Recent work on Beauville surfaces, structures and groups

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Introduction

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A Beauville group gives a complex surface (a **Beauville surface**), roughly speaking, via the quotient $(\mathcal{C}_1 \times \mathcal{C}_2)/G$ where $\mathcal{C}_1$, $\mathcal{C}_2$ are compact Riemann surfaces, genus≥ 2 and the action is 'really nice' ($\mathcal{C}_i \to \mathcal{C}_i/G \cong \mathbb{P}_1(\mathbb{C})$ ramified at 3 points etc).

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- They have uses, for example, the Friedman-Morgan conjecture; action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on regular dessins.

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If G is an abelian group, then G is a Beauville group if and only if $G = (\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z})$ where $n > 1$ and gcd(n,6)=1.

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- If G and H are Beauville groups that have coprime order, then clearly $G \times H$ will be a Beauville group.
- The study of nilpotent Beauville groups thus reduces to the study of Beauville p-groups.
- The above theorem gives infinitely many (non-abelian) examples if $p \geq 5$ - just set $n = p^r$.

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Open Problem

Construct infinitely many Beauville 3-groups!

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- Special cases had been settled earlier by various people (Bauer, Catanese, Grunewald, Fuertes, Jones, González-Diez...)

Real matters

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A Beauville structure $\{\{x_1, y_1\}, \{x_2, y_2\}\} \subset G \times G$ of a Beauville group G is *strongly real* if there exist automorphisms $\phi_1, \phi_2 \in Aut(G)$ and elements $g_1, g_2 \in G$ such that for $i = 1, 2$

$$\begin{aligned} g_i\phi_i(x_i)g_i^{-1} &= x_i^{-1} \text{ and} \\ g_i\phi_i(y_i)g_i^{-1} &= y_i^{-1}. \end{aligned}$$

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Conjecture

An almost simple group is a Beauville group if and only if it is a strongly real Beauville group.

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- (F., '14+) ${}^2B_2(2^{2r+1})$ for $r > 1$
- (F., '14+) G simple with $|G| \leq 100\,000\,000$

Thanks for listening!