Extension of the classification of high rank polytopes

M. Elisa Fernandes

Universidade de Aveiro, Portugal

joint work with

Dimitri Leemans and Mark Mixer

SIGMAP 14

A **string C-group** is a group generated by involutions

$$\langle \rho_0, \ldots, \rho_{r-1} \rangle$$

satisfying (1) and (2).

A **string C-group** is a group generated by involutions

$$\langle \rho_0, \dots, \rho_{r-1} \rangle$$

satisfying (1) and (2).

(1) The **commuting property**

$$|i - j| > 1 \Rightarrow (\rho_i \rho_j)^2 = 1$$

A **string C-group** is a group generated by involutions

$$\langle \rho_0, \ldots, \rho_{r-1} \rangle$$

satisfying (1) and (2).

(1) The **commuting property**

$$|i-j| > 1 \Rightarrow (\rho_i \rho_j)^2 = 1$$

(2) The **intersection condition**

$$\langle \rho_j \mid j \in J \rangle \cap \langle \rho_j \mid j \in K \rangle = \langle \rho_j \mid j \in J \cap K \rangle, \quad J, K \subseteq \{0, ..., r-1\};$$

A **string C-group** is a group generated by involutions

$$\langle \rho_0, \ldots, \rho_{r-1} \rangle$$

satisfying (1) and (2).

(1) The **commuting property**

$$|i-j| > 1 \Rightarrow (\rho_i \rho_j)^2 = 1$$

(2) The **intersection condition**

$$\langle \rho_j \mid j \in J \rangle \cap \langle \rho_j \mid j \in K \rangle = \langle \rho_j \mid j \in J \cap K \rangle, \quad J, K \subseteq \{0, ..., r-1\};$$

The **type** is $\{p_1, \ldots, p_{r-1}\}$ where p_i is the order of $\rho_{i-1}\rho_i$, $i \in \{1, \ldots, r-1\}$.

A **string C-group** is a group generated by involutions

$$\langle \rho_0, \ldots, \rho_{r-1} \rangle$$

satisfying (1) and (2).

(1) The **commuting property**

$$|i-j| > 1 \Rightarrow (\rho_i \rho_j)^2 = 1$$

(2) The **intersection condition**

$$\langle \rho_j \mid j \in J \rangle \cap \langle \rho_j \mid j \in K \rangle = \langle \rho_j \mid j \in J \cap K \rangle, \quad J, K \subseteq \{0, ..., r-1\};$$

The **type** is $\{p_1, \ldots, p_{r-1}\}$ where p_i is the order of $\rho_{i-1}\rho_i$, $i \in \{1, \ldots, r-1\}$.

A string C-group has **connected Coxeter diagram (CD)** if
$p_i \neq 2, \ i \in \{1, \ldots, r-1\}$

A **string C-group** is a group generated by involutions

$$\langle \rho_0, \dots, \rho_{r-1} \rangle$$

satisfying (1) and (2).

(1) The **commuting property**

$$|i-j| > 1 \Rightarrow (\rho_i \rho_j)^2 = 1$$

(2) The **intersection condition**

$$\langle \rho_j \mid j \in J \rangle \cap \langle \rho_j \mid j \in K \rangle = \langle \rho_j \mid j \in J \cap K \rangle, \quad J, K \subseteq \{0, ..., r-1\};$$

The **type** is $\{p_1, \dots, p_{r-1}\}$ where p_i is the order of $\rho_{i-1}\rho_i$, $i \in \{1, \dots, r-1\}$.

A string C-group has **connected Coxeter diagram (CD)** if $p_i \neq 2,\ i \in \{1, \dots, r-1\}$

The **dual** of a string C-group is obtained by reversing the order of the generators.

A **string C-group** is a group generated by involutions

$$\langle \rho_0, \ldots, \rho_{r-1} \rangle$$

satisfying (1) and (2).

(1) The **commuting property**

$$|i-j| > 1 \Rightarrow (\rho_i \rho_j)^2 = 1$$

(2) The **intersection condition**

$$\langle \rho_j \mid j \in J \rangle \cap \langle \rho_j \mid j \in K \rangle = \langle \rho_j \mid j \in J \cap K \rangle, \quad J, K \subseteq \{0, ..., r-1\};$$

The **type** is $\{p_1, \ldots, p_{r-1}\}$ where p_i is the order of $\rho_{i-1}\rho_i$, $i \in \{1, \ldots, r-1\}$.

A string C-group has **connected Coxeter diagram (CD)** if
$p_i \neq 2, \ i \in \{1, \ldots, r-1\}$

The **dual** of a string C-group is obtained by reversing the order of the generators.

r-polytope := string C-group of rank r with CD.

Highest rank of polytopes for primitive groups

Extension of the classification of high rank polytopes

[2000; **Whiston**] The size of an independent generating set is at most $n-1$. Moreover, an independent set of size $n-1$ generates S_n.

Highest rank of polytopes for primitive groups

Extension of the classification of high rank polytopes

[2000; **Whiston**] The size of an independent generating set is at most $n-1$. Moreover, an independent set of size $n-1$ generates S_n.

[2011; **Fernandes, Leemans, Mixer**]

- For every $n \geq 12$ there exist polytope for A_n with rank $\lfloor \frac{n-1}{2} \rfloor$.

Conjecture: The highest rank of a polytope for A_n is $\lfloor \frac{n-1}{2} \rfloor$.

Highest rank of polytopes for primitive groups

Extension of the classification of high rank polytopes

[2000; **Whiston**] The size of an independent generating set is at most $n-1$. Moreover, an independent set of size $n-1$ generates S_n.

[2011; **Fernandes, Leemans, Mixer**]

- For every $n \geq 12$ there exist polytope for A_n with rank $\lfloor \frac{n-1}{2} \rfloor$.

Conjecture: The highest rank of a polytope for A_n is $\lfloor \frac{n-1}{2} \rfloor$.

[2002; **Maróti**] The size of a primitive group of degree n which is not A_n nor S_n is at most $50n^{\sqrt{n}}$.

Highest rank of polytopes for primitive groups

Extension of the classification of high rank polytopes

[2000; **Whiston**] The size of an independent generating set is at most $n-1$. Moreover, an independent set of size $n-1$ generates S_n.

[2011; **Fernandes, Leemans, Mixer**]

- For every $n \geq 12$ there exist polytope for A_n with rank $\lfloor \frac{n-1}{2} \rfloor$.

Conjecture: The highest rank of a polytope for A_n is $\lfloor \frac{n-1}{2} \rfloor$.

[2002; **Maróti**] The size of a primitive group of degree n which is not A_n nor S_n is at most $50n^{\sqrt{n}}$.

[2013; **Conder**] The size of a r-polytope is at least $\frac{4^r}{2}$ when $r > 9$.

Highest rank of polytopes for primitive groups

Extension of the classification of high rank polytopes

[2000; **Whiston**] The size of an independent generating set is at most $n-1$. Moreover, an independent set of size $n-1$ generates S_n.

[2011; **Fernandes, Leemans, Mixer**]

- For every $n \geq 12$ there exist polytope for A_n with rank $\lfloor \frac{n-1}{2} \rfloor$.

Conjecture: The highest rank of a polytope for A_n is $\lfloor \frac{n-1}{2} \rfloor$.

[2002; **Maróti**] The size of a primitive group of degree n which is not A_n nor S_n is at most $50n^{\sqrt{n}}$.

[2013; **Conder**] The size of a r-polytope is at least $\frac{4^r}{2}$ when $r > 9$.

For $r > 9$ we have, $r \leq \log_4(100n^{\sqrt{n}})$.

Highest rank of polytopes for primitive groups

Extension of the classification of high rank polytopes

[2000; **Whiston**] The size of an independent generating set is at most $n-1$. Moreover, an independent set of size $n-1$ generates S_n.

[2011; **Fernandes, Leemans, Mixer**]

- For every $n \geq 12$ there exist polytope for A_n with rank $\lfloor \frac{n-1}{2} \rfloor$.

Conjecture: The highest rank of a polytope for A_n is $\lfloor \frac{n-1}{2} \rfloor$.

[2002; **Maróti**] The size of a primitive group of degree n which is not A_n nor S_n is at most $50n^{\sqrt{n}}$.

[2013; **Conder**] The size of a r-polytope is at least $\frac{4^r}{2}$ when $r > 9$.

For $r > 9$ we have, $r \leq \log_4(100n^{\sqrt{n}})$.

[**FLM**] Let r be the rank of a transitive primitive polytope of degree $n \geq 12$, which is neither A_n nor S_n.

$$r \leq \frac{n-3}{2}$$

[2000; **Whiston**] For $n \geq 7$, an independent generating set for a transitive imprimitive group of degree n has size at most $n-3$.

[2000; **Whiston**] For $n \geq 7$, an independent generating set for a transitive imprimitive group of degree n has size at most $n-3$.

[**FLM**] Let r be the rank of a transitive imprimitive string C-group .

$$r \leq m + k + 1,$$

where k is the size of a block and m is the number of blocks for an embedding of the string C-group into $S_k \wr S_m$ having a maximal k.

Highest rank of string C-group for imprimitive groups

[2000; **Whiston**] For $n \geq 7$, an independent generating set for a transitive imprimitive group of degree n has size at most $n-3$.

[**FLM**] Let r be the rank of a transitive imprimitive string C-group .

$$r \leq m + k + 1,$$

where k is the size of a block and m is the number of blocks for an embedding of the string C-group into $S_k \wr S_m$ having a maximal k.

We assume Γ is embedded into $S_k \wr S_m$ with k being the maximal.

L- generators that independently generate the block action;

C- generators commuting with every element of L;

R- remaining generators of Γ.

[2000; **Whiston**] For $n \geq 7$, an independent generating set for a transitive imprimitive group of degree n has size at most $n-3$.

[**FLM**] Let r be the rank of a transitive imprimitive string C-group .

$$r \leq m + k + 1,$$

where k is the size of a block and m is the number of blocks for an embedding of the string C-group into $S_k \wr S_m$ having a maximal k.

We assume Γ is embedded into $S_k \wr S_m$ with k being the maximal.

L- generators that independently generate the block action;

C- generators commuting with every element of L;

R- remaining generators of Γ.

- $|L| \leq m-1$ and $|L| = m-1 \Rightarrow |R| \leq 2$

[2000; **Whiston**] For $n \geq 7$, an independent generating set for a transitive imprimitive group of degree n has size at most $n-3$.

[**FLM**] Let r be the rank of a transitive imprimitive string C-group .

$$r \leq m + k + 1,$$

where k is the size of a block and m is the number of blocks for an embedding of the string C-group into $S_k \wr S_m$ having a maximal k.

We assume Γ is embedded into $S_k \wr S_m$ with k being the maximal.

L- generators that independently generate the block action;

C- generators commuting with every element of L;

R- remaining generators of Γ.

- $|L| \leq m-1$ and $|L| = m-1 \Rightarrow |R| \leq 2$
- $|C| \leq k-1$

[2000; **Whiston**] For $n \geq 7$, an independent generating set for a transitive imprimitive group of degree n has size at most $n-3$.

[**FLM**] Let r be the rank of a transitive imprimitive string C-group .

$$r \leq m + k + 1,$$

where k is the size of a block and m is the number of blocks for an embedding of the string C-group into $S_k \wr S_m$ having a maximal k.

We assume Γ is embedded into $S_k \wr S_m$ with k being the maximal.

L- generators that independently generate the block action;

C- generators commuting with every element of L;

R- remaining generators of Γ.

- $|L| \leq m-1$ and $|L| = m-1 \Rightarrow |R| \leq 2$
- $|C| \leq k-1$
- $|R| \leq 4$

Number of polytopes , up to duality, for S_n $(5 \leq n \leq 14)$

G\r	3	4	5	6	7	8	9	10	11	12	13
S_5	4	**1**	0	0	0	0	0	0	0	0	0
S_6	2	4	**1**	0	0	0	0	0	0	0	0
S_7	35	7	**1**	**1**	0	0	0	0	0	0	0
S_8	68	36	11	**1**	**1**	0	0	0	0	0	0
S_9	129	37	7	**7**	**1**	**1**	0	0	0	0	0
S_{10}	413	203	52	13	**7**	**1**	**1**	0	0	0	0
S_{11}	1221	189	43	25	**9**	**7**	**1**	**1**	0	0	0
S_{12}	3346	940	183	75	40	**9**	**7**	**1**	**1**	0	0
S_{13}	7163	863	171	123	41	35	**9**	**7**	**1**	**1**	0
S_{14}	23126	3945	978	303	163	54	35	**9**	**7**	**1**	**1**

Polytopes of rank $r \geq n - 4$

Extension of the classification of high rank polytopes

[2011; **FL**]

- For $n \geq 5$, the n-simplex is the unique $(n-1)$-polytope for S_n.
- For $n \geq 7$, there is, up to duality, a unique $(n-2)$-polytope S_n.
- There exists at least one r-polytope for each rank $r \in \{3, \ldots, n-1\}$ for S_n.

[2011; **FL**]

- For $n \geq 5$, the n-simplex is the unique $(n-1)$-polytope for S_n.
- For $n \geq 7$, there is, up to duality, a unique $(n-2)$-polytope S_n.
- There exists at least one r-polytope for each rank $r \in \{3, \ldots, n-1\}$ for S_n.

[**FML**]

Polytopes of rank $r \geq n-4$

Extension of the classification of high rank polytopes

[2011; **FL**]

- For $n \geq 5$, the n-simplex is the unique $(n-1)$-polytope for S_n.
- For $n \geq 7$, there is, up to duality, a unique $(n-2)$-polytope S_n.
- There exists at least one r-polytope for each rank $r \in \{3, \ldots, n-1\}$ for S_n.

[**FML**]

- For $n \geq 9$, there are, up to duality, 7 polytopes of rank $n-3$ for S_n.

Polytopes of rank $r \geq n - 4$

Extension of the classification of high rank polytopes

[2011; **FL**]

- For $n \geq 5$, the n-simplex is the unique $(n-1)$-polytope for S_n.
- For $n \geq 7$, there is, up to duality, a unique $(n-2)$-polytope S_n.
- There exists at least one r-polytope for each rank $r \in \{3, \ldots, n-1\}$ for S_n.

[**FML**]

- For $n \geq 9$, there are, up to duality, 7 polytopes of rank $n-3$ for S_n.
- For $n \geq 11$, there are, up to duality, 9 polytopes of rank $n-4$ for S_n.

Polytopes of rank $r \geq n - 4$

[2011; **FL**]

- For $n \geq 5$, the n-simplex is the unique $(n-1)$-polytope for S_n.
- For $n \geq 7$, there is, up to duality, a unique $(n-2)$-polytope S_n.
- There exists at least one r-polytope for each rank $r \in \{3, \ldots, n-1\}$ for S_n.

[**FML**]

- For $n \geq 9$, there are, up to duality, 7 polytopes of rank $n-3$ for S_n.
- For $n \geq 11$, there are, up to duality, 9 polytopes of rank $n-4$ for S_n.
- If Γ is a r-polytope for a transitive group of degree n with $r \geq n-4$ and $n \geq 11$, then

$$\Gamma \cong S_n.$$

Let $\Gamma = \langle \rho_0, \ldots, \rho_{r-1} \rangle$ be a r-polytope for a transitive permutation group of degree $n \geq 15$ with $r \geq n - 4$.

Let $\Gamma = \langle \rho_0, \ldots, \rho_{r-1} \rangle$ be a r-polytope for a transitive permutation group of degree $n \geq 15$ with $r \geq n - 4$.

The **parabolic subgroup** Γ_i is the group generated by $\{\rho_j \mid j \neq i\}$.

Let $\Gamma = \langle \rho_0, \ldots, \rho_{r-1} \rangle$ be a r-polytope for a transitive permutation group of degree $n \geq 15$ with $r \geq n - 4$.

The **parabolic subgroup** Γ_i is the group generated by $\{\rho_j \mid j \neq i\}$.

All parabolic subgroups are intransitive.

CPR graphs

Extension of the classification of high rank polytopes

- We construct all possible permutation graphs for Γ when all Γ_i are intransitive.

- We construct all possible permutation graphs for Γ when all Γ_i are intransitive.

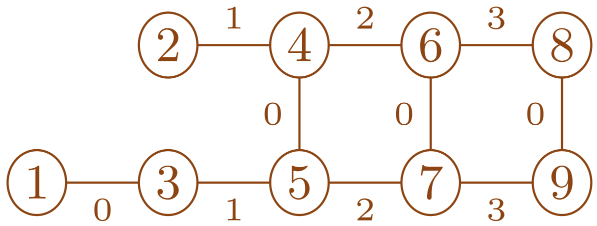
[2008; **Pellicer**] A **CPR graph** is a permutation representation of a permutation string C-group .

- We construct all possible permutation graphs for Γ when all Γ_i are intransitive.

[2008; **Pellicer**] A **CPR graph** is a permutation representation of a permutation string C-group .

Example: CPR graph of 4-polytope of type (10,3,3) for A_9

$$\rho_0 = (13)(45)(67)(89) \quad \rho_1 = (24)(35) \quad \rho_2 = (46)(57) \quad \rho_3 = (68)(79).$$

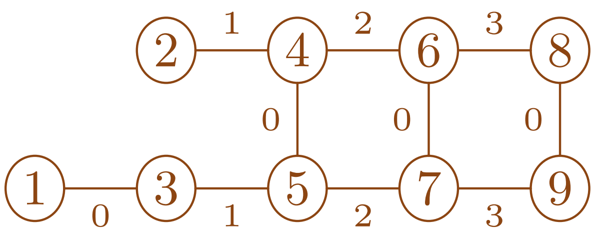


- We construct all possible permutation graphs for Γ when all Γ_i are intransitive.

[2008; **Pellicer**] A **CPR graph** is a permutation representation of a permutation string C-group .

Example: CPR graph of 4-polytope of type (10,3,3) for A_9

$$\rho_0 = (13)(45)(67)(89) \quad \rho_1 = (24)(35) \quad \rho_2 = (46)(57) \quad \rho_3 = (68)(79).$$



A CPR graphs has the following properties:

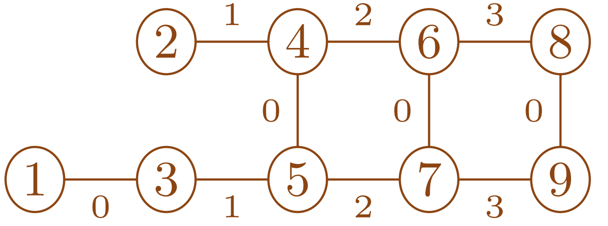
CPR graphs

- We construct all possible permutation graphs for Γ when all Γ_i are intransitive.

[2008; **Pellicer**] A **CPR graph** is a permutation representation of a permutation string C-group .

Example: CPR graph of 4-polytope of type (10,3,3) for A_9

$$\rho_0 = (13)(45)(67)(89) \quad \rho_1 = (24)(35) \quad \rho_2 = (46)(57) \quad \rho_3 = (68)(79).$$



A CPR graphs has the following properties:

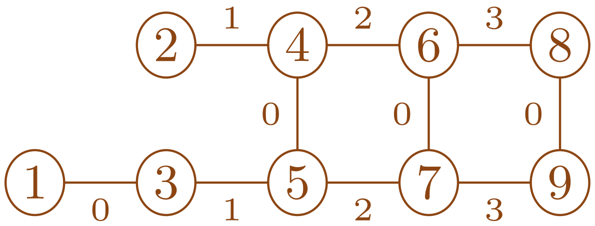
(1) The graph induced by the i-edges is a matching;

- We construct all possible permutation graphs for Γ when all Γ_i are intransitive.

[2008; **Pellicer**] A **CPR graph** is a permutation representation of a permutation string C-group .

Example: CPR graph of 4-polytope of type (10,3,3) for A_9

$$\rho_0 = (13)(45)(67)(89) \quad \rho_1 = (24)(35) \quad \rho_2 = (46)(57) \quad \rho_3 = (68)(79).$$



A CPR graphs has the following properties:

(1) The graph induced by the i-edges is a matching;

(2) If $|i - j| \geq 2$, then the i and j-edges are either parallel edges, double edges or form an alternating i, j-squares.

CPR graphs

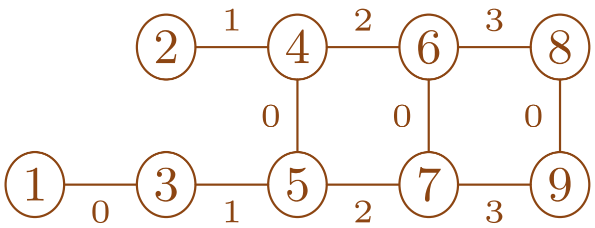
Extension of the classification of high rank polytopes

- We construct all possible permutation graphs for Γ when all Γ_i are intransitive.

[2008; **Pellicer**] A **CPR graph** is a permutation representation of a permutation string C-group .

Example: CPR graph of 4-polytope of type (10,3,3) for A_9

$$\rho_0 = (13)(45)(67)(89) \quad \rho_1 = (24)(35) \quad \rho_2 = (46)(57) \quad \rho_3 = (68)(79).$$



A CPR graphs has the following properties:

(1) The graph induced by the i-edges is a matching;

(2) If $|i-j| \geq 2$, then the i and j-edges are either parallel edges, double edges or form an alternating i,j-squares.

A CPR graph is **linear** if and only if adjacent edges have consecutive labels.

Fracture graph

Fracture graph

Let Γ_i be intransitive for all $i \in \{0, \ldots, r-1\}$ and $\mathcal{G}$ be its CPR graph.

Fracture graph

Extension of the classification of high rank polytopes

Let Γ_i be intransitive for all $i \in \{0, \ldots, r-1\}$ and $\mathcal{G}$ be its CPR graph.

A **fracture graph** $\mathcal{F}$ of Γ is a graph with n vertices and with one edge of each label, as follows:

$$\mathcal{F} \text{ has an } i\text{-edges } \{a, b\} \text{ if } a \text{ and } b \text{ are in different } \Gamma_i\text{-orbits.}$$

Fracture graph

Let Γ_i be intransitive for all $i \in \{0, \ldots, r-1\}$ and $\mathcal{G}$ be its CPR graph.

A **fracture graph** $\mathcal{F}$ of Γ is a graph with n vertices and with one edge of each label, as follows:

$$\mathcal{F} \text{ has an } i\text{-edges } \{a, b\} \text{ if } a \text{ and } b \text{ are in different } \Gamma_i\text{-orbits.}$$

Any fracture graph of $\mathcal{G}$ has the following properties:

Fracture graph

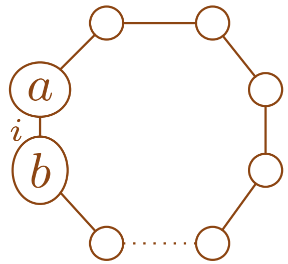
Let Γ_i be intransitive for all $i \in \{0, \ldots, r-1\}$ and $\mathcal{G}$ be its CPR graph.

A **fracture graph** $\mathcal{F}$ of Γ is a graph with n vertices and with one edge of each label, as follows:

$$\mathcal{F} \text{ has an } i\text{-edges } \{a, b\} \text{ if } a \text{ and } b \text{ are in different } \Gamma_i\text{-orbits.}$$

Any fracture graph of $\mathcal{G}$ has the following properties:

- A fracture graph has no cycles.



a and b are in the same Γ_i-orbit. Contradition!!!

Fracture graph

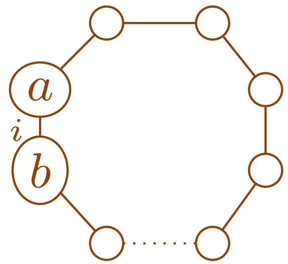
Let Γ_i be intransitive for all $i \in \{0, \ldots, r-1\}$ and $\mathcal{G}$ be its CPR graph.

A **fracture graph** $\mathcal{F}$ of Γ is a graph with n vertices and with one edge of each label, as follows:

$$\mathcal{F} \text{ has an } i\text{-edges } \{a, b\} \text{ if } a \text{ and } b \text{ are in different } \Gamma_i\text{-orbits.}$$

Any fracture graph of $\mathcal{G}$ has the following properties:

- A fracture graph has no cycles.



a and b are in the same Γ_i-orbit. Contradition!!!

- A fracture graph has c connected components if and only if $r = n - c$.

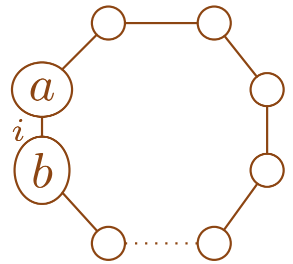
Let Γ_i be intransitive for all $i \in \{0, \ldots, r-1\}$ and $\mathcal{G}$ be its CPR graph.

A **fracture graph** $\mathcal{F}$ of Γ is a graph with n vertices and with one edge of each label, as follows:

$$\mathcal{F} \text{ has an } i\text{-edges } \{a, b\} \text{ if } a \text{ and } b \text{ are in different } \Gamma_i\text{-orbits.}$$

Any fracture graph of $\mathcal{G}$ has the following properties:

- A fracture graph has no cycles.



a and b are in the same Γ_i-orbit. Contradition!!!

- A fracture graph has c connected components if and only if $r = n - c$.

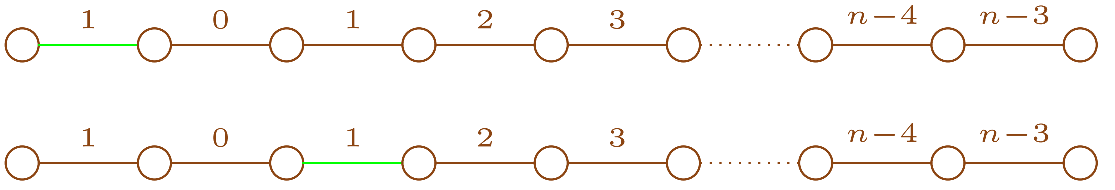
- If $\mathcal{G}$ has an alternating square then at least two vertices of the square are in different components of any fracture graph.

Fracture graphs of polytopes of rank $\geq n-4$

For $r = n-1$, a fracture graph of $\mathcal{G}$ has only one component and is linear. There is only one possibility, corresponding to the CPR-graph of the $(n-1)$**-simplex**.

For $r = n-1$, a fracture graph of $\mathcal{G}$ has only one component and is linear. There is only one possibility, corresponding to the CPR-graph of the $(n-1)$**-simplex**.

For $r = n-2$, a fracture graph of $\mathcal{G}$ has two components and is linear. Up to duality, there are two possibilities for $\mathcal{F}$ corresponding to unique possibility for $\mathcal{G}$:

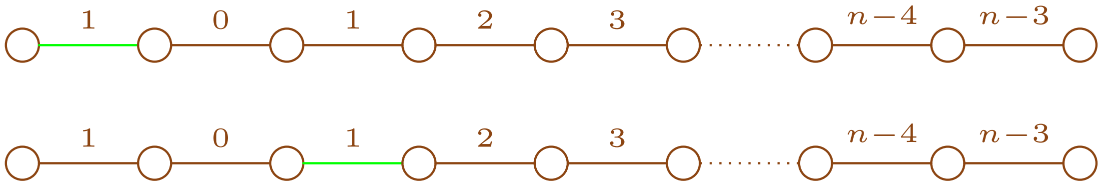


Fracture graphs of polytopes of rank $\geq n-4$

Extension of the classification of high rank polytopes

For $r = n-1$, a fracture graph of $\mathcal{G}$ has only one component and is linear. There is only one possibility, corresponding to the CPR-graph of the $(n-1)$**-simplex**.

For $r = n-2$, a fracture graph of $\mathcal{G}$ has two components and is linear. Up to duality, there are two possibilities for $\mathcal{F}$ corresponding to unique possibility for $\mathcal{G}$:

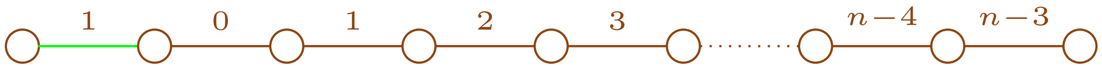


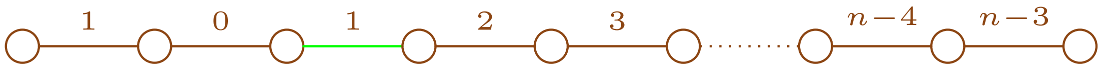
For $r = n-3$ or $n-4$:

Fracture graphs of polytopes of rank $\geq n-4$

For $r = n-1$, a fracture graph of $\mathcal{G}$ has only one component and is linear. There is only one possibility, corresponding to the CPR-graph of the **$(n-1)$-simplex**.

For $r = n-2$, a fracture graph of $\mathcal{G}$ has two components and is linear. Up to duality, there are two possibilities for $\mathcal{F}$ corresponding to unique possibility for $\mathcal{G}$:





For $r = n-3$ or $n-4$:

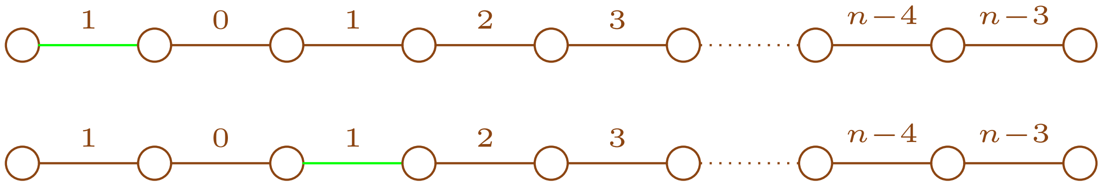
- A fracture graph of $\mathcal{G}$ has either 3 or 4 components, resp..

Fracture graphs of polytopes of rank $\geq n-4$

Extension of the classification of high rank polytopes

For $r = n-1$, a fracture graph of $\mathcal{G}$ has only one component and is linear. There is only one possibility, corresponding to the CPR-graph of the $(n-1)$**-simplex**.

For $r = n-2$, a fracture graph of $\mathcal{G}$ has two components and is linear. Up to duality, there are two possibilities for $\mathcal{F}$ corresponding to unique possibility for $\mathcal{G}$:



For $r = n-3$ or $n-4$:

- A fracture graph of $\mathcal{G}$ has either 3 or 4 components, resp..

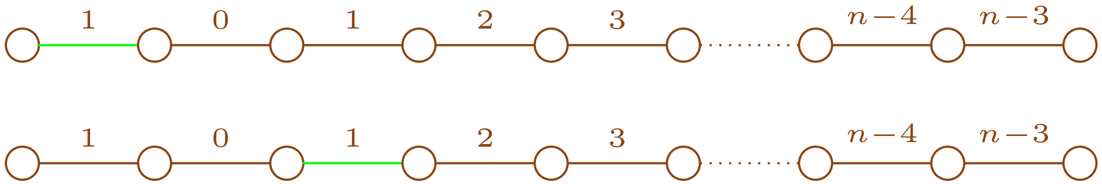
- A fracture graph of $\mathcal{G}$ is linear or has exactly one vertex of degree 3.

Fracture graphs of polytopes of rank $\geq n-4$

Extension of the classification of high rank polytopes

For $r = n-1$, a fracture graph of $\mathcal{G}$ has only one component and is linear. There is only one possibility, corresponding to the CPR-graph of the $(n-1)$-**simplex**.

For $r = n-2$, a fracture graph of $\mathcal{G}$ has two components and is linear. Up to duality, there are two possibilities for $\mathcal{F}$ corresponding to unique possibility for $\mathcal{G}$:



For $r = n-3$ or $n-4$:

- A fracture graph of $\mathcal{G}$ has either 3 or 4 components, resp..

- A fracture graph of $\mathcal{G}$ is linear or has exactly one vertex of degree 3.

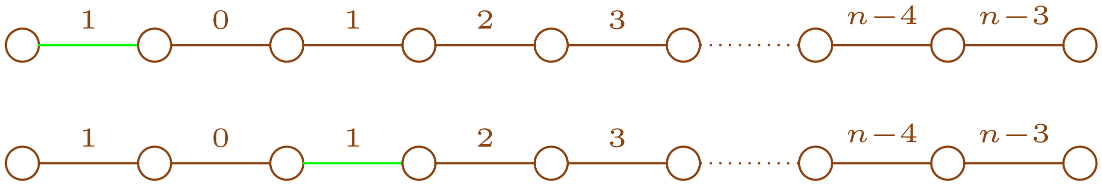
- We list all possibilities of connecting the 3 (or 4) components of $\mathcal{F}$ and we find all possibilities for $\mathcal{G}$.

Fracture graphs of polytopes of rank $\geq n-4$

Extension of the classification of high rank polytopes

For $r = n-1$, a fracture graph of $\mathcal{G}$ has only one component and is linear. There is only one possibility, corresponding to the CPR-graph of the $(n-1)$**-simplex**.

For $r = n-2$, a fracture graph of $\mathcal{G}$ has two components and is linear. Up to duality, there are two possibilities for $\mathcal{F}$ corresponding to unique possibility for $\mathcal{G}$:



For $r = n-3$ or $n-4$:

- A fracture graph of $\mathcal{G}$ has either 3 or 4 components, resp..

- A fracture graph of $\mathcal{G}$ is linear or has exactly one vertex of degree 3.

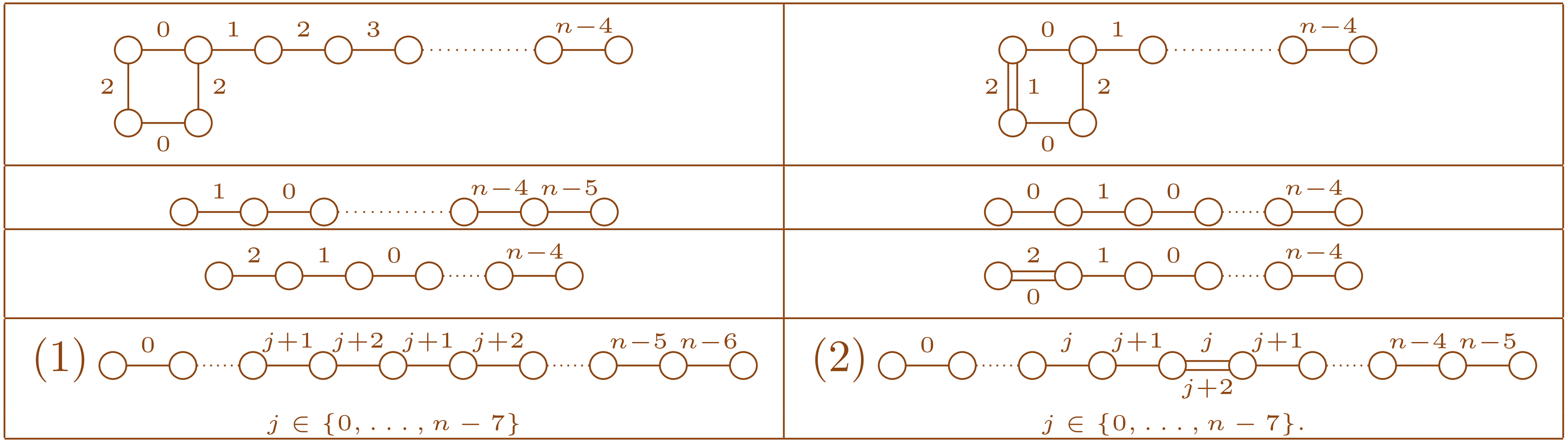
- We list all possibilities of connecting the 3 (or 4) components of $\mathcal{F}$ and we find all possibilities for $\mathcal{G}$.

- Not all string groups generated by involution obtained are string C-groups .

case $r = n - 3$

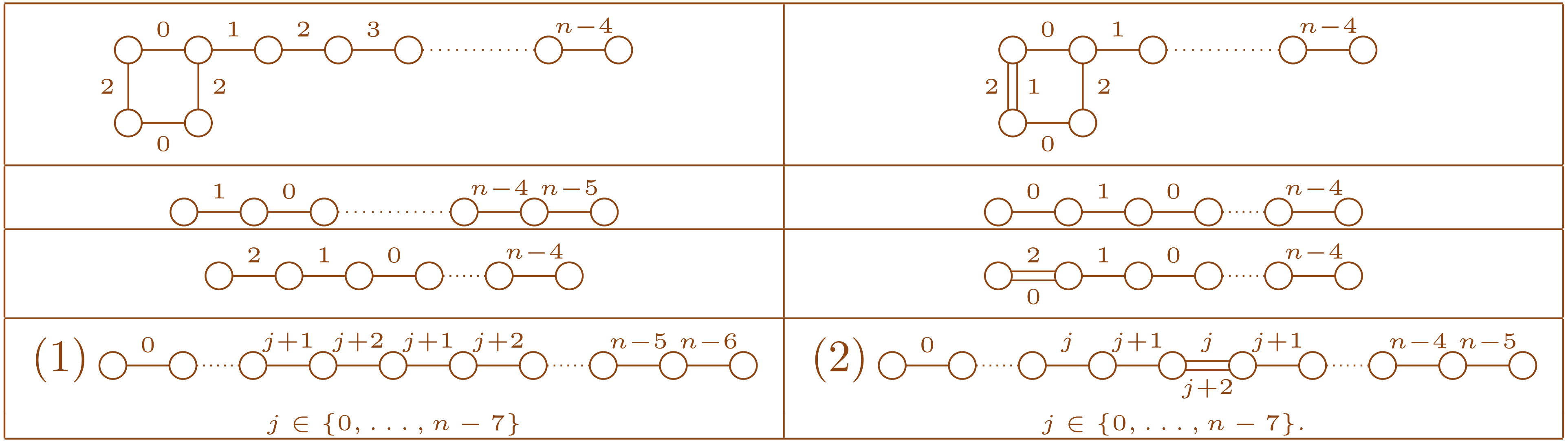
case $r = n - 3$

We get the following possibilities for $\mathcal{G}$:



case $r = n - 3$

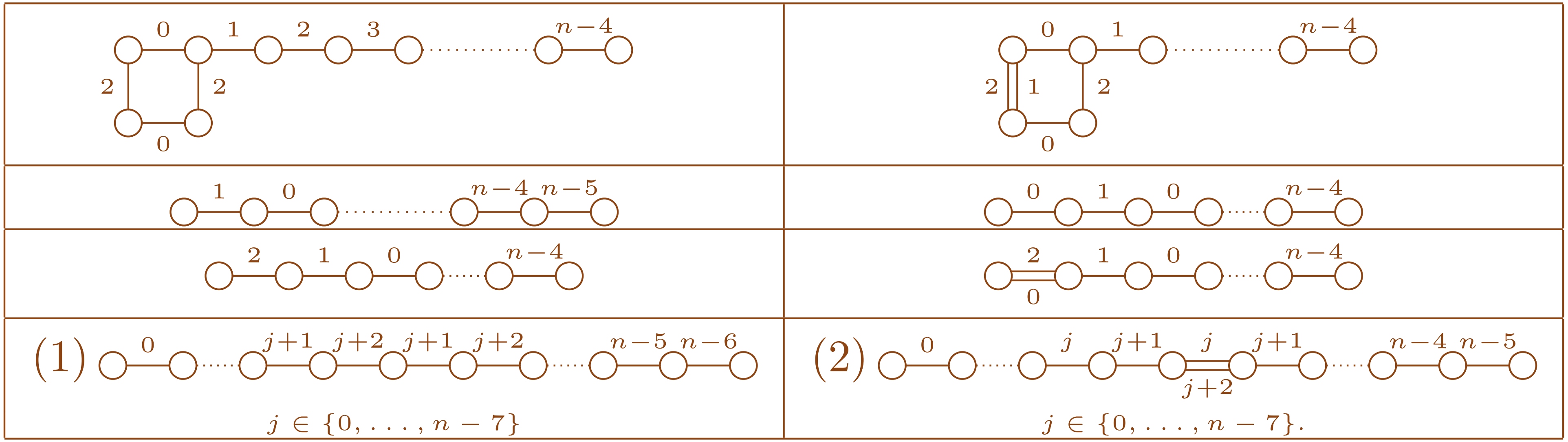
We get the following possibilities for $\mathcal{G}$:



(1) is not the graph of a string C-group for every $j \in \{0, \ldots, n-7\}$.

case $r = n - 3$

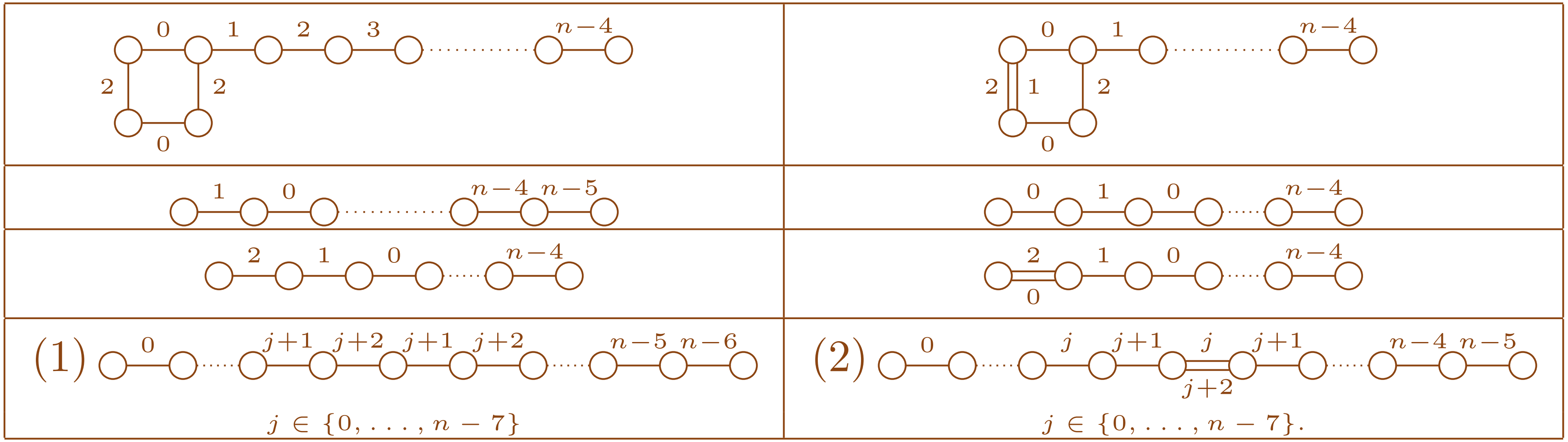
We get the following possibilities for $\mathcal{G}$:



(1) is not the graph of a string C-group for every $j \in \{0, \ldots, n-7\}$.

(2) is a permutation graph of a string C-group if and only if $j = 0$.

We get the following possibilities for $\mathcal{G}$:



(1) is not the graph of a string C-group for every $j \in \{0, \ldots, n-7\}$.

(2) is a permutation graph of a string C-group if and only if $j = 0$.

The remaining graphs are graphs of string C-groups Γ and $\Gamma \cong S_n$.

The 7 $(n-3)$-polytopes and the 9 $(n-4)$-polytopes

Extension of the classification of high rank polytopes