fuchsian groups, the problem of moduli for Riemann surfaces remained unresolved until the theory of complex analytic deformations was set up, first by Teichmüller, in outline from 1938 to 1943 , and in rigorous detail by the school of Lars Ahlfors and Lipman Bers in the 1950s. The latter will not concern us here; that (now standard) material can be found in many sources, including the collected works of these two authors, [1] and [2].

2 Hurwitz's theorem reconsidered.

A brief note in a paper of C.L. Siegel ([7]) led Murray Macbeath to formulate a fundamental new approach to the study of Riemann surface automorphisms in the late 1950's. In 1893, A. Hurwitz showed that, for values of the genus $g \geq 2$, the maximum number of automorphisms of a surface is $84(g-1)$, a bound attained by the famous Klein quartic curve of genus 3, with automorphism group the simple group $PSL(2, \mathbb{F}_7)$ of order 168. This had been placed by Klein(1879) in the appropriate setting of the non-Euclidean plane geometry; for more details of that fascinating story, see [3]. Soon after, Poincaré began his own study of these discrete groups, which he called Fuchsian, later brought into great prominence by his realisation that the groups emerging from his study of uniformisation by differential equations are these same groups of isometries of the hyperbolic plane,

$$\mathcal{H} = \{z = x + iy \in \mathbb{C} : y > 0\}, \quad \text{with the Poincaré metric} \quad ds_h = \frac{|ds|}{2y}.$$

The (sense-preserving) isometry group of $\mathcal{H}$ is the projective matrix group $G = PSL(2, \mathbb{R})$ acting transitively by fractional linear transformations: if A is a 2×2 matrix, the corresponding mapping is

$$T_A : z \mapsto \frac{az+b}{cz+d}, \text{ when } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The invariant Haar measure of G induces an invariant notion of area μ in the homogeneous space $\mathcal{H} \cong G/PSO(2)$, known as the *Gauss-Bonnet area measure*; it coincides (up to a multiplicative constant) with the non-Euclidean area element induced by the Poincaré metric ds_h.

For a discrete subgroup Γ of the Lie group G, the action on $\mathcal{H}$ is properly discontinuous and there is a *fundamental domain*, a (Borel-measurable) subset F with the characteristic properties:-

(i) $F \cap \gamma F = \emptyset$ for $\gamma \in \Gamma$with $\gamma \neq \mathrm{Id}$;

(ii) $\bigcup_{\gamma \in \Gamma} \gamma F = \mathcal{H}$.

For Γ a co-compact lattice, i.e. such that the quotient orbit space $S = \mathcal{H}/\Gamma$ is a compact Riemann surface, the *Gauss-Bonnet area* $\mu(F)$ of any fundamental domain for Γ is a fixed positive number, independent of the choice of fundamental set. In particular, by the Gauss-Bonnet Theorem, the area of a fundamental set for a genus g closed surface is $4\pi(g-1)$. Siegel showed in [7] that over the range of all possible lattice groups in G there is a smallest positive value for the area.

Theorem 2.1 *For all co-compact Fuchsian groups, the minimum value of the invariant area is* $\pi/21$.

Note: this value corresponds to the triangle group $\langle x, y : x^2 = y^3 = (xy)^7 = 1\rangle$.

Now we let $K < \Gamma$ be a subgroup of finite index in the Fuchsian lattice group. Choosing a finite set of coset representatives, $\gamma_1, \ldots, \gamma_n$, we see that the union of these translates $\Cup_{j=1}^{n} \gamma_j F$ of a given fundamental set F for Γ forms a fundamental set F_K for K, and invariance of the measure proves that

Theorem 2.2 *The index of the subgroup K, $[\Gamma : K]$, is equal to the quotient $\mu(F_K)/\mu(F)$.*

This is the *Gauss-Bonnet Index Theorem* for Fuchsian groups. A simple consequence of this fact and the result of Siegel is the Hurwitz theorem: choose a Fuchsian cocompact group $K \cong \pi_1(S)$ and let Γ denote the group of all possible lifts to the universal covering of S of automorphisms of s. Then Γ contains K as a normal subgroup, and since $|\mathrm{Aut}(S)| = [\Gamma : K]$, we obtain at once the following

Corollary 2.3 *The order of any automorphism group acting on a genus g Riemann surface is at most $84(g-1)$.*

Of course this is by no means the end of the story: the same line of reasoning produces the Riemann-Hurwitz branching formula involving the genus of $\mathcal{H}/\Gamma$ and the orders of the maximal periodic generators of Γ, and there is a natural extension to subgroups of finite index in arbitrary non-Euclidean crystallographic groups, leading to a vast catalogue of results recording the patterns of group actions on hyperbolic surfaces.

Macbeath's paper [4] fills in the details of the above proof, and unveils a method (the *'Macbeath trick' mentioned by Marston Conder in his conference talk here) for constructing from a single finite index normal subgroup $K \triangleleft \Gamma$ an infinite sequence of other finite index subgroups of Γ. The original exposition of this approach to Fuchsian groups and surface automorphisms was presented in lecture notes by Macbeath at Dundee in 1961; they can be obtained in pdf format by emailing me. A very pleasant account of the whole story can be found in [6].*

References

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