Recent developments in the study of regular maps

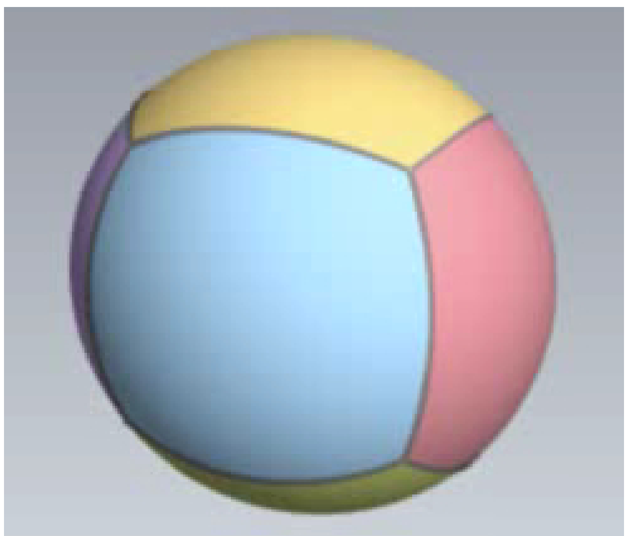
Marston Conder
University of Auckland
m.conder@auckland.ac.nz

[Joint work with lots of other people]

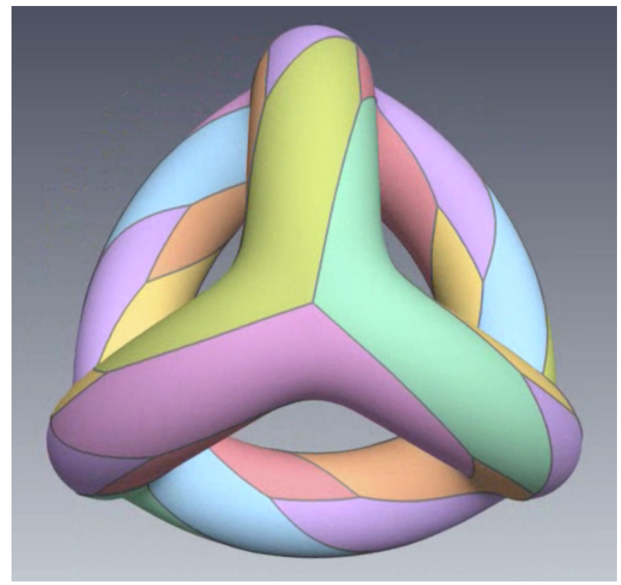
Lecture dedicated to the late Murray Macbeath
(1923–2014)

Regular maps

Regular maps are symmetric embeddings of (connected) graphs or multigraphs on surfaces:



Q_3 on sphere



Klein map (genus 3)

Some key points

A flag of a map is an incident vertex-edge-face triple (v, e, f).

Apart from some weird exceptions, every automorphism of a map M is uniquely determined by its effect on a given flag, and it follows that $|\mathrm{Aut}\, M| \le 4|E|$ where E is the edge set.

A map M is (fully) regular if $\mathrm{Aut}\, M$ is transitive on flags.

A map M on an orientable surface is orientably-regular if $\mathrm{Aut}\, M$ is transitive on arcs, and in that case $|\mathrm{Aut}^{+} M| \le 2|E|$. Moreover, such a map M is either reflexible (when $\mathrm{Aut}\, M$ is transitive on flags), or otherwise chiral (irreflexible).

Transitivity, type and genus

If M is a regular or orientably-regular map, then its underlying graph or multigraph is vertex-transitive, edge-transitive and face-transitive. In particular, every face has the same number of edges (say k) and every vertex has the same valency (say m). In this case we say that M has type $\{k,m\}$.

If M is orientable with $|V|$ vertices, $|E|$ edges and $|F|$ faces,

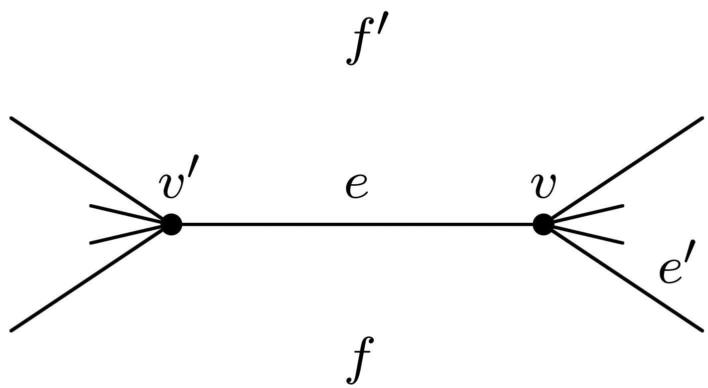
then $$m|V| = 4|E| = k|F| = |\mathrm{Aut}\,M|$$

so its genus g is given by the Euler-Poincaré formula:

$$2-2g = \chi = |V| - |E| + |F| = |\mathrm{Aut}\,M| \left(\frac{1}{2m} - \frac{1}{4} + \frac{1}{2k}\right).$$

If M is a regular map of type $\{k, m\}$, then for any given flag (v, e, f) there exist automorphisms a, b and c such that

- a fixes (v, e) but takes f to the other incident face f'
- b fixes (e, f) but takes v to the other incident vertex v'
- c fixes (v, f) but takes e to the other incident edge e'



These generate the automorphism group $\mathrm{Aut}\, M$, and satisfy the full $(2, k, m)$ triangle group relations

$$a^2 = b^2 = c^2 = (ab)^2 = (bc)^k = (ac)^m = 1.$$

Conversely ...

Given any group epimorphism $\theta\colon \Delta(2,k,m) \to G$ with torsion-free kernel K, a (fully) regular map M may be constructed with automorphism group G:

Take as vertices of M the right cosets of $K\langle a,c\rangle$, edges the right cosets of $K\langle a,b\rangle$, and faces the right cosets of $F = K\langle b,c\rangle$, and let incidence be non-empty intersection. Then $\Delta(2,k,m)$ acts on M by right multiplication, inducing the automorphism group $\Delta(2,k,m)/K \cong G = \mathrm{Aut}\,M$.

Thus regular maps of type $\{k,m\}$ correspond to non-degenerate homomorphic images of the full $(2,k,m)$ triangle group.

Dual map

If M is regular of type $\{k, m\}$ then its geometric dual M^D (obtainable by interchanging the roles of vertices and faces) is also regular, of type $\{m, k\}$, with $\mathrm{Aut}\, M^D \cong \mathrm{Aut}\, M$.

Note: this is not the same as the dual of a 3-polytope!

Petrie dual

Take a walk on a regular map M, and at every vertex met, turn immediately left or immediately right, in an alternating fashion. This is called a Petrie polygon. The Petrie polygons of M are the faces of a new map M^P, the Petrie dual of M.

If M is regular of type $\{k, m\}$ with Petrie polygons of length q, then its Petrie dual M^P is also regular, of type $\{q, m\}$, with Petrie polygons of length k, and $\mathrm{Aut}\, M^P \cong \mathrm{Aut}\, M$.

(Standard) Coxeter/Wilson operators

The mappings $M \mapsto M^D$ and $M \mapsto M^P$ are operators on the family of all regular maps. Each has order 2. The composite operators DP and PD are 'triality' operators, of order 3. The map M^{DPD} ($= M^{PDP}$) is called the 'opposite' of M.

If M has type $\{k, m\}_q$ (where $q =$ length of Petrie polygons)

then

- M^D has type $\{m, k\}_q$
- M^P has type $\{q, m\}_k$
- M^{DP} has type $\{q, k\}_m$
- M^{PD} has type $\{m, q\}_k$
- M^{DPD} has type $\{k, q\}_m$

so the operators D and P generate a group of order 6 and give all permutations of the three parameters k, m and q.

Correspondence with automorphisms of Aut M

The operators D and P can also be viewed as transforming the generating triple (a, b, c) for the automorphism group of the regular map M. These generators satisfy the relations

$$a^2 = b^2 = c^2 = (ab)^2 = (bc)^k = (ac)^m = (abc)^q = 1.$$

The dual operator D takes (a, b, c) to (b, a, c), swapping bc with ac, and taking abc to $bac = (cba)^c$.

The Petrie dual P takes (a, b, c) to (a, ab, c), preserving ac, and swapping bc with abc.

The triality operator DP takes (a, b, c) to (ab, a, c), and so induces a 3-cycle $bc \mapsto ac \mapsto abc \mapsto bc$.

The opposite DPD takes (a, b, c) to (ab, b, c), preserving bc, and swapping ac with abc.

Coxeter/Wilson 'hole' operators

Given a d-valent map M, and any integer e coprime to d, we may define the power map M^e by taking M and replacing the cyclic rotation of edges at each vertex on the surface with the eth power of that rotation.

This stems from the concept of 'holes' for a regular map, introduced by Coxeter (1937) and extended by Wilson (1979) and also by Nedela and Škoviera (1997) in their work on exponents of orientable maps.

If M is regular, then so is M^e, with the same underlying graph, and the same automorphism group. Taking $M \mapsto M^e$ takes the vertex-stabilizing automorphism ac to $(ac)^e$, and the canonical generating triple (a, b, c) to $(a, b, a(ac)^e)$.

Kaleidoscopic maps with trinity symmetry

If the map M is self-dual and self-Petrie, then M is invariant under all six of the 'standard' Wilson operators (in the group generated by D and P), and we say M has trinity symmetry.

If the map M is isomorphic to all of its power maps M^e (for e coprime to the valence), then M is invariant under all of the Wilson 'hole' operators, and we say M is kaleidoscopic.

Maps that are regular, kaleidoscopic and have trinity symmetry are in a sense the most highly symmetric of all.

Questions: Do such maps exist? and for what valences? How large is the group generated by all the map operators?

Examples

- A 2-cycle embedded on the sphere, with type $\{2,2\}_2$
- A regular map of type $\{4,4\}_4$ on the torus
- A regular map of type $\{6,6\}_6$ on a surface of genus 10

Theorem [Archdeacon, Conder & Siráň (2010)]

For every positive integer n, there exists a kaleidoscopic regular map of type $\{2n,2n\}_{2n}$ with trinity symmetry on an orientable surface of genus $n^3 - 2n^2 + 1$.

The existence of this family was conjectured by Steve Wilson as a PhD student in 1976, without the extra kaleidoscopic assumption. The theorem can be proved with the help of some combinatorial group theory.

New examples from old

We also have a construction that takes an orientable regular kaleidoscopic map of degree d with trinity symmetry, and produces from it a regular kaleidoscopic map of degree dn with trinity symmetry, for every positive integer n.

Question: Is there an example of odd valence?

Answer: Yes! There's an example of type $\{15, 15\}_{15}$ on a non-orientable surface (of large genus), with automorphism group $A_5 \times A_5 \times A_5$ [MC, 2010]

Open question: Are there any with odd prime valence?

How large can the operator group be?

For any regular map with trinity symmetry, the standard Wilson operators D and P generate a group of order 6.

For any kaleidoscopic regular map of valence k, the Wilson hole operators H_e generate a group of order $\phi(k)$, where ϕ is Euler's ϕ-function.

Theorem [MC, Young Soo Kwon & Jozef Siráň (2011)]
Let $\omega(n)$ be the number of prime divisors of n. Then for the kaleidoscopic regular map M_n of type $\{2n, 2n\}_{2n}$ with trinity symmetry, the set of all the Wilson operators generates a group of order $6(\phi(2k))^3/2^i$ where $i = \omega(n)$ if $n \not\equiv 0$ mod 4, $i = \omega(n) + 1$ if $n \equiv 4$ mod 8, or $i = \omega(n) + 2$ if $n \equiv 0$ mod 8.

Question: Is the order of the operator group bounded by the valence?

Answer: No!

Theorem [MC, Young Soo Kwon & Jozef Siráň (2011)]

For every positive integer n, there exists a kaleidoscopic regular map of type $\{8, 8\}_8$ with trinity symmetry, having automorphism group of order $128n^{16}$, and operator group of order divisible by $48n$.

Hence even for fixed valence, the operator group can be arbitrarily large.

Regular Cayley maps for cyclic groups

If the underlying graph of the orientably-regular map M is a Cayley graph for some group A, or equivalently, if M admits an orientation-preserving group A that acts regularly on the vertices of M, then M is a *regular Cayley map* for A

— e.g. the standard regular embedding of the cube Q_3 in the sphere is a regular Cayley map for the dihedral group D_4, while a regular embedding of Q_3 in the torus is a regular Cayley map for the abelian group $C_2 \times C_2 \times C_2$.

With either definition, the embedding prescribes an order on the generating set X. Certain kinds of orderings give RCMs that are balanced, anti-balanced, or t-balanced for some t.

Classification by 'CD structure'

Define C to be the commutator subgroup of the rotation group $G = \text{Aut}^+M$, and D the normal subgroup generated by all conjugates of the edge-reversing generator x in G.

Then $C = G'$ is cyclic (by some group theory), and so every subgroup of C is normal in G. On the other hand, Y is core-free in G, so it follows that $C \cap Y = \{1\}$.

Next, G/D is cyclic (generated by Dy), so $C = G' \subseteq D$, and $G = DY$. Also G/D can be obtained from $G/G' = G/C$ by 'killing' Cx, so C has index at most 2 in D.

By considering two cases ($G = CY$ and $G \neq CY$), we get the following:

- If n is odd, then $G = CY$ with $C \cap Y = \{1\}$ and $C \cong C_n$, and $G = \langle v, y \,|\, v^n = y^s = 1,\, yvy^{-1} = v^r \rangle$, where $r \in \mathbb{Z}_n$ is a root of -1 mod n, of multiplicative order s, and $x = vy^{s/2}$.

- If $n = 2m$, then $G = DY$ with $D \cap Y = \{1\}$ and $D \cong D_m$, and $G = \langle x, v, y \,|\, v^m = x^2 = y^s = 1,\, xvx = v^{-1},\, yvy^{-1} = v^r,\, yxy^{-1} = xv \rangle$, where r is a unit in $\mathbb{Z}_m$, and s is the order of the automorphism of $\langle x, v \rangle$ induced by conjugation by y.

These are necessary conditions. We must also consider the values of r that make them sufficient for a group with the given presentation to be the group of some RCM for C_n, which we denote as $M(n, r)$.

Complete classification of RCMs for C_n

[MC & Tom Tucker (2009), published in TAMS 366 (2014)]

For odd n, every regular Cayley map for C_n is isomorphic to $M(n, r)$ for exactly one root r of -1 mod n.

For $n = 2m$ (even), every regular Cayley map for C_{2m} is isomorphic to $M(2m, r)$ for exactly one unit r mod m with the property that if b is the largest divisor of m coprime to $r - 1$, then either $b = 1$, or r is a root of -1 mod b of multiplicative order $2k$ where k is coprime to m/b.

We also know the types, genus, reflexibility and number of these maps (for given n), and can tell when each has a balanced (or t-balanced) representation.

Regular maps with simple underlying graphs

Let us call a map simple if its underlying graph is simple.

- Every Platonic map on the sphere is simple and reflexible
- Almost all orientably-regular maps on the torus are simple
- Just one of the ten orientably-regular maps of genus 2 is simple, namely the one of type $\{8,3\}$ ... and hence there is no orientably-regular map M of genus 2 such that both M and its dual $M*$ are simple
- 7 of the 20 orientably-regular maps of genus 3 are simple
- There is no (fully) regular map of genus 20 that is simple (but there exists a simple chiral map of genus 20).

On the other hand ...

The census of regular maps [on MC's website] shows that for every small g ($0 \le g \le 301$), there is at least one simple orientably-regular map of genus g.

For example, when $g = p + 1 = 20, 32, 38, 44, 62, 68, 74, 80$ or 98, there is no simple regular orientable map of genus g, but there exists at least one chiral example.

Question(s): Just how many g can we cover? and how?

One nice family (covering genera $\equiv 0 \bmod 3$)

It is well known that an infinite family of regular maps of type $\{3n, 4\}$ can be constructed as cyclic regular coverings of the octahedral map (e.g. Maclachlan (1969)).

For any positive integer n, form the semi-direct product $U = C_{3n} \rtimes S_4$ of a cyclic group $C_{3n} = \langle\, w \mid w^{3n} = 1 \,\rangle$ by the symmetric group S_4, with conjugation of C_{3n} by S_4 given by $w^u = w^{-1}$ whenever $u \in S_4 \setminus A_4$ (i.e. whenever u is odd).

Then the elements $R = w(1,2,3)$ and $S = (1,4,3,2)$ have orders $3n$ and 4, with $R^3 = w^3$ and $(RS)^2 = (w(3,4))^2 = 1$, so the subgroup $G = \langle R, S\rangle$ has order $24n$ and is the rotation group of an orientably-regular map of type $\{3n, 4\}$, characteristic $\chi = 8 - 12n + 6n = 8 - 6n$ and genus $g = 3n - 3$.

And this map is simple, since $\langle y^2\rangle$ is not normal in $\langle x, y\rangle$.

Another family (covering genera $\equiv 1 \bmod 4$)

Take the group $U = \langle\, r, s \mid s^4 = (rs)^2 = [r^2, s^2] = 1 \,\rangle$.

In this group, the third relation is equivalent to assuming that r^4 is inverted under conjugation by s. It follows that r^4 generates a cyclic normal subgroup K of index 32 in U, with quotient U/K isomorphic to the group H from earlier.

Also by Reidemeister-Schreier theory, the normal subgroup K is infinite. Hence for each positive integer n, we can factor out the normal subgroup generated by r^{4n}, and get a quotient G which is an extension of C_n by H. This quotient group G is the rotation group of an orientably-regular map of type $\{4n, 4\}$ and genus $g = 4n - 3$, again with simple underlying graph.

Another family (covering genera $\equiv 3 \bmod 4$)

Take the group $U = \langle\, r, s \mid s^4 = (rs)^2 = [r^2, s^2] = 1 \,\rangle$ from the previous case, and this time, for any positive integer n, factor out the cyclic normal subgroup K generated by r^{8n}.

In the resulting quotient, the elements r^{4n} and $[r, s^2]$ are central involutions, and therefore so is their product $r^{4n}[r, s^2]$.

The quotient of U by the central subgroup N generated by this third involution has order $32n$, generated by two elements of orders $8n$ and 4 with product of order 2. Thus $G = U/N$ is the rotation group of a simple orientably-regular map of type $\{8n, 4\}$, with characteristic $\chi = 4 - 16n + 8n$, and genus $g = 4n - 1$.

Families covering 83.3% of all genera

Class	Genus	Type	$\lvert \mathrm{Aut}^{\mathrm{o}} M \rvert$
$g \equiv 0 \bmod 3$	$g = 3n - 3$	$\{3n, 4\}$	$24n$
$g \equiv 1 \bmod 4$	$g = 4n - 3$	$\{4n, 4\}$	$32n$
$g \equiv 3 \bmod 4$	$g = 4n - 1$	$\{8n, 4\}$	$32n$
$g \equiv 4 \bmod 18$	$g = 18\ell+4$	$\{6(3\ell+1), 6\}$	$36(3\ell+1)$
$g \equiv 10 \bmod 18$	$g = 18\ell+10$	$\{6(3\ell+2), 6\}$	$36(3\ell+2)$
$g \equiv 16 \bmod 18$	$g = 18\ell-2$	$\{18\ell, 6\}$	108ℓ

These families cover all genera except for $g \equiv 2 \bmod 6$ ...
in order words, $15/18 = 5/6$ of all positive integers.

All the maps in the first five families are reflexible (and hence fully regular), while those in the sixth family are chiral.

Theorem [MC & Jicheng Ma (2012)]

For every $g \not\equiv 2$ mod 6, there exists at least one orientably-regular map of genus g with simple underlying graph. Moreover, if also $g \not\equiv 16$ mod 18, then there exists at least one fully regular map of genus g with simple underlying graph.

In a sense this is the best possible for the fully regular case, since we know there are infinitely many genera $g \equiv 2$ mod 6 (with $g = p + 1$ for prime $p \equiv 1$ mod 6) for which there exists no fully regular map of genus g with simple underlying graph, and also we now know that the same holds for many $g \equiv 16$ mod 18 (such as $g = 142, 178, 214$ and 250).

Open question: What about genus $g \equiv 2$ mod 6?

Conjecture [MC]: There exists an orientably-regular map of genus g with simple underlying graph, for every $g \geq 0$.

Additional comment: Jicheng Ma and Wei-Juan Zhang are now applying the same covering techniques to regular and chiral polytopes (for ranks greater than 3).

Polytopal regular maps [MC & Deborah Oliveros]

A fully regular map is called polytopal if it can also be viewed as a regular 3-polytope.

This means that the Hasse diagram showing the partial ordering of vertices, edges and faces (given by incidence) is strongly flag-connected and satisfies a diamond condition.

In algebraic terms, this can be shown to be equivalent to the vertex-stabiliser $V = \langle b, c \rangle$, edge-stabiliser $E = \langle a, c \rangle$ and face-stabiliser $F = \langle a, b \rangle$ satisfying the intersection condition

$$\langle a, b \rangle \cap \langle b, c \rangle = \langle b \rangle,$$

which itself requires that $G = \langle a, b, c \rangle$ has no non-trivial cyclic normal subgroup contained in $\langle ab \rangle$ or $\langle bc \rangle$.

As a consequence, we have the following ...

Corollary 1: If G is a finite simple group, or more generally, has no non-trivial cyclic normal subgroups (e.g. A_n or S_n for some n), then G is the automorphism group of a regular 3-polytope of type $\{k, m\}$ whenever G is a smooth quotient of the full $(2, k, m)$ triangle group.

Corollary 2: For every non-negative integer g, there exists a polytopal regular map on an orientable surface of genus g.

Proof. There exists a family of groups G_n of order $16n$ (for $n \in \mathbb{Z}^+$), with each G_n being a smooth quotient of the full $(2, 4, 2n)$ triangle group, and they satisfy the IC since they have no cyclic normal subgroups of the kind not allowed.

These are 'Accola-Maclachlan' maps $\mathcal{AM}_n$ (of genus $n-1$).

Orientably-regular simple maps with nilpotent automorphism groups

This topic was covered by Roman Nedela on Tuesday.

Theorem: There exists a known sequence $(f_1, f_2, f_3, \ldots)$ of positive integers such that for every integer $c \geq 1$ there exists a unique nilpotent regular map $\mathcal{M}_c$ of class c with simple underlying graph, with 2^{1+f_c} vertices, and type $\{2^c, 2^{c-1}\}_{2^c}$.

Furthermore, this map $\mathcal{M}_c$ is 'universal', in the sense that every nilpotent regular map of class at most c with simple underlying graph is a quotient of $\mathcal{M}_c$. The proof indirectly uses the standard regular embedding of $K_{m,m}$ when $m = 2^c$, to confirm the existence and order of $\mathrm{Aut}^+(\mathcal{M}_c)$.

The sequence (f_n) can be defined as follows:
Set $h_1 = 0$, and then for $n \geq 2$ take

$$h_n = \frac{1}{n} \sum_{d|n} \mu(n/d) \Big(\sum_{0 \leq i \leq d/3} (-1)^i \frac{d}{d-2i} \binom{d-2i}{i} 2^{d-3i} \Big),$$

$$g_n = h_1 + h_2 + \cdots + h_n = \sum_{j=1}^{n} h_n,$$

$$f_n = g_1 + g_2 + \cdots + g_n = \sum_{i=1}^{n} g_n,$$

where μ is the number-theoretic Mobius function.

The first few terms of these sequences are given below:

h_n : 0, 1, 1, 1, 2, 2, 4, 5, 8, 11, 18, 25, 40, ...

g_n : 0, 1, 2, 3, 5, 7, 11, 16, 24, 35, 53, 78, 118, ...

f_n : 0, 1, 3, 6, 11, 18, 29, 45, 69, 104, 157, 235, 353, ...

Chiral maps/polyhedra of given type

By an amazing piece of work of Murray Macbeath (1969), it is known that for every hyperbolic pair (k, m) of positive integers (with $1/k + 1/m < 1/2$), there exist infinitely many orientably-regular maps of type $\{k, m\}$ (with rotation groups PSL$(2, p)$ for various primes p). All of these maps are reflexible, and hence full regular.

Question [Singerman (1992)]: What about chiral maps?

Theorem [Bujalance, MC & Costa (2010)]: For every $\ell \geq 7$, all but finitely many A_n are the automorphism group of an orientably-regular but chiral map of type $\{3, \ell\}$.

New Theorem (2014):

For every hyperbolic pair (k, m), there exist infinitely many orientably-regular but chiral maps of type $\{k, m\}$.

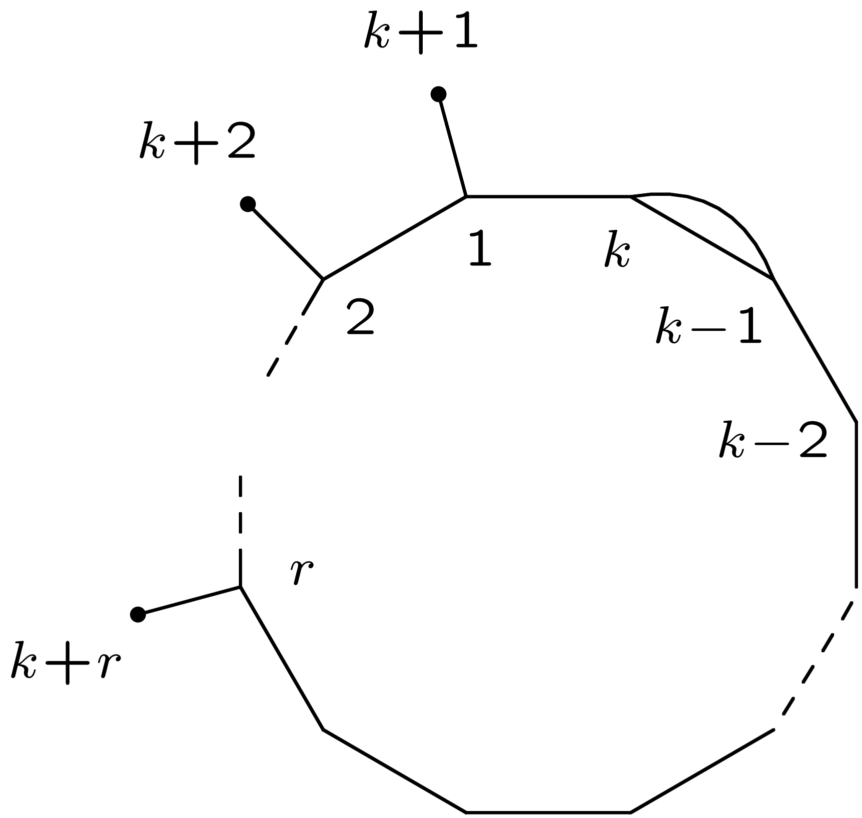
One 'base' example for each type can be found by

- constructing permutation representations of the ordinary $(2, k, m)$ triangle group [MC, Hucíková, Nedela & Širáň],

or by

- using group representations and the theory of differentials on Riemann surfaces [Jones].

Example:



Generating permutations for type $\{k,m\}$ in S_n

where $n = m+1 = k+r$ when $k+2 \le m \le 2k-4$

Then infinitely many more examples of each type $\{k,m\}$ can be constructed by the 'Macbeath trick' for abelian p-covers:

Suppose $G \cong \Delta^+(2,k,m)/K$ is the automorphism group of a chiral map M of type $\{k,m\}$ and genus g, where K is the fundamental group of the carrier surface of M. Then K has presentation $\langle\, a_1, b_1, \ldots, a_g, b_g \mid [a_1,b_1][a_2,b_2]\ldots[a_g,b_g] = 1\,\rangle$.

Now for any prime p not dividing $|G|$, let $K^{(p)}$ be the subgroup of $\Delta^+(2,k,m)$ generated by the pth powers of all elements of K. Then $[K,K]K^{(p)}$ has index p^{2g} in K, and is characteristic in K and therefore normal in $\Delta^+(2,k,m)$, and a little extra theory shows that $\Delta^+(2,k,m)/[K,K]K^{(p)}$ is the automorphism group of a chiral cover of M.

Other consequences

1) **Chiral maps with simple underlying graphs**:

The automorphism groups of the 'base' examples of both kinds are 'almost-simple', and in particular, have no cyclic normal subgroups. It follows that the vertex- and face-stabilisers are core-free in the automorphism group of the map, and hence for every hyperbolic pair (k, m), there exist at least two orientably-regular maps of type $\{k, m\}$, one reflexible and one chiral, such that both the map and its dual have simple underlying graph.

In fact there are infinitely many of each kind, and the same is known for the toroidal case (with $1/k + 1/m = 1/2$).

2) **Chiral polyhedra of every hyperbolic type**:

Also in each of these maps, every edge has two vertices and every edge lies in two faces, and therefore the maps are abstract polyhedra. Thus we have the following as well:

For every pair (k,m) of integers with $1/k+1/m \le 1/2$, there exist infinitely many regular and infinitely many orientably-regular but chiral polyhedra of type $\{k,m\}$.

3) We can also use similar methods to prove that for every hyperbolic pair (k,m), there exists a **fully regular map** of type $\{k,m\}$ with orientation-preserving automorphism group isomorphic to A_n or S_n for some n. This strengthens a theorem of Mushtaq and Servatius (1993) and part of a much bigger theorem on Fuchsian groups by Everitt (2000).

Some other important recent developments

- Chirality groups and regular coverings of regular oriented hypermaps [Breda, Rodrigues & Fernandes (2011)]
- Self-dual and self-Petrie-dual regular maps [Richter, Širáň & Wang (2012)]
- Regular maps with $4p$ vertices [Zhou & Feng (2012)]
- Non-orientable regular embeddings of Hamming graphs [Jones & Kwon (2012)]
- Non-orientable regular maps over linear fractional groups [Jones, Mačaj & Širáň (2013)
- Characterisations and Galois conjugacy of generalised Paley maps [Jones (2013)]
- Orientably regular maps with Euler characteristic divisible by few primes [Gill (2013)]

THANK YOU

THANK YOU