2-Arc-Transitive regular covers of $K_{n,n} - nK_2$ having the covering transformation group $\mathbb{Z}_p^3$

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1. Regular Coverings of Graphs

Graph Covering: **A graph X is called a covering of a graph Y with the projection $p : X \to Y$ if there is a surjection $p : V(X) \to V(Y)$ such that $p|_{N(x)} : N(x) \to N(y)$ is a bijection for any $y \in V(Y)$ and $x \in p^{-1}(y)$.**

X: Covering graph; Y: base graph;

Vertex fibre: $p^{-1}(v)$, $v \in V(Y)$;

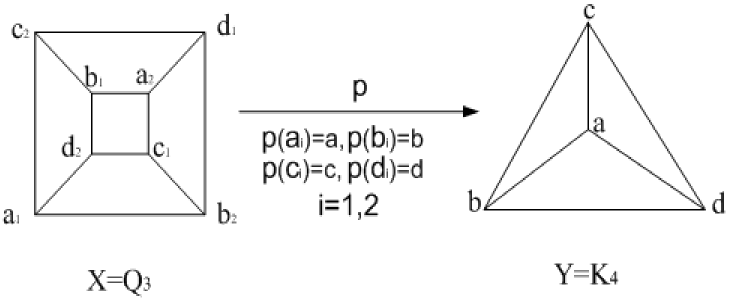
Edge fibre: $p^{-1}(e)$, $e \in E(Y)$;

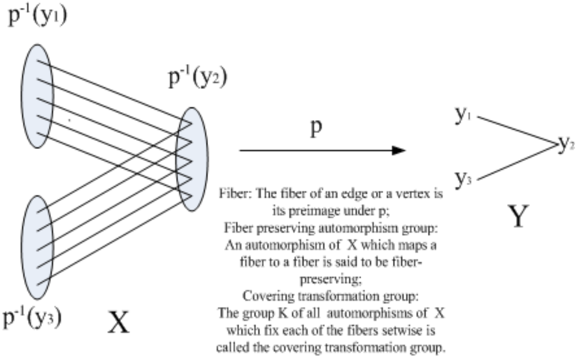
Fibre-preserving automorphism $\sigma \in \mathrm{Aut}(X)$**: maps a fibre to a fibre**

Covering transformation group K**:**

$$K = \{\sigma \in \mathrm{Aut}(X) \mid \sigma \text{ fix every fibre setwise.}\}$$

Regular covering: if K **acts regularly on each fibre (**X **is connected)**


c2
d1
b1
a2
d2
c1
a1
b2
X=Q3
p
p(ai)=a, p(bi)=b
p(ci)=c, p(di)=d
i=1,2
c
a
b
d
Y=K4



p-1(y1)
p-1(y2)
p-1(y3)
X
p
y1
y2
y3
Y
Fiber: The fiber of an edge or a vertex is its preimage under p;
Fiber preserving automorphism group: An automorphism of X which maps a fiber to a fiber is said to be fiber-preserving;
Covering transformation group: The group K of all automorphisms of X which fix each of the fibers setwise is called the covering transformation group.

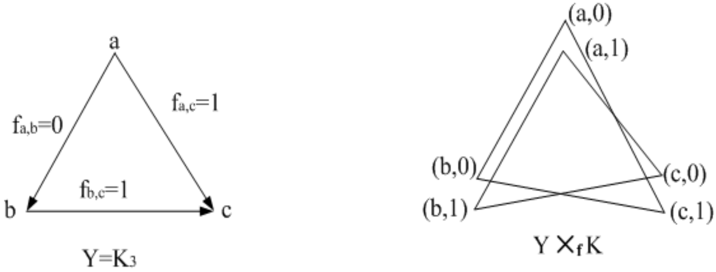
Voltage Graphs and Liftings

Voltage assignment f**: graph** Y**, finite group** K
a function $f : A(Y) \to K$ **s. t.** $f_{u,v} = {f_{v,u}}^{-1}$ **for each** $(u, v) \in A(Y)$.

*Voltage graph***:** (Y, f)

Derived graph $Y \times_f K$**: vertex set** $V(Y) \times K$**,**
arc-set $\{((u, g), (v, f_{u,v})g) \mid (u, v) \in A(Y),\ g \in K\}$.

$$f: A(Y) \longrightarrow K=\{1,0\}$$



Remark:

1. Derived graph $Y \times_f K$ is a covering of Y;

2. Derived graph is conn iff voltages on all closed walks generate K.

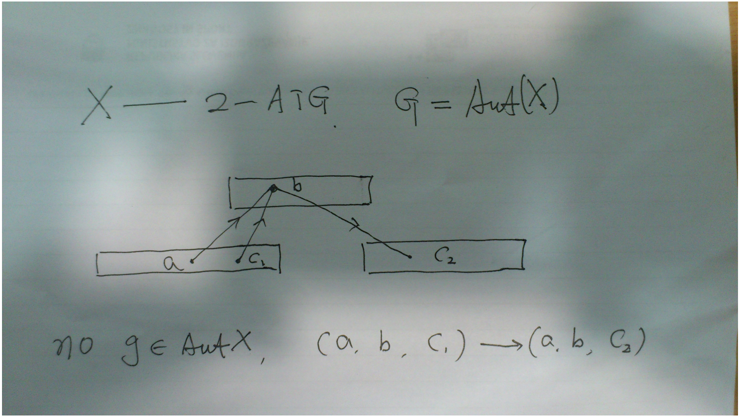
3. Each connected regular covering can be reconstructed by a derived graph.

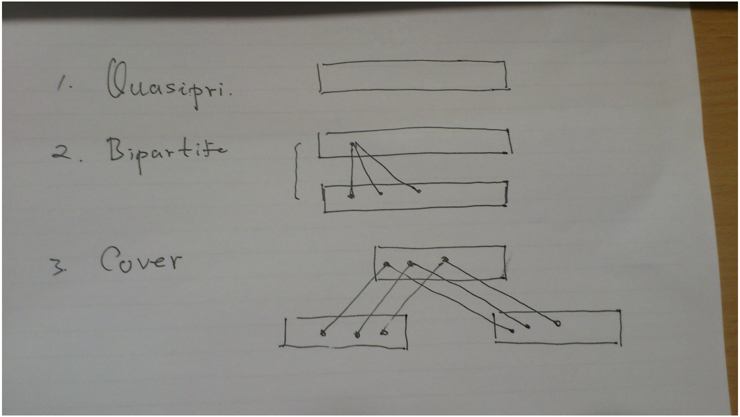
Lifting: $\alpha \in \mathrm{Aut}\,(Y)$ **lifts to an automorphism** $\bar{a} \in \mathrm{Aut}\,(X)$ **if** $\alpha p = p\bar{a}$.

Question: Given a graph Y**, a group** K **and** $H \leq \mathrm{Aut}\,(Y)$**, find all the connected regular coverings** $Y \times_f K$ **on which** H **lifts.**

For a graph X, an *s-arc* **of X is a sequence $(v_0, v_1, \ldots, v_s)$ of $s+1$ vertices such that $(v_i, v_{i+1}) \in A(Y)$ and $v_i \neq v_{i+2}$.**

X is said to be *2-arc-transitive* **if $\mathrm{Aut}\, X$ acts transitively on the set of 2-arcs of X.**


X — 2-ATG G = Aut(X)
b
a
c1
c2
no g ∈ Aut X, (a, b, c1) → (a, b, c2)


1. Quasipri.
2. Bipartite
3. Cover

Praeger's Reduction Theorem

*J. London Math. Soc. (2)***47***(1993), 227-239.*

Theorem

Every finite connected 2-arc-transitive graph is one of the following:

(1) *Quasiprimitive Type: every non-trivial normal subgroup of* $\mathrm{Aut}\,X$ *acts transitively on* $V(X)$,

(1) *Bipartite Type: every non-trivial normal subgroup of* $\mathrm{Aut}\,X$ *has at most two orbits on* $V(X)$ *and at least one of normal subgroups of* $\mathrm{Aut}\,X$ *has exactly two orbits on* $V(X)$.

(3) *Covering Type: There exists a normal subgroup of* $\mathrm{Aut}\,X$ *which has at least three orbits on* $V(X) \to$ *regular covers of graphs in (1) and (2).*

C.E. Praeger, An O'Nan-Scott theorem for finite quasiprimitive permutation groups and an application to 2-**arc transitive graphs,** *J. London Math. Soc.* **(2)47(1993), 227-239.**

Every primitive group must be quasiprimitve, but a quasiprimitve group is not necessarily primitive.

Example:

$G = \mathbf{PSL}(2,7)$

acts on the edge set E of point-line incident graph of $\mathrm{PG}(2,2)$, where $|E| = 21$

for any edge e, we have $G_e = D_8 \leq S_4 \leq G$ and so G is an imprimitive group on E.

R.W.Baddeley, Two-arc transitive graphs and twisted wreath products, J. Alg. Combin. 2 (1993), 215–237.

A.A. Ivanov and C.E. Praeger, On finite affine 2**-arc-transitive graphs,** *Europ. J. Combin.* **14 (1993), 421–444.**

X.G. Fang and C.E. Praeger, On graphs admitting arc-transitive actions of almost simple groups, *J. Algebra* **205 (1998), 37-52.**

X.G. Fang and C.E. Praeger, Finite two-arc-transitive graphs admitting a Suzuki simple group, *Comm. Algebra* **27(1999), 3727-3754.**

C. H. Li, The finite vertex-primitive and vertex-biprimitive s**-transitive graphs for** $s \geq 4$**, Trans. Amer. Math. Soc. 353 (2001), 3511–3529.**

Locally primitive graphs...and so on

Bipartite Type

A reduction theorem for this case was given by Praeger (1993).

C. E. Praeger, On a reduction theorem for finite, bipartite, 2-**arc-transitive graphs, Australas J. Combin. 7 (1993), 21-36.**

In sense of group theory, it induces two directions:

(1) Study Quasiprimitive type

$\rightarrow$ **to study finite G, which is a simple group, almost simple group, Quasisimple group, primitive group, Quasiprimitive and so on, and to study the suborbit structures of the related permutation representations:**

$H = G_\alpha$ **for** $\alpha \in V$

$[G : H]$**=the set of right coset of** H **in** G

the neighbor of α**=**HgH

the induced action of H **on** HgH **is a 2-transitive group.**

Bipartite type:

$A = \mathrm{Aut}\,(X)$

G **is the subgroup** A **fixing two biparts setwise and** G **be one of group as in (1)**

$H = G_\alpha$ **and** $R = G_\beta$ **for** $\{\alpha, \beta\}$ **is an edge**

$[G : H]$, $[G : R]$**=the set of right coset of** H **and** R **in** G**, resp.**

the neighbor of α**=**RgH

the induced action of H **on** RgH **is a 2-transitive group.**

(2) Study regular covers of Quasiprimitive or Bipartite type

$\rightarrow$ to study the group extensions of the above groups, in many cases, it is related central extension theory as well as Schur Multiplier theory, (ordinary and in most cases, modular) representations of almost simple groups and so on.

One of our long term topics is to classify covers whose fibre preserving group acts 2-arc-transitively, by

given base graphs (2-ATG of either Quasiprimitive type or Bipartite type)

given covering transformation groups (Z_p^n, ablian groups and nonabelian groups)

3. Some Classifications and Methods

1. Present some classifications of 2-arc-transitive regular covers with given base graphs and given covering transformation groups.

2. Show our general methods for classifying the 2-arc-transitive covers and for constructing voltage graphs.

3.1 Classify 2-arc-transitive regular covers of complete graphs with covering transformation group $\mathbb{Z}_p^k$.

Problem: Classify regular covers of complete graphs having the covering transformation group $K = \mathbb{Z}_p^k$ and whose fibre-preserving group acts 2-arc -transitively.

Motivation: If $1 \neq H \lhd \lhd K$, then $X \to X_1$ and $X_1 \to K_n$

$X = X_1 \times_{f_2} (H)$ **and** $X_1 = K_n \times_{f_1} (K/H)$

First need to determine the minimal covers, that is K is a characteristically simple group. Therefore, we choose $K = \mathbb{Z}_p^k$, an abelian characteristically simple group.

S.F.Du, D.Marušič and A.O.Waller, On 2-arc-transitive covers of complete graphs, J. Combin. Theory, B 74 (1998), 276–290.

Theorem

If $Y = K_n$ *and* K *is cyclic, then*
(i) $K \cong \mathbb{Z}_2$ *and* $X = K_{n,n} - nK_2$*;*
(ii) $K \cong \mathbb{Z}_4$ *and* $X \cong X_1(4, q)$*, where* $q = n - 1$*;*

If $Y = K_n$ *and* $K = \mathbb{Z}_p^2$*, then*
$K \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ *and* $X \cong X_2(4, q)$*, where* $q = n - 1$*.*

Definitions for $X_1(4,q)$ and $X_2(4,q)$:

Let $Y = K_n$ **with** $V(Y) = \mathrm{PG}(1,q) = F(q) \cup \{\infty\}$ **where** $F(q)^* = \langle\theta\rangle$

$X_1(4,q) = Y \times_f \mathbb{Z}_4$**, where** $q \equiv 3(mod\, 4)$ **and** $q \geq 3$**:**

$$f(x,y) = \begin{cases} 0\,, & \infty \ \in \{x,y\} \\ 1\,, & y-x \in \langle\theta^2\rangle \\ 3\,, & y-x \in \langle\theta^2\rangle\theta \end{cases}$$

$X_2(4,q) = Y \times_f \mathbb{Z}_2^2$, **where** $q \equiv 1 (mod\, 4)$ **and** $q \geq 5$**:**

$$f(x,y) = \begin{cases} (0,0)\,, & \infty \ \in \{x,y\} \\ (1,0)\,, & y-x \in \langle \theta^2 \rangle \\ (0,1)\,, & y-x \in \langle \theta^2 \rangle \theta \end{cases}$$

Remark: for all both covers, $\mathrm{PGL}(2,q)$ **lifts and so** X **is a 2-ATG.**

S.F.Du, J.H.Kwak and M.Y. Xu, On 2-arc-transitive covers of complete graphs with covering transformation group Z_p^3, J. Combin. Theory, B 93 (2005), 73–93.

Theorem

If $Y = K_n$ and $K = \mathbb{Z}_p^3$, then we have

(i) $X_1(p) = K_4 \times_f \mathbb{Z}_p^3$,

(ii) $X_2(p) = K_5 \times_f \mathbb{Z}_p^3$ for $p = 5$ or $p \equiv \pm 1 \pmod{10}$,

(iii) $X_3(p) = K_{1+p} \times_f \mathbb{Z}_p^3$ for $p \geq 5$,

(iii) $X_4(3) = K_8 \times_f \mathbb{Z}_2^3$.

$k = 4$

Open

3.2 Classify 2-arc-transitive circulant and dihedrant

For a Cayley graph, its automorphism group contains a vertex-regular subgroup.

Cayley graphs of cyclic and dihedral groups are called *Circulant* **and** *Dihedrants*, **respectively.**

B.Alspach, M.D.E.Conder, D.Marušič and M.Y.Xu, A classification of 2-arc-transitive circulants, J. Alg. Combin., 5 (1996), 83–86.

The proof is combinatorial and is independent on CFSG.

D. Marušič, On 2-arc-transitivity of Cayley graphs, J. Combin. Theory, B 87 (2003), 162–196.

S.F. Du, A. Malnič and D. Marušič, Classification of 2-arc-transitive dihedrants, J. Combin. Theory, B, 98(6), (2008), 1349-1372

Theorem

Let $n \geq 3$ and let X be a connected 2-arc-transitive Cayley graph of a dihedral group of order $2n$. Then one of the following occurs: 3mm

(i) *base graph: C_{2n}, n a prime; K_{2n}; $K_{n,n}$; $B(H_{11})$ or $B'(H_{11})$; $B(PG(d,q))$ or $B'(PG(d,q))$, or*

(ii) $K_{n,n} - nK_2$ *or* K^{2d}_{q+1}.

$Y = K_{q+1,q+1} - (q+1)K_2$,

$V(Y) = \{i, j' \mid i, j \in \mathrm{PG}(1,q)\}$

$E(Y) = \{i, j' \mid i \neq j, i, j \in \mathrm{PG}(1,q)\}$.

$K_{q+1}^{2d} = (K_{q+1,q+1} - (q+1)K_2) \times_f \mathbb{Z}_d$, **where**

$$f_{\infty',i} = f_{\infty,j'} = \overline{0} \text{ for } i, j \neq \infty;$$

$$f_{i,j'} = \overline{h} \text{ if } j - i = \theta^h, \text{ for } i, j \neq \infty,$$

where $F_q^* = \langle\theta\rangle$, $d \mid (q-1)$ **and** $d \geq 2$.

3.3. 2-Arc-Transitive Metacyclic Covers of Complete Graphs

W. Q. Xu, S. F. Du, J. H. Kwak and M. Y. Xu,
2-Arc-Transitive Metacyclic Covers of Complete Graphs, to J. Combina Theory (B), 2012 (revised version).

Theorem

Let X be a connected regular cover of the complete graph K_n ($n \geq 4$) whose covering transformation group K is nontrivial metacyclic and whose fibre-preserving automorphism group acts 2-arc-transitively on X. Then X is isomorphic to one of covers:

(1) $K_{n,n} - nK_2$;

(2) $n = 4$, $AT_D(4,6)$ *with* $K \cong D_6$;

(3) $n = 4$, $AT_Q(4,12)$ *with* $K \cong Q_{12}$;

(4) $n = 5$, $AT_D(5,6)$ *with* $K \cong D_6$;

(5) $n = 1 + q \geq 4$, $AT_Q(1+q, 2d)$ *with* $K \cong Q_{2d}$, *where* $d \mid q - 1$ *and* $d \nmid \frac{1}{2}(q-1)$;

(6) $n = 1 + q \geq 6$, $AT_D(1+q, 2d)$ *with* $K \cong D_{2d}$, *where* $d \mid \frac{1}{2}(q-1)$ *and* $d \geq 2$.

where Q_{2d} is the generalized quaternion group of order $2d$ and D_{2d} is the dihedral group of order $2d$. Note that $Q_4 \cong \mathbb{Z}_4$ and $D_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

W.Q. Xu and S.F. Du, 2-Arc-trantive cyclic covers of $K_{n,n} - nK_2$**,** *J.Algebr. Comb.* **39(2014), 883-902.**

Theorem

Let X be a connected regular cover of the complete bipartite graph minus a matching $K_{n,n} - nK_2$ ($n \geq 4$) with a nontrivial cyclic covering transformation group of order d, whose fibre-preserving automorphism group acts 2-arc-transitively. Then one of the following holds:

(1) $n = 4$ *and* $X \cong K_4^4$ *for* $d = 2$; $X_1(3)$ *for* $d = 3$; *or* $X(6)$ *for* $d = 6$;

(2) $n = 5$ *and* $X \cong X_2(3)$ *for* $d = 3$;

(3) $n = q + 1 \geq 6$ *and* $X \cong K_{q+1}^{2d}$, *where* $d \mid q - 1$ *and* $d \geq 2$.

4. 2-Arc-Transitive regular covers of $K_{n,n} - nK_2$ having the covering transformation group $\mathbb{Z}_p^3$

S.F. Du and W.Q. Xu, 2-arc-trantive regular covers of $K_{n,n} - nK_2$ having the covering transformation group $\mathbb{Z}_p^3$, submitted to ***Journal of the Australian Mathematical Society*****, 2013.**

Theorem

Let X be a connected regular cover of the complete bipartite graph minus a matching $K_{n,n} - nK_2$ ($n \geq 3$) with a covering transformation group K isomorphic to $\mathbb{Z}_p^3$, whose fiber-preserving automorphism group acts 2-arc-transitively. Then one of the following holds:

(1) $n = 4$ *and* $X \cong X_1(4, p)$;

(2) $n = 5$ *and* $X \cong X_{21}(5, p)$ *for* $p \equiv \pm 1 (\mathrm{mod}\ 10)$ *or* $X_{22}(5, 5)$ *for* $p = 5$;

(3) $n = p + 1 \geq 6$ *and* $X \cong X_{31}(p + 1, p)$ *for* $p \geq 5$, *or* $X_{32}(6, 5)$ *for* $p = 5$;

(4) $n = 8$ *and* $X \cong X_4(8, 2)$ *for* $p = 2$.

$Y := K_{n,n} - nK_2$ **and** $K \cong \mathbb{Z}_p^3$**, voltage assignment** f**, where** $V(Y) = \{i, i' \mid 1 \le i \le n\}$**,** $E(Y) = \{ij' \mid i \ne j, i, j' \in V(Y)\}$ **and** $K = V^+(3, p)$**.**

(1) $n = 4$ **and** $X_1(4, p) = Y \times_f K$**, where**

$$f_{12'} = f_{13'} = f_{14'} = f_{24'} = f_{21'} = f_{31'} = f_{41'} = (0, 0, 0),$$
$$f_{23'} = (1, 0, 0),\ f_{42'} = (0, 1, 0),\ f_{34'} = (0, 0, 1),$$
$$f_{43'} = (0, 1, -1),\ f_{32'} = (-1, 1, 0).$$

(2) $n = 5$, $p = \pm 1 \pmod{10}$ **and** $X_{21}(5, p) = Y \times_f K$, **where**

$$
\begin{array}{ll}
f_{1,2'} = (0, 2t, 1 - 2t), & f_{1,3'} = (2t, 1 - 2t, 0), \\
f_{1,4'} = (1 - 2t, 0, 2t), & f_{1,5'} = (-1, -1, -1), \\
f_{2,3'} = (1 - 2t, 0, -2t), & f_{2,4'} = (2t, 2t - 1, 0), \\
f_{2,5'} = (-1, 1, 1), & f_{3,4'} = (0, -2t, 1 - 2t), \\
f_{3,5'} = (1, 1, -1), & f_{4,5'} = (1, -1, 1), \\
f_{i,j'} = f_{i',j}, \quad i, j \in \{1, 2, 3, 4, 5\}, & \textit{where} \quad t = \frac{1+\sqrt{5}}{4} \in \mathbb{F}_p^*.
\end{array}
$$

$n = p = 5$ **and** $X_{22}(5, 5) = Y \times_f K$, **where**

$$
\begin{array}{ll}
f_{1,2'} = (0, -1, 0), & f_{1,3'} = (3, -1, 2), \\
f_{1,4'} = (2, 3, -1), & f_{1,5'} = (0, 1, 2), \\
f_{2,3'} = (0, -1, 3), & f_{2,4'} = (3, 0, 1), \\
f_{2,5'} = (2, 2, -1), & f_{3,4'} = (0, -1, 1), \\
f_{3,5'} = (3, 1, 2), & f_{4,5'} = (0, -1, -1), \\
f_{i,j'} = f_{i',j}, & i, j \in \{1, 2, 3, 4, 5\}.
\end{array}
$$

(3) **Label** $V(Y) = \{i, j' \mid i, j \in \mathrm{PG}(1, p)\}$ **and** $E(Y) = \{ij' \mid i, j' \in V(Y), i \neq j\}$.

$n = 1 + p$, $p \geq 5$ **and** $X_{31}(p+1, p) = Y \times_f K$, **where** $f_{\infty, i'} = f_{\infty', i} = (0, 1, 2i)$, **and** $f_{i,j'} = f_{i',j} = \left(\frac{1}{i-j}, \frac{i+j}{i-j}, \frac{2ij}{i-j}\right)$ **for all** $i \neq j$ **in** $\mathbb{F}_p$.

$n = 6$, $p = 5$ **and** $X_{32}(6, 5) = Y \times_f K$, **where** $f_{\infty, i'} = f_{\infty', i} = \left(-i, -i^2, i^3\right)$, $f_{i,j'} = (0, \pm 2, \pm 2(i+j))$ **for** $(i-j)^2 = \mp 1$, **where** $i, j \in \mathbb{F}_5$.

(4) **Let** $\Omega = \mathrm{PG}(2,2)$**;**

$V = V(\Omega)$: **the characteristic functions** $\chi(\Delta)$, $\Delta \in P(\Omega)$;

V **is a 7-dimensional PSL**$(2,7)$**-module by natural action;**

V_1: **the subspaces of** V **generated by** $\{i, j, i+j \mid i, j \in \Omega, i \neq j\}$

V_2: **the subspaces of** V **generated by** $\{i, j, k, i+j, i+k, j+k, i+j+k \mid i, j, k \in \Omega\}$.

Let $Y = K_{8,8} - 8K_2$,
$V(Y) = \{i, j' \mid i, j \in V(3,2)\}$
$E(Y) = \{ij' \mid i, j' \in V(Y), i \neq j\}$
$K = (V_1/V_2, +)$
$n = 8$, $p = 2$ **and** $X_4(8,2) = Y \times_f K$, **where**

$f_{0,j'} = \overline{0} := V_2$ **and** $f_{i,j'} = \overline{\chi}_{\{i,j,i+j\}} := \chi_{\{i,j,i+j\}} + V_2$ **for all** $i \neq j$ **in** Ω.

Outline of the proof

base graph $Y = K_{n,n} - nK_2$**,**

covering graph X**,**

covering transformation group $K = Z_p^3$

A**:** 2**-arc-transitive subgroup of** $\mathrm{Aut}\,(Y)$

G**: subgroup of** A **fixing two biparts setwise,**

G **is 3-transitive on both biparts**

G is the following:

(1) $\mathrm{soc}(G)$ **is** 4**-transitive;**

(2) $\mathrm{soc}(G) = M_{22}$ **or** A_5**, 3-transitive but not 4-transitive;**

(3) **PSL**$(2, q) \leq G \leq \mathrm{P\Gamma L}(2, q)$**;**

(4) $G = \mathrm{AGL}(m, 2)$ **with** $m \geq 3$ **or** $\mathbb{Z}_2^4 \rtimes A_7$**.**

Main problem is to determine the group extension

$1 \to K \to \tilde{G} \to G$

$1 \to K \to \tilde{A} \to A$

The key point in graph (regular-) covering theory is the lifting problem of automorphisms in the base graphs, which is related to several branches.

Lifting criterion for any covering group K :

Theorem

Let $X = Y \times_f K$ be a regular covering. Then $\alpha \in \mathrm{Aut}(Y)$ lifts if and only if, for each closed walk W in Y, we have $f_{W^\alpha} = 1$ iff $f_W = 1$.

see A. Malnič, Group actions, coverings and lifts of automorphisms, Discrete Math. 182(1998), 203-218.

Lifting criterion for abelian covering group K :

Theorem

(Du, Kwak, Marusic, Waller, Xu) Let $X = Y \times_f K$ be a connected regular cover of a graph Y, where K is abelian. If $\alpha \in \mathrm{Aut}\, Y$ is an automorphism one of whose liftings $\tilde{\alpha}$ centralizes K, then $f_{W^\alpha} = f_W$ for any closed W of Y.

Linear criteria for liftings of automorphisms for elementary abelian covering group $K = Z_p^n$

S.F. Du, J.H. Kwak and M.Y. Xu, Linear criteria for lifting of automorphisms in elementary abelian regular coverings, *Linear Alegebra and Its Applications*, **373, 101-119(2003).**

Malnic, Aleksander; Potocnik, Primoz. Invariant subspaces, duality, and covers of the Petersen graph, *European J. Combin.* **27 (2006), no. 6, 971–989.**

Linear criteria for liftings of automorphisms for abelian covering group

1. Conder, Marston D. E.; Ma, Jicheng. Arc-transitive abelian regular covers of the Heawood graph. J. Algebra 387 (2013), 243–267.

2. Conder, Marston D. E.; Ma, Jicheng. Arc-transitive abelian regular covers of cubic graphs. J. Algebra 387 (2013), 215–242.

Example:

Coset Graphs:

Definition

G group; $H \leq G$, core-free; $D = HdH$ double coset
Coset graph $X = X(G; H, D)$:
$V = [G : H]$: right cosets of G relative H
$E = \{(Hd, Hdg) \mid d \in D, g \in G\}$

Lemma

(1) G acts-transitively on X;
(2) Every G-arc-transitive graph is isomorphic to a coset graph.
(3) X is conn iff $G = \langle D \rangle$;
(4) X is undirected iff $D = D^{-1}$.

Bicoset Graph:

Definition

G be a group, $L, R \leq G$, $D = RdL$
Bicoset graph $X = (G, L, R; D)$:
$V(X) = [G : L] \cup [G : R]$
$E(X) = \{\{Lg, Rdg\} \mid g \in G, d \in D\}$.

Lemma

(i) *G acts edge-transitively;*

(iii) *X is connected if and only if G is generated by elements of* $D^{-1}D$;

(iv) *Every G-edge-tran graph is isomorphic to a Bicoset graph.*

Let $X = Y \times_f K$ and let $G \le \mathrm{Aut}(Y)$. Suppose that G lifts to A. Then $A/K \cong G$.

Step 1: **Determine the group A by the group theoretical tools(group extension, representation theory and so on);**

Step 2: **Determine the permutation representations of A relative to all the possible subgroups H (point stabilizer);**

Step 3: **Determine the corresponding suborbit structure of the above representations so that obtain the coset graphs;**

Step 4: **Find the voltage assignment from these coset graphs.**

Example 1

Example

Let $Y = K_{1+p}$ where $V(Y) = PG(1, p) = GF(p) \cup \{\infty\}$ and let $K = (V(3, p), +)$. Find all the regular coverings $X = Y \times_f K$ such that $\mathrm{PGL}(2, p) \leq \mathrm{Aut}\,(Y)$ lifts.

Solution: **(1) Define $X(p) =: K_{1+p} \times_f Z_p^3$ as follows:**
$f_{\infty,j} = (0, 1, 2j)$,
$f_{i,j} = \left(\frac{1}{i-j}, \frac{i+j}{i-j}, \frac{2ij}{i-j}\right)$ **for all $i \neq j$ in $GF(p)$.**

(2) $X'(5) = K_6 \times_f Z_p^3$ **as follows:**

....

Proof for Example 1

$A/K \cong \mathrm{PGL}(2,p)$ **for** $p \geq 5$ **and** $n = 1 + p$.
Take a fibre F **and a vertex** $v \in F$**. Then** $A_F = A_v K$**.**
Since $(|A : A_F|, |K|) = (1 + p, p^3) = 1$ **and** K **is an abelian normal subgroup of** A**, we know that** K **has a complement in** A **which is isomorphic to** $\mathrm{PGL}(2,p)$**, that is**

$$A \cong Z_p^3 \rtimes \mathrm{PGL}(2,p)$$

Modular p- Representations of 2-dimensional linear groups:

1. Brauer and C. Nesbitt, On the modular characters of groups, Annals of Math, 42(2), 556-590.

2.R. Burkhardt, Die Zerlegungsmartrizen de Gruppen $PSL(2, p^f)$, J. Algebra of Algebra, 40(1976), 75-96

SL$(2, p)$ **has** p **irreducible modular** p- **Representations**

PSL$(2, p)$ **has** $\frac{p+1}{2}$ **irreducible modular** p- **Representations with degrees** $1, 3, 5, \cdots, p$

Degree 3:
$V_3 = \langle x^i y^j \mid i+j=2 \rangle$ **homogeneous space over** F_p
$g = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$
Define $G = \mathbf{PSL}(2,p)$**-module** V_3 **extended by**
$g(x^i y^j) = (a_{11}x + a_{12}y)^i (a_{21} + a_{22}y)^j$

Let $G = PGL(2,p)$**. Define two** G**-modules** V_3 **extended by**
$g(x^i y^j) = det(g)^{-1}(a_{11}x + a_{12}y)^i (a_{21} + a_{22}y)^j$
and
$g(x^i y^j) = det(g)^{\frac{p-1}{2}-1}(a_{11}x + a_{12}y)^i (a_{21} + a_{22}y)^j$

Take a base in V_3, we get two homomorphisms ϕ of $\mathrm{PGL}(2,p)$ into $\mathrm{GL}(3,p)$

$$\phi : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (ad-bc)^{-1} \begin{pmatrix} a^2 & 2ab & 2b^2 \\ ac & ad+bc & 2bd \\ c^2/2 & cd & d^2 \end{pmatrix}.$$

$$\phi : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (ad-bc)^{\frac{p-1}{2}-1} \begin{pmatrix} a^2 & 2ab & 2b^2 \\ ac & ad+bc & 2bd \\ c^2/2 & cd & d^2 \end{pmatrix}.$$

Note: The first case will give the covers $X(p)$ the second will gives the covers $X'(5)$.

Step 2: Determination of conjugacy class of point stabilizers

Take a subgroup $H_1 = \langle t_1 \rangle \rtimes \langle a_1 \rangle \cong \mathbb{Z}_p \rtimes \mathbb{Z}_{p-1}$ **of** $\mathrm{PGL}(2,p)$**, where**

$$t_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad a_1 = \begin{pmatrix} \theta & 0 \\ 0 & 1 \end{pmatrix}$$

for a generator θ **of** $GF(p)^*$**. Let** $PG(1,p) = \{\infty, 0, 1, \ldots, p-1\}$ **be the projective line over** $GF(p)$**, where we identify** $\langle (0,1) \rangle$ **and** $\langle (1,\ell) \rangle$ **with** ∞ **and** ℓ**, respectively. Then,** H_1 **fixes** $\infty \in PG(1,p)$ **and** t_1^i **maps** ℓ **into** $\ell + i$**. Furthermore, we have** $H := \phi(H_1) = \langle t \rangle \rtimes \langle a \rangle$**, where** $t = \phi(t_1)$ **and** $a = \phi(a_1)$**, and for any** i**,**

$$t^i = \phi(t_1^i) = \begin{pmatrix} 1 & 2i & 2i^2 \\ 0 & 1 & 2i \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad a^i = \phi(a_1^i) = \begin{pmatrix} \theta^i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \theta^{-i} \end{pmatrix}.$$

Lemma

Let $M = K \rtimes H$. Then, M has only one conjugate class of subgroups L satisfying $\langle a \rangle \leq L \cong H$ and $L \cap K = 1$.

Proof **Note that**
$|M| = |K \rtimes H| = |(K \rtimes \langle t\rangle) \rtimes \langle a\rangle| = p^4(p-1)$. **Let** $P = K \rtimes \langle t\rangle$. **Then,** P **is a** p**-group of order** p^4. **Since** $p \geq 5$ **by assumption,** P **is a regular** p**-group (for the definition of regular** p**-groups. Since** $\Phi(P) \leq K$ **and the order of** t **is** p**,** P **has exponent** p**. Clearly,** M **has only one conjugacy class of subgroups isomorphic to** $\langle a\rangle$. **Assume that** L **is a subgroup of** M **such that** $\langle a\rangle \leq L \cong H$ **and** $L \cap K = 1$. **Then, we may assume that** $L = \langle kt\rangle \rtimes \langle a\rangle$ **for some** $k = (x, y, z) \in K$. **Suppose that** $(kt)^a = (kt)^i$. **Then, we have** $(kt)^a = k^a t^a = (\theta x, y, \theta^{-1} z)t^{\theta^{-1}}$

$$\begin{aligned}(kt)^i &= (kk^{t^{-1}}k^{t^{-2}}\cdots k^{t^{-i+1}})t^i\\ &= ((x,y,z)+(x,-2x+y,2x-2y+z)+\cdots\\ &\quad +(x,-2(i-1)x+y,2(i-1)^2x-2(i-1)y+z))t^i\\ &= (ix,-(i-1)ix+iy,\tfrac{(i-1)i(2i-1)}{3}x-(i-1)iy+iz)t^i.\end{aligned}$$

Thus, we get $i=\theta^{-1}$ **and**

$$(\theta x,y,\theta^{-1}z)=(ix,-(i-1)ix+iy,\frac{(i-1)i(2i-1)}{3}x-(i-1)y+iz).$$

From these two equations, we have $\theta x=ix=\theta^{-1}x$ **and so** $\theta^2x=x$**. Since** $p\geq 5$**, we get** $\theta^2\neq 1$**, and so** $x=0$ **and** $y=0$ **by the second equation again. Hence,** $k=(0,0,z)$ **for any** $z\in GF(p)$**, that means** k **has** p **possibilities. For each** k**, we get an** $L=\langle kt\rangle\rtimes\langle a\rangle$**; in particular,** $L=H$ **when** $z=0$**. Furthermore, these** p **subgroups are conjugate in** M**.**

In fact, for any $k = (0, 0, z)$**, by taking** $k' = (0, \frac{z}{2}, 0)$, **we have**

$$\begin{aligned}(kt)^{k'} &= k(k')^{-1}tk' = k(k')^{-1}(k')^{t^{-1}}t \\ &= ((0,0,z) - (0,\tfrac{z}{2},0) + (0,\tfrac{z}{2},-z))t = (0,0,0)t = t\end{aligned}$$

$$a^{k'} = {k'}^{-1}ak' = {k'}^{-1}(k')^{a^{-1}}a = \left((0,-\frac{z}{2},0) + (0,\frac{z}{2}0)\right)a = a,$$

which forces $L^{k'} = H$, **completing the proof.** □

Step 3: Determination of suborbits of A relative to H

Lemma

Let $[A : H]$ be the set of right cosets of H in A. Then, in its right multiplication action on $[A : H]$, A has $p-1$ suborbits of length p not contained in $[M : H]$, which correspond to the $p-1$ double cosets $Hg(0, y, 0)H$ for any $y \in GF(p)^*$ and $g = \phi(g_1)$, where

$$g_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Proof Suppose that the double coset D corresponds to a suborbit of A of length p relative to H not contained in $[M : H]$. Since H has only one conjugacy class of subgroups of order $p-1$, a must fix a point in this suborbit.
Noting that T is 2-transitive on $[T : H]$, we may choose $D = HgkH$ such that $Hgk = Hgka$, in other words, $Hg = Hga^{-1}k^a$, which forces that $Hg = Hga^{-1}$ and $k^a = k$. Hence, we may fix $g = \phi(g_1)$. Assume $k = (x, y, z)$. From $(\theta x, y, \theta^{-1}z) = k^a = k = (x, y, z)$, we have $x = z = 0$ as $\theta \neq \pm 1$, and so $k = (0, y, 0)$, where $y \neq 0$. Therefore, we get $p-1$ choices for k and so for D. □

Now, $M = K \rtimes H = A_F$ **for a fibre** F**. For any** $u \in F$**, we have** $M_u \cong H$ **and** $M_u \cap K = 1$. **Since** M **has only one conjugacy class of subgroups isomorphic to** $\langle a \rangle$**, there exists a vertex** $v \in F$ **such that** $\langle a \rangle \leq M_v$**. By Lemma ??,** M_v **is conjugate to** H **in** M. **It follows that** H **fixes a vertex in** F**. Therefore,** X **is isomorphic to one of** $X(A, H, D)$**, where** $D = Hg(0, y, 0)H$ **is as in Lemma 0.6. Moreover, it is easy to see that the** $p - 1$ **graphs corresponding to the** $p - 1$ **choices for** D **are isomorphic to each other, by changing the basis of** $V(3, p)$**. Now, we may choose** $k = (0, 1, 0)$. **Note that**

$$g = \phi(g_1) = \begin{pmatrix} 0 & 0 & 2 \\ 0 & -1 & 0 \\ 1/2 & 0 & 0 \end{pmatrix}.$$

Since $(gk)^2 = 1$**, we get** $D = HgkH = H(gk)^{-1}H = D^{-1}$**. So,** $X(A, H, D)$ **is an undirected graph. Clearly,** A **acts** 2**-arc-transitively on** $X(A, H, D)$**, because** T **is** 3**-transitive on** $V(K_p)$

Step 5: Determination of the voltage assignment

Lemma

$X(A, H, D) \cong X(p)$, and its group of fibre-preserving automorphisms acts 2-arc-transitively.

Proof **Considering the action of** $\mathrm{PGL}(2, p)$ **on** $PG(1, p)$, **one can easily check that for** $\ell \in GF(p)^*$ **both** $g_1 t_1^{\ell} g_1 t_1^{i}$ **and** $g_1 t_1^{i-\ell^{-1}}$ **map** ∞ **to** $i - \ell^{-1}$, **respectively. Since** $(\mathrm{PGL}(2, p))_\infty = H_1$, **we have that for any** $i \in GF(p)$, $H_1 g_1 t_1^{\ell} g_1 t_1^{i} = H_1 g_1 t^{i-\ell^{-1}}$ **and so under the homomorphism** ϕ **mentioned before** $Hgt^{\ell}gt^{i} = Hgt^{i-\ell^{-1}}$. **In addition,** $(Hg)gt^{i} = H$.

By the arguments before the lemma, we know that in the coset graph $X(A, H, D)$, H **is adjacent to** $Hgkt^{\ell}$ **for any** $\ell \in GF(p)$. **Hence, for any** $i \in GF(p)$, Hgt^{i} **is adjacent to** $Hgkt^{\ell}gt^{i} = Hgt^{\ell}gt^{i}k^{(t^{\ell}gt^{i})}$ **for any** $\ell \in GF(p)$.

If $\ell = 0$**, then**

$$Hgt^{\ell}gt^{i}k^{(t^{\ell}gt^{i})} = H(0,1,0)gt^{i} = H(0,-1,-2i).$$

Hence, Hgt^{i} **is adjacent to** $H(0,-1,-2i)$ **for any** $i \in GF(p)$**, or equivalently,** H **is adjacent to** $Hgt^{j}(0,1,2j)$ **for any** $j \in GF(p)$**.**
Assume $\ell \in GF(p)^{*}$ **and let** $i - \ell^{-1} = j$**. Then,**

$$\begin{aligned} Hgt^{\ell}gt^{i}k^{(t^{\ell}gt^{i})} &= Hgt^{i-\ell^{-1}}(0,1,0)t^{\ell}gt^{i} \\ &= Hgt^{i-\ell^{-1}}(\ell, 2i\ell - 1, 2i^{2}\ell - 2i) = Hgt^{j}\left(\frac{1}{i-j}, \frac{i+j}{i-j}, \frac{2ij}{i-j}\right.\end{aligned}$$

Hence, Hgt^{i} **is adjacent to** $Hgt^{j}\left(\frac{1}{i-j}, \frac{i+j}{i-j}, \frac{2ij}{i-j}\right)$ **for any** $i \neq j \in GF(p)$.
Considering the action of $\mathrm{PGL}(2,p)$ **on** $PG(1,p)$**, we may define a bijection from** $[\mathrm{PGL}(2,p) : H_1]$ **to** $PG(1,p)$ **by sending** H_1 **to** ∞ **and** $H_1g_1t^{i}$ **to** i**. Accordingly, we may define a bijection from** $[T : H]$ **to** $PG(1,p)$ **by sending** H **to** ∞ **and** Hgt^{i} **to** i**.**

Finally, we may define a map σ from $V(X(A, H, D))$ to $V(X(p)) = PG(1, p) \times K$ by sending Hk to (∞, k) and $Hgt^i k$ to (i, k). In viewing the above arguments and the definition of $X(p)$, we find that σ is an isomorphism from $X(A, H, D)$ to $X(p)$. Moreover, since A acts 2-arc-transitively on $X(A, H, D)$, it follows that for the graph $X(p)$, its group of fibre-preserving automorphisms acts 2-arc-transitively. □

Lemma

For each cover in $X(q)$, the group of fibre-preserving automorphisms acts 2-arc-transitively.

Proof Recall that $V(K_{1+q})$ is identified with the projective line $PG(1,q) = GF(q) \cup \{\infty\}$. We will adopt the usual computations between ∞ and the elements in $GF(q)$, that is, $\infty + i = \infty$ for $i \in GF(q)$; $\infty i = \infty$ for $i \in GF(q)^*$; and $\frac{\infty}{\infty} = 1$. Let K be the corresponding additive group of $V(3,q)$. Then, $X(q) = K_{1+q} \times_f K$ is defined by $f_{i,j} = \left(\frac{1}{i-j}, \frac{i+j}{i-j}, \frac{2ij}{i-j}\right)$ for all $i \neq j$ in $PG(1,q)$.

To prove the lemma, it suffices to show that $\mathrm{PGL}(2,q)$ lifts. For a computation, we identify the element ∞ and any $i \in GF(q)$ in $PG(1,q)$ with $\langle(1,0)\rangle$ and $\langle(i,1)\rangle$ respectively. For a matrix g in $\mathrm{GL}(2,q)$, we denote by $\overline{g}$ the image of g in $\mathrm{PGL}(2,p^\ell)$ under the natural homomorphism.

Then, the action of $\overline{g} \in \mathrm{PGL}(2, p^\ell)$ on ∞ and any $i \in PG(1, p^\ell)$ can be written respectively as follows:

$$\infty^{\overline{g}} := \langle g(1,0)\rangle \quad \textbf{and} \quad i^{\overline{g}} := \langle g(i,1)\rangle.$$

Let

$$g_1 = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}, \; g_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; g_3 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \; g_4 = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$$

where x is a primitive element in $GF(q)$. Then, all of these elements generate $\mathrm{PGL}(2, q)$. In addition, it is easy to check that

$$i^{\overline{g_1}} = ix^2, \quad i^{\overline{g_2}} = i+1, \quad i^{\overline{g_3}} = \frac{i}{i+1}, \quad i^{\overline{g_4}} = ix,$$

where $i \in PG(1, q)$. In what follows, we show that for $1 \le k \le 4$, $\overline{g_k}$ lifts.
Let W be a closed walk in Y with $f_W = 0$, and for any arc $(i, j) \in A(Y)$, let $\ell_{i,j}$ has the same notation as above.

Now, we get

$$f_W = \sum_{(i,j)\in A(Y)} \ell_{i,j} f_{i,j} = \sum_{(i,j)\in A(Y)} \ell_{i,j}\left(\frac{1}{i-j}, \frac{i+j}{i-j}, \frac{2ij}{i-j}\right) = \mathbf{0}.$$

Therefore, we have

$$\sum_{(i,j)\in A(Y)} \frac{\ell_{i,j}}{i-j} = 0, \quad \sum_{(i,j)\in A(Y)} \frac{(i+j)\ell_{i,j}}{i-j} = 0, \quad \sum_{(i,j)\in A(Y)} \frac{2ij\ell_{i,j}}{i-j} = 0.$$

Also, we have

Now, we get

$$\begin{aligned} f_{W^{\overline{g_1}}} &= \sum_{(i,j)\in A(Y)} \ell_{i,j} f_{i^{\overline{g_1}},j^{\overline{g_1}}} \\ &= \sum_{(i,j)\in A(Y)} \ell_{i,j} f_{ix^2,ix^2} \\ &= \sum_{(i,j)\in A(Y)} \ell_{i,j} \left(\frac{1}{ix^2 - jx^2}, \frac{ix^2 + jx^2}{ix^2 - jx^2}, \frac{2ix^2jx^2}{ix^2 - jx^2} \right) \\ &= \left(x^{-2} \sum_{(i,j)\in A(Y)} \frac{\ell_{i,j}}{i-j}, \sum_{(i,j)\in A(Y)} \frac{(i+j)\ell_{i,j}}{i-j}, x^2 \sum_{(i,j)\in A(Y)} \frac{2ij\ell_{i,j}}{i-j} \right) \\ &= \mathbf{0}. \end{aligned}$$

Similarly, we get that $f_{W^{\overline{g_k}}} = \mathbf{0}$, for $k = 2, 3$ and 4. By Proposition 8, $\overline{g_k}$ lifts, and so $\mathrm{PGL}(2, q)$ lifts. □

Thank You Very Much !