The Grothendieck-Teichmüller group and equivariant dessins

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SIGMAP 2014

Introduction

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There is an inclusion

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow \lim_{G} \mathcal{GT}(G) .$$

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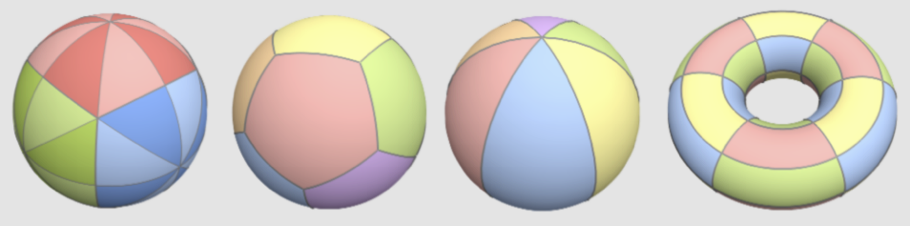

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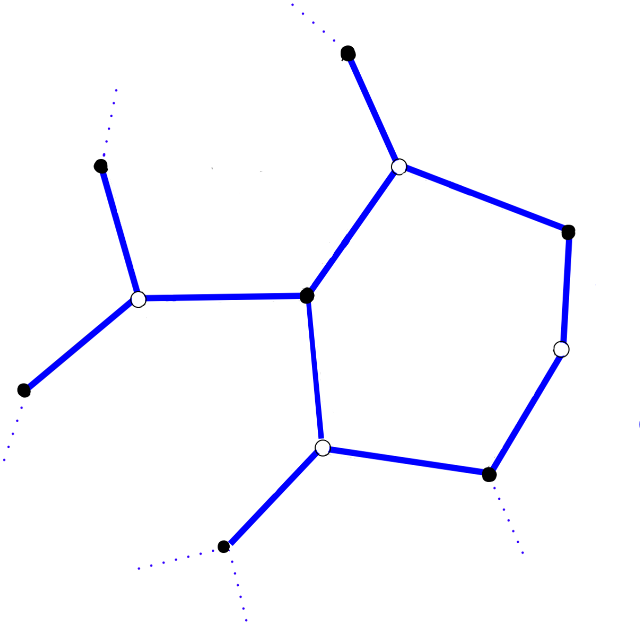

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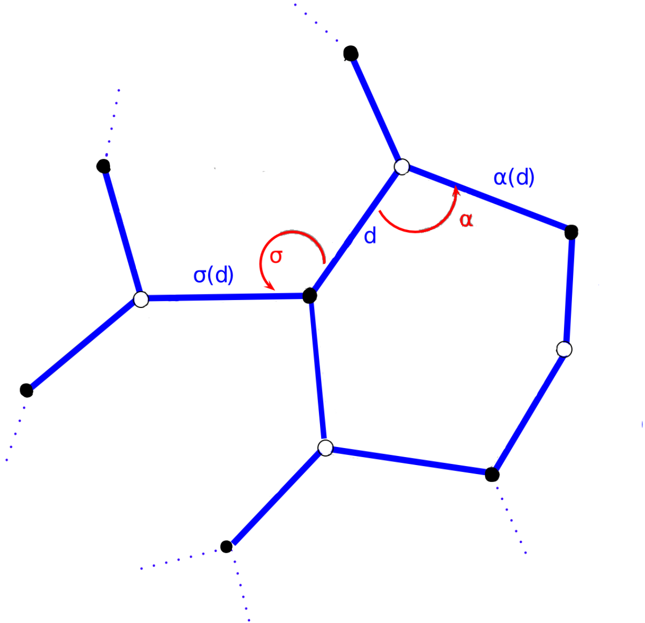

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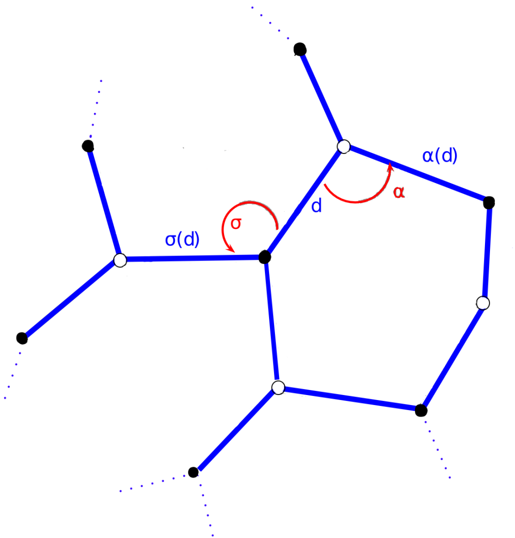
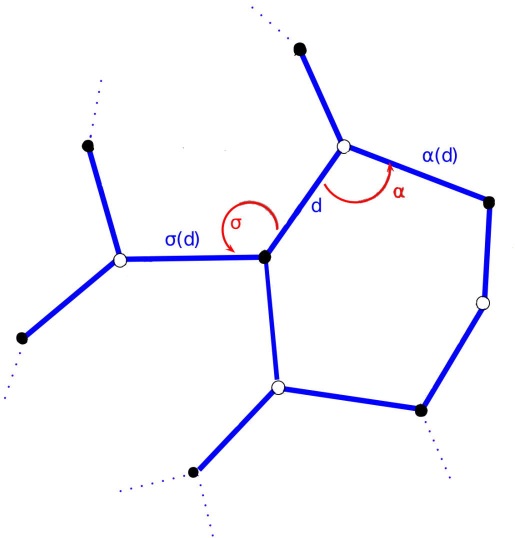


Graph embedded on a surface
$\Longleftrightarrow$
two permutations σ and α on a finite set.



Graph embedded on a surface
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When the set is $G = \langle \sigma, \alpha \rangle$ itself, the dessin is called regular.

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$$f\colon C \longrightarrow \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}\,.$$

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Example

A regular dessin (G, σ, α) becomes (G, σ', α').

Construction of the covering group of G

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Given a finite group G, consider $F_2 = \langle \sigma, \alpha \rangle$ and

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One can show that the image of this monomorphism actually lies in $\lim_G \mathcal{GT}(G)$, to be defined next.

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We also define $\mathcal{GT}_1(G)$ to be the subgroup of $\mathcal{GT}(G)$ for which we can take $k = 1$.

First results

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Theorem

1. When G is a p-group, so is $\mathcal{GT}_1(G)$.
2. When G is nilpotent, so is $\mathcal{GT}_1(G)$.
3. The quotient $\mathcal{GT}(G)/\mathcal{GT}_1(G)$ is abelian.

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When G is simple, the group $\mathcal{GT}_1(G)$ is isomorphic to a permutation group which is the intersection of

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Corollary

A simple factor occuring in $\mathcal{GT}_1(G)$ must be isomorphic to either

- C_2,
- C_3,
- a subquotient of $Out(G)$,
- an alternating group A_s where $s \leq \frac{m^2}{|G|}$.

($m =$ size of the largest conjugacy class of G)

Explicit computations

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- $\mathcal{GT}_1(PSL_2(\mathbb{F}_4))$ is trivial.
- $\mathcal{GT}_1(PSL_2(\mathbb{F}_7)) \cong C_2^3 \times D_8^2$.
- $\mathcal{GT}_1(PSL_2(\mathbb{F}_9)) \cong C_2^{12} \times D_8$.
- $\mathcal{GT}_1(PSL_2(\mathbb{F}_8))$ is trivial.
- $\mathcal{GT}_1(PSL_2(\mathbb{F}_{11})) \cong C_2^{27} \times D_8^7$.
- $\mathcal{GT}_1(PSL_2(\mathbb{F}_{13})) \cong C_2^{54} \times D_8^{17}$.
- $\mathcal{GT}_1(PSL_2(\mathbb{F}_{17})) \cong C_2^{104} \times D_8^{50}$.
- $\mathcal{GT}_1(PSL_2(\mathbb{F}_{19})) \cong C_2^{133} \times D_8^{74}$.
- $\mathcal{GT}_1(PSL_2(\mathbb{F}_{16}))$ is trivial.

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The direct product of the simple factors of $\mathcal{GT}_1(M_{11})$ is

$$C_2^{465} \times C_3^{46} \times A_5^{10} \times A_6^{9} \times A_7^{10} \times A_8^{4} \times A_9^{4} \times A_{10}^{5} \times A_{11}^{5} \times A_{12} \times A_{14}^{2} \times A_{15}^{4} \times A_{16} \times A_{17}^{3} \times A_{18}^{12} \times A_{19} \times A_{20}^{2} \times A_{23} \times A_{28} \times A_{31} \times A_{33}^{2} .$$

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Order : $2^{1141} \cdot 3^{407} \cdot 5^{165} \cdot 7^{98} \cdot 11^{43} \cdot 13^{34} \cdot 17^{23} \cdot 19^8 \cdot 23^5 \cdot 29^3 \cdot 31^3 =$

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Remark

- Regular dessins with underlying group G are in bijection with $\mathcal{P}/Aut(G)$, where $\mathcal{P} = \{(\sigma, \alpha) \in G \times G$ such that $G = \langle \sigma, \alpha \rangle\}$.

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- G-equivariant regular dessins with underlying group G are in bijection with $\mathcal{P}/Inn(G)$.

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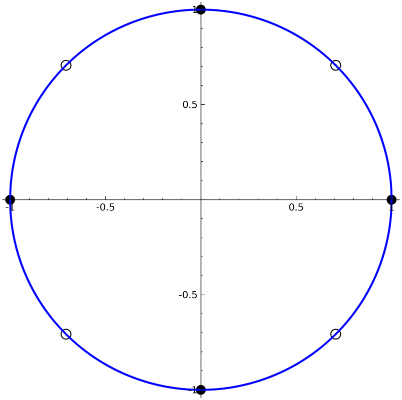

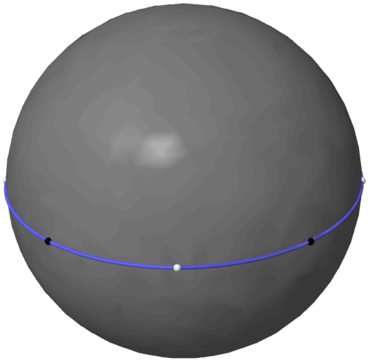

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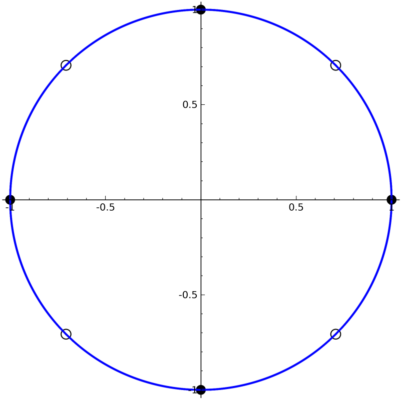

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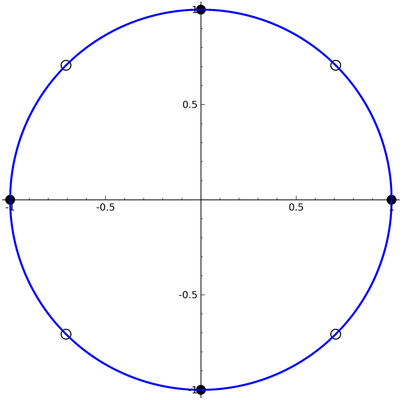

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However there are two distinct C_4 actions, permuted by $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Thank you !

arxiv : 1309.1968