C-groups of $\mathrm{PSL}(2, q)$ and $\mathrm{PGL}(2, q)$

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Coxeter groups

C-groups

String C-groups of $\mathrm{PSL}(2, q)$ and $\mathrm{PGL}(2, q)$

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Coxeter groups

A (finitely generated) Coxeter group (W, S) is a group W together with a set $S = \{r_0, \ldots, r_{n-1}\}$ that admits the following presentation

$$G = \langle r_0, \ldots, r_{n-1} \mid (r_i r_j)^{m_{ij}} = 1 \rangle \quad (1)$$

where

- $m_{ii} = 1$;
- $2 \leq m_{ij} \leq \infty$, for $i \neq j$.

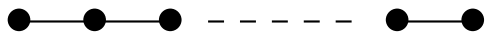
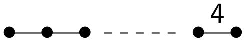
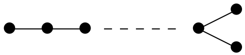
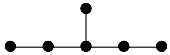
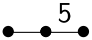
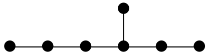
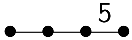
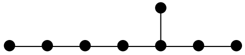
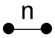
In other words, W is a group generated by involutions $r_0, \ldots, r_{n-1}$; the only relations between the generators are the orders of their pairwise products.

Coxeter groups

The diagram of the Coxeter group (W, S) is an undirected labelled graph such that

- vertices are indexed by involutions $r_0, \ldots, r_{n-1}$;
- the pair $\{r_i, r_j\}$ is an edge iff $m_{ij} \geq 3$ (i.e. iff r_i and r_j do not commute);
- the edge $\{r_i, r_j\}$ is labelled with the order $p_{ij} = o(r_i r_j)$.

Finite Coxeter groups

A_n		$B_n = C_n$	
D_n		F_4	
E_6		H_3	
E_7		H_4	
E_8		$I_2(n)$	

C-groups

A *C-group of rank n* is a pair (G, S) such that G is a group and $S := \{\rho_0, \ldots, \rho_{n-1}\}$ is a generating set of involutions of G that satisfy the following property for all subsets $I, J \subseteq \{0, \ldots, n-1\}$.

$$\langle \rho_i \mid i \in I \rangle \cap \langle \rho_j \mid j \in J \rangle = \langle \rho_k \mid k \in I \cap J \rangle \tag{2}$$

C-groups

Let $(W, \{r_0, \ldots, r_{n-1}\})$ be a Coxeter group and let $(G, \{\rho_0, \ldots, \rho_{n-1}\})$ be a C-group of same diagram.

Then there exists a surjective homomorphism

$$\sigma : W \to G : r_i \mapsto \rho_i$$

such that $\sigma : \langle r_i, r_j \rangle \mapsto \langle \rho_i, \rho_j \rangle$ is an isomorphism. C-groups are smooth quotients of Coxeter groups.

String C-groups

C-groups with a string diagram are called string C-groups. They are also called abstract regular polytopes.

- Natural generalization of ‘usual’ regular polytopes.
- Natural language between combinatorial object and algebraic structures.
- Many recent results
  - in the context of finite simple groups
  - from a combinatorial viewpoint
  - from a topological viewpoint

Polytopes for $\mathrm{PSL}(2, q)$ and $\mathrm{PGL}(2, q)$

Lemma (Sjerve & Cherkassof, Leemans & Vauthier, Leemans & Schulte)

Let $q = p^k$ be a prime power and let $G = \mathrm{PSL}(2, q)$. Then G may be generated by three involutions, two of which commute, if and only if $q \neq 2, 3, 7$ or 9. Moreover, if G is the full automorphism group of a regular polytope of rank 4, then $q = 11$ or 19. Finally, G is not the full automorphism group of a regular polytope of rank 5 and higher.

Lemma (Sjerve & Cherkassof, Leemans & Schulte)

Let $q = p^k$ be a prime power and let $G = \mathrm{PGL}(2, q)$. Then G may be generated by three involutions, two of which commute, if and only if $q \neq 2$. Moreover, if G is the full automorphism group of a regular polytope of rank 4, then $G = \mathrm{PGL}(2, 5) \cong S_5$. Finally, G is not the full automorphism group of a regular polytope of rank 5 and higher.

Dropping the string condition

- Natural generalization
- C-groups are smooth quotients of Coxeter groups
- Related to independent generating sets
- Flag-transitive, residually connected, thin geometries are C-groups
  Drawback: in rank ≥ 4, the converse is not necessarily true: a C-group may not be a regular geometry anymore.
- Results in the case of $\mathrm{Sz}(q)$ (Connor & Leemans, 2013).

Minimax sets of $\mathrm{PSL}(2, q)$

Theorem (Saxl & Whiston, 2002)

Let $G \cong \mathrm{PSL}(2, q)$ *for some prime power* $q = p^d$.

- *If* $d = 1$ *then* $\mu(G) \leq 4$. *Moreover* $\mu(G) = 3$ *unless* $p \equiv \pm 1 \mod 8$ *or* $p \equiv \pm 1 \mod 10$.
- *If* $d \geq 2$ *then* $\mu(G) \leq \max(6, \pi + 2)$ *where* $\pi = \pi(d)$ *is the number of distinct prime divisors of* d.

Theorem (Jambor, 2013)

Let p *be a prime. The group* $\mathrm{PSL}(2, p)$ *has a minimax set of size four if and only if* $p \in \{7, 11, 19, 31\}$. *More precisely, up to automorphisms there are two minimax sets of size four for* $\mathrm{PSL}(2, 7)$, *fourteen for* $\mathrm{PSL}(2, 11)$, *three for* $\mathrm{PSL}(2, 19)$ *and one for* $\mathrm{PSL}(2, 31)$.

C-groups related to $\mathrm{PSL}(2, q)$ and $\mathrm{PGL}(2, q)$

Theorem (Connor, Jambor & Leemans, 2014)

Let $G \cong \mathrm{PSL}(2, q)$ for some prime power $q \geq 4$. The C-rank of G is 4 if and only if $q \in \{7, 9, 11, 19, 31\}$. Otherwise it is 3. The list of C-groups of rank 4 is given in Table 1.
Let $G \cong \mathrm{PGL}(2, q)$ for some prime power $q \geq 4$. The C-rank of G is 4 if and only if $q = 5$. Otherwise it is 3. The list of C-groups of rank 4 is given in Table 2.

C-groups of rank 4 for $\mathrm{PSL}(2, q)$

q	$\rho_0\rho_1$	$\rho_0\rho_2$	$\rho_0\rho_3$	$\rho_1\rho_2$	$\rho_1\rho_3$	$\rho_2\rho_3$	Max. Para.	FT
7	4	2	3	3	2	4	S_4, S_4, S_4, S_4	yes
	4	2	3	3	2	3	S_4, S_4, S_4, S_4	yes
9	3	2	3	3	3	5	A_5, A_5, $E_9 \rtimes C_2$, S_4	no
	3	2	5	4	2	3	S_4, A_5, A_5, S_4	no
11	3	2	2	5	2	3	A_5, D_{20}, D_{20}, A_5	yes
	5	2	2	3	5	3	A_5, D_{12}, A_5, A_5	yes
	5	2	2	3	5	3	A_5, D_{12}, A_5, A_5	yes
	3	3	3	5	5	5	A_5, A_5, A_5, A_5	no
	3	5	5	5	5	3	A_5, A_5, A_5, A_5	yes
19	5	2	2	3	2	5	A_5, D_{20}, D_{20}, A_5	yes
	5	2	2	3	3	5	A_5, D_{20}, A_5, A_5	yes
	3	3	5	5	3	3	A_5, A_5, A_5, A_5	yes
31	3	2	3	3	2	5	A_5, A_5, S_4, S_4	yes

Table : C-groups of rank 4 for $\mathrm{PSL}(2, q)$

C-groups of rank 4 of $\mathrm{PGL}(2, q)$

$\rho_0\rho_1$	$\rho_0\rho_2$	$\rho_0\rho_3$	$\rho_1\rho_2$	$\rho_1\rho_3$	$\rho_2\rho_3$	Max. Para.	FT
3	2	2	3	2	3	S_4, D_{12}, D_{12}, S_4	yes
3	2	2	3	3	3	S_4, D_{12}, S_4, S_4	yes
3	3	3	3	3	3	S_4, S_4, S_4, S_4	yes

Table : C-groups of rank 4 for $\mathrm{PGL}(2, 5)$

Strategy of the proof of the main result

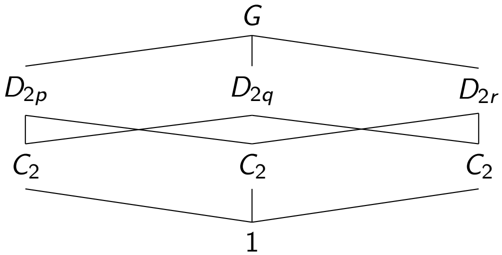


Figure : The boolean lattice of a C-group of rank 3

Strategy of the proof of the main result

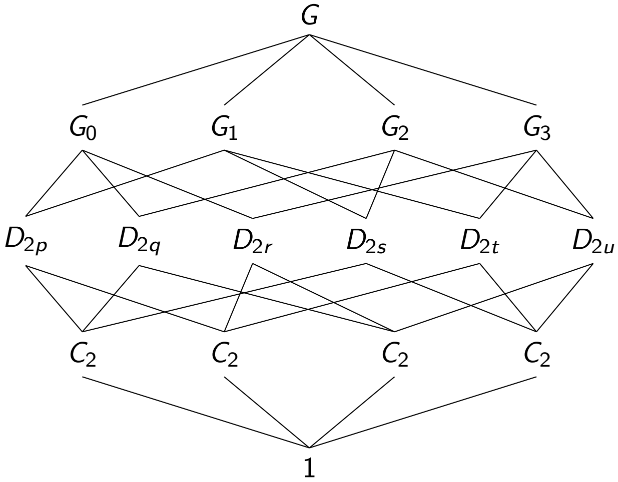


Figure : The boolean lattice of a C-group of rank 4

Strategy of the proof of the main result

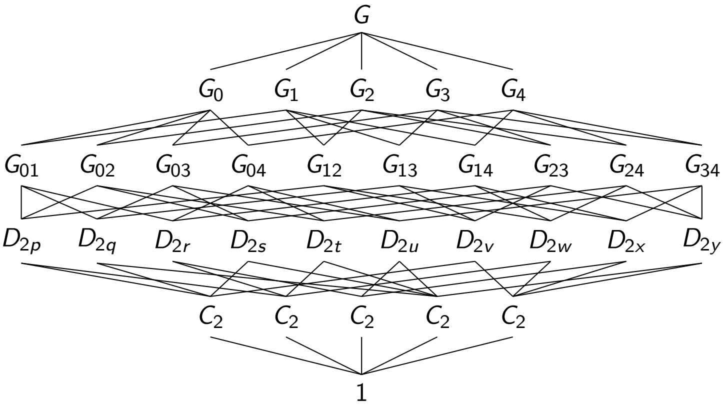


Figure : The boolean lattice of a C-group of rank 5

Strategy of the proof of the main result

1. Select possible subgroups of $\mathrm{PSL}(2, q)$ and $\mathrm{PGL}(2, q)$ that could be maximal parabolic subgroups (namely subfield subgroups, A_5, S_4, elementary abelian 2–groups or simply even dihedral groups).
2. Use that information and the intersection property to bound the rank of a C-group representation for $\mathrm{PSL}(2, q)$ and $\mathrm{PGL}(2, q)$.
3. Build the list of possible diagrams of a C-group representation of $\mathrm{PSL}(2, q)$ and $\mathrm{PGL}(2, q)$ using C-groups representations of possible maximal parabolic subgroups.
4. Use the L_2–quotient algorithm to build the actual C-groups representations.

Strategy of the proof of the main result

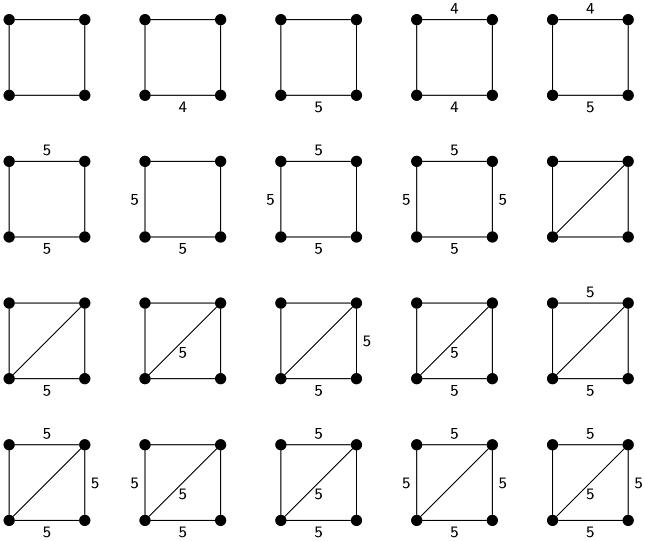


Table : Square diagrams (1)

Strategy of the proof of the main result

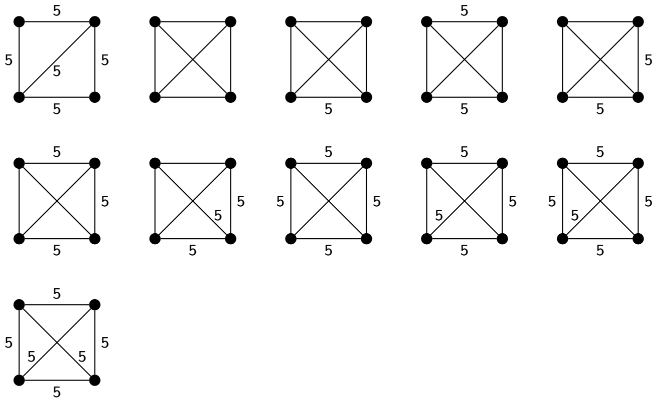


Table : Square diagrams (2)

Strategy of the proof of the main result

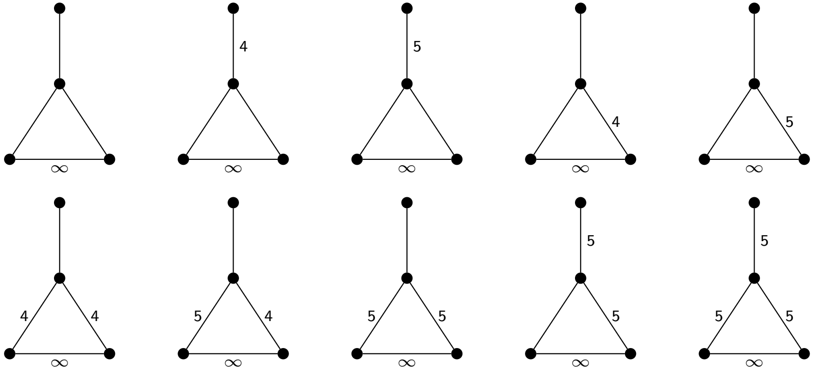


Table : Cherry diagrams

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